On non-modular n-distributive lattices I. Lattices of convex sets

A. P. HUHN

1. Introduction. A lattice is called n-distributive if it satisfies the identity

$$(1)\qquad x\wedge \bigvee_{i=0}^{n} y_i = \bigvee_{j=0}^{n} [x\wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i].$$

A lattice satisfying the dual of (1) is called dually n-distributive. The class of n-distributive (respectively, dually n-distributive) lattices is denoted by Δ_n (respectively, ∇_n). n-distributive lattices were introduced to describe dimension like properties of modular lattices. Here we present some examples of non-modular n-distributive lattices. E^{n-1} denotes the $(n-1)$-dimensional Euclidean space and $\mathfrak{L}(E^{n-1})$ denotes its lattice of convex sets. Our first result describes how $\mathfrak{L}(E^{n-1})$ is situated in the classes Δ_m and ∇_m.

Theorem 1.1. $\mathfrak{L}(E^{n-1})\in(\Delta_n\setminus\Delta_{n-1})\cap(\nabla_n\setminus\nabla_{n-1})$.

The proof of n-distributivity in Section 2 is based on Carathéodory's theorem, while the dual n-distributivity is derived from Helly's theorem.

In Section 3 we strengthen part of this result. Let F denote the class of finite lattices.

Theorem 1.2. $\mathfrak{L}(E^{n-1})\in \mathrm{HSP}(\Delta_n\cap F)$.

In other words, $\mathfrak{L}(E^{n-1})$ is in the lattice variety (equational class) generated by the finite n-distributive lattices. The intuitive reason for Theorem 1.2 is that, if we restrict the operation of convex closure to a finite subset H of E^{n-1}, then this closure system has an n-distributive lattice of closed sets by Carathéodory's theorem, and this lattice resembles $\mathfrak{L}(E^{n-1})$ as H becomes large. We note that $\mathfrak{L}(E^{n-1})$ is also in the

Received June 5, 1985.

class $\mathrm{HSP}(\nabla_n \cap F)$. The proof of this theorem involves more geometry and will be published separately together with other Helly-type results.

Notice that the above sketch of the proof of Theorem 1.2 gives rise to a high variety of n-distributive lattices: associated with any finite subset of E^{n-1} there is an n-distributive lattice. The example given by the following theorem is of different character. Let $\bar{\mathfrak{L}}(E^{n-1})$ denote the lattice of closed convex sets of E^{n-1}. In Section 4 we prove:

Theorem 1.3. $\bar{\mathfrak{L}}(E^{n-1}) \in (\Delta_n \setminus \Delta_{n-1}) \cap (\nabla_n \setminus \nabla_{n-1})$.

Carathéodory's theorem provides also a new aspect to the study of modular n-distributive lattices. In Section 5 we characterize complete, complemented, modular, completely n-distributive lattices among all projective geometries as those satisfying a Carathéodory type condition. (Completely n-distributive lattices are defined in Section 5 in analogy with completely distributive lattices.) An unexpected consequence of our characterization is that this class of lattices (as well as the corresponding class of projective geometries) is self-dual.

Finally, in Section 6 we prove the following fact on modular n-distributive lattices:

Theorem 1.4. *Every modular n-distributive lattice is a member of* $\mathrm{HSP}(\Delta_n \cap F)$.

It is now natural to ask whether there are any further examples of non-modular n-distributive lattices in other branches of mathematics. It is not hard to show that the partition lattice of an $(n+1)$-element set is in $(\Delta_n \setminus \Delta_{n-1}) \cap (\nabla_n \setminus \nabla_{n-1})$. This example will be developed further in Part II of this paper, where graphs with an n-distributive (respectively, dually n-distributive) contraction lattice are characterized. Partition lattices occur as special cases, as they are the contraction lattices of complete graphs.

In an independent paper [3] Horst Gerstmann also considers nonmodular n-distributive lattices, defines complete and infinite n-distributive laws and characterizes the different sorts of n-distributivity of the closed sets of a closure space in terms of properties of the closure operator. Gerstmann's generalized distributive laws cover, beside the n-distributive laws, the concepts of (von Neumann) $\wedge$-continuity and of Scott-continuity.

2. The lattice of convex sets. We first quote the two classical theorems that are in the centre of this paper.

Helly's theorem. *Let $\mathscr{C}$ be a finite family of convex subsets of E^{n-1}. If any n elements of $\mathscr{C}$ have a non-empty intersection, then the intersection of the whole family $\mathscr{C}$ is not empty.*

Carathéodory's theorem. *Let H be a subset of E^{n-1} and let p be a point in E^{n-1}. If p is in the convex closure of H, then it is in the convex closure of an n element subset of H.*

We first prove that $\mathfrak{L}(E^{n-1})$ is n-distributive. Let $X, Y_0, Y_1, \ldots, Y_n \in \mathfrak{L}(E^{n-1})$. Let p be a point of E^{n-1} and assume that

$$p \in X \wedge \bigvee_{i=0}^{n} Y_i$$

(where the $\wedge$ and $\vee$ are the operations of $\mathfrak{L}(E^{n-1})$). Then, by Carathéodory's theorem there are n elements of the set union $\bigcup_{i=0}^{n} Y_i$, say $p_0, p_1, \ldots, p_{n-1}$, such that p is an element of their convex closure. If $p_j \in Y_{i_j}$, $j=0, 1, \ldots, n-1$, then p is also in $\bigvee_{j=0}^{n-1} Y_{i_j}$. Of course, $p \in X$, hence

$$p \in \bigvee_{j=0}^{n} [X \wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} Y_i],$$

that is,

$$X \wedge \bigvee_{i=0}^{n} Y_i \subseteqq \bigvee_{j=0}^{n} [X \wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} Y_i].$$

The reverse inclusion is obvious.

Now we prove that the dual n-distributive law holds in $\mathfrak{L}(E^{n-1})$. Let $X, Y_0, Y_1, \ldots, Y_n \in \mathfrak{L}(E^{n-1})$. Let

$$p \in \bigwedge_{j=0}^{n} [X \vee \bigwedge_{\substack{i=0\\ i\neq j}}^{n} Y_i].$$

Then there exist points $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ such that

$$x_j \in X, \quad y_j \in \bigwedge_{\substack{i=0\\ i\neq j}}^{n} Y_i, \quad j = 0, 1, \ldots, n$$

and p is a convex linear combination of each pair x_j, y_j. Now a trivial induction over k yields that, whenever y is a convex linear combination of $y_0, y_1, \ldots, y_k$ $(k \leqq n)$ then there is a convex linear combination x of $x_0, x_1, \ldots, x_k$ such that p is a convex linear combination of x and y.

We are ready to apply Helly's theorem. Let Y_i' be the convex closure of $\{y_0, \ldots \ldots, y_{i-1}, y_{i+1}, \ldots, y_n\}$. Then

$$y_j \in \bigwedge_{\substack{i=0\\ i\neq j}}^{n} Y_i', \quad j = 0, 1, \ldots, n.$$

By Helly's theorem, the intersection of the Y_i' is not empty. Let

$$y \in \bigwedge_{i=0}^{n} Y_i'.$$

y is a convex linear combination of, say, $y_0, y_1, \ldots, y_{n-1}$. Applying our last observation, there is an x in the convex closure of $x_0, x_1, \ldots, x_{n-1}$ (hence also in X) such that p is in the convex closure of x and y:

$$p \in X \vee \bigwedge_{i=0}^{n} Y_i' \subseteqq X \vee \bigwedge_{i=0}^{n} Y_i,$$

as claimed.

Finally, $\mathfrak{L}(E^{n-1})$ is not $(n-1)$-distributive, as the following counterexample shows: Let S be a simplex, let $x \in S$ such that x is not contained in any $(n-2)$-dimensional face of S, and let $y_0, y_1, \ldots, y_{n-1}$ be the extremal points of S. Then

$$\{x\} \wedge \bigvee_{i=0}^{n-1} \{y_i\} = \{x\} \neq \emptyset = \bigvee_{j=0}^{n-1} \Big[\{x\} \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n-1} \{y_i\}\Big].$$

$\mathfrak{L}(E^{n-1})$ is not dually $(n-1)$-distributive either: Let X be a closed halfspace disjoint from S (S is also closed) and let $Y_0, Y_1, \ldots, Y_{n-1}$ be the $(n-2)$-dimensional faces of S. Then

$$X \vee \bigwedge_{i=0}^{n-1} Y_i = X \vee \emptyset = X,$$

which is a proper part of

$$\bigwedge_{j=0}^{n-1} \Big[X \vee \bigwedge_{\substack{i=0 \\ i \neq j}}^{n-1} Y_i\Big] = \bigwedge_{j=0}^{n-1} \big[X \vee \{y_j\}\big].$$

3. On the variety generated by all finite n-distributive lattices. In this section we prove Theorem 1.2 via the following three lemmas.

Lemma 3.1. $\mathfrak{L}(E^{n-1}) \in \mathrm{HSP}(\mathfrak{L}_{\mathrm{fin}}(E^{n-1}))$. *where* $\mathfrak{L}_{\mathrm{fin}}(E^{n-1})$ *denotes the set of all those convex sets of* E^{n-1} *that are the convex closures of a finite set of points.*

Proof. Every element of $\mathfrak{L}(E^{n-1})$ is a join of atoms and every atom of $\mathfrak{L}(E^{n-1})$ is compact by Carathéodory's theorem. Thus $\mathfrak{L}(E^{n-1})$ is algebraic. Furthermore, its compact elements are exactly the elements of $\mathfrak{L}_{\mathrm{fin}}(E^{n-1})$. Hence $\mathfrak{L}(E^{n-1})$ is isomorphic to the ideal lattice of $\mathfrak{L}_{\mathrm{fin}}(E^{n-1})$, whence it is in the variety generated by $\mathfrak{L}_{\mathrm{fin}}(E^{n-1})$.

In the above proof we implicitely made use of the fact that $\mathfrak{L}_{\mathrm{fin}}(E^{n-1})$ is a sublattice of $\mathfrak{L}(E^{n-1})$, that is, the intersection of two convex polytopes is a convex polytope, otherwise we could not have spoken of the *lattice* $\mathfrak{L}_{\mathrm{fin}}(E^{n-1})$.

Now let H be any finite subset of E^{n-1}, and let $\mathfrak{L}(H)$ denote the set of all those subsets X of H which are of the form $X=C\cap H$ with $C\subseteqq E^{n-1}$ convex. Clearly

$$\mathfrak{L}(H) = \{X(\subseteqq H) \mid X = (\operatorname{conv} X)\cap H\},$$

where "conv" denotes the operator associating with any set its convex hull. Now it is clear that $\mathfrak{L}(H)$ is a lattice relative to the inclusion and its operations $\vee^H$ and $\wedge^H$ are as follows.

$$X\vee^H Y = (\operatorname{conv} X \vee \operatorname{conv} Y)\cap H,$$

$$X\wedge^H Y = (\operatorname{conv} X \wedge \operatorname{conv} Y)\cap H = X\cap Y,$$

where $\vee$ and $\wedge$ are the operations in $\mathfrak{L}(E^{n-1})$.

Lemma 3.2. *$\mathfrak{L}(H)$ is n-distributive.*

Proof. Assume that $X, Y_0, Y_1, \ldots, Y_n \in \mathfrak{L}(H)$, $p\in H$, and

$$p\in X\wedge^H \bigvee_i{}^H Y_i.$$

As in the proof of Theorem 1.1, Carathéodory's theorem and the descriptions of $\vee^H$ and $\wedge^H$ before the Lemma yield that there is a $j\in\{0, 1, \ldots, n\}$ such that

$$p\in \bigvee_{\substack{i \\ i\neq j}}{}^H Y_i,$$

that is,

$$p\in \bigvee_j{}^H [X\wedge^H \bigvee_{\substack{i \\ i\neq j}}{}^H X_i],$$

proving the lemma.

The following lemma finishes the proof of Theorem 1.2.

Lemma 3.3. $\mathfrak{L}_{\mathrm{fin}}(E^{n-1})\in \mathbf{HSP}(\mathfrak{L}(H) | H\subseteqq E^{n-1}, |H|<\aleph_0)$.

Proof. Let $\mathscr{H}=\{H | H\subseteqq E^{n-1}, |H|<\aleph_0\}$. Let

$$L = \prod_{H\in\mathscr{H}} \mathfrak{L}(H),$$

and let M consist of all $a\in L$ for which there is a $P\in\mathfrak{L}_{\mathrm{fin}}(E^{n-1})$ with the property that for some $H_0\in\mathscr{H}$ and for all $H\in\mathscr{H}$ containing H_0, we have $a(H)=H\cap P$. If $a\in M$ and P has the above property, then P is called a support of a. The support of a is uniquely determined. Indeed, if $P\neq P'\in\mathfrak{L}_{\mathrm{fin}}(E^{n-1})$, $H_0, H_0'\in\mathscr{H}$, $a(H)=P\cap H$ for all $H_0\subseteqq H\in\mathscr{H}$ and $a(H)=P'\cap H$ for all $H_0'\subseteqq H\in\mathscr{H}$ then extend $H_0\cup H_0'$ to an $H\in\mathscr{H}$ that contains an element from the symmetric difference $P\triangle P'$. For this H we have $a(H)=P\cap H\neq P'\cap H=a(H)$, a contradiction.

We first prove that M is a sublattice of L. Let $a, b \in M$, let P_a and P_b be the supports of a and b, respectively, and choose H_a and H_b such that

$$a(H) = H \cap P_a \quad \text{if} \quad H_a \subseteqq H \in \mathscr{H}$$

and

$$b(H) = H \cap P_b \quad \text{if} \quad H_b \subseteqq H \in \mathscr{H}.$$

Let $H_0 \in \mathscr{H}$ contain the sets H_a and H_b and the sets of extremal points of P_a and of P_b. Then we have

$$\operatorname{conv}(H \cap P_a) = P_a, \quad \operatorname{conv}(H \cap P_b) = P_b$$

whenever $H_0 \subseteqq H \in \mathscr{H}$. Compute the values of $a \vee b$ and $a \wedge b$ at H (H as above).

$$(a \vee b)(H) = a(H) \vee^H b(H) = (H \cap P_a) \vee^H (H \cap P_b) =$$
$$= (\operatorname{conv}(H \cap P_a) \vee \operatorname{conv}(H \cap P_b)) \cap H = (P_a \vee P_b) \cap H.$$

Clearly $P_a \vee P_b \in \mathfrak{L}_{\text{fin}}(E^{n-1})$, whence $a \vee b \in M$,

$$(a \wedge b)(H) = a(H) \wedge^H b(H) = (H \cap P_a) \cap (H \cap P_b) = H \cap (P_a \wedge P_b).$$

Applying that $P_a \wedge P_b \in \mathfrak{L}_{\text{fin}}(E^{n-1})$, we obtain that $a \wedge b \in M$.

We have also obtained that the map $M \to \mathfrak{L}_{\text{fin}}(E^{n-1})$, $a \mapsto P_a$ is a lattice homomorphism. For any $P \in \mathfrak{L}_{\text{fin}}(E^{n-1})$, P is the support of the choice function a defined by $a(H) = P \cap H$. Hence $\mathfrak{L}_{\text{fin}}(E^{n-1})$ is a homomorphic image of M, which completes the proof.

4. The lattice of closed convex sets. In this section we prove Theorem 1.3. The operations of $\overline{\mathfrak{L}}(E^{n-1})$ will be denoted as sum and product. Obviously, $XY = X \wedge Y$ and $X + Y$ is the topological closure of $X \vee Y$ if $X, Y \in \overline{\mathfrak{L}}(E^{n-1})$. Choose a point

$$p \in X \sum_{i=0}^{n} Y_i,$$

where $X, Y_0, Y_1, \ldots, Y_n \in \overline{\mathfrak{L}}(E^{n-1})$. Then $p \in X$ and $p = \lim_{m \to \infty} p_m$ for some $\{p_m\}_{m \in \mathbf{N}} \subseteqq \bigvee_{i=0}^{n} Y_i$. By Carathéodory's theorem, for every $m \in N$ there is a $j(m) \in \{0, 1, \ldots, n\}$ such that $p_m \in \bigvee_{\substack{i=0 \\ i \neq j(m)}}^{n} Y_i$. For at least one $k \in \{0, 1, \ldots, n\}$, $k = j(m)$ for infinitely many $m \in N$. Therefore, the subsequence $\{p_m\}_{j(m)=k}$ of $\{p_m\}_{m \in \mathbf{N}}$ is infinite and converges to p. Besides $p_m \in \bigvee_{\substack{i=0 \\ i \neq k}}^{n} Y_i$. Hence

$$p \in X \sum_{\substack{i=0 \\ i \neq k}}^{n} Y_i.$$

Thus

$$X \sum_{i=0}^{n} Y_i \subseteq \sum_{k=0}^{n} [X \sum_{\substack{i=0 \\ i\neq k}}^{n} Y_i].$$

To prove the dual n-distributivity, we need a lemma.

Lemma 4.1. *Let* $p, q, r \in E^{n-1}$. *Then, for any* $u \in \operatorname{conv}\{p, r\}$, $v \in \operatorname{conv}\{q, s\}$, *and* $x \in \operatorname{conv}\{p, q\}$, *there exist* $y \in \operatorname{conv}\{r, s\}$ *and* $z \in \operatorname{conv}\{u, v\}$ *such that* $z \in \operatorname{conv}\{x, y\}$.

Proof. We may assume that $u \notin \{p, r\}$ and $v \notin \{q, s\}$ as otherwise the statement is trivial. The conditions of the lemma show that there exist real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ such that

$$q = \alpha_1 s + \alpha_2 v, \quad \alpha_1 + \alpha_2 = 1, \quad \alpha_1 \leqq 0,$$

$$p = \beta_1 r + \beta_2 u, \quad \beta_1 + \beta_2 = 1, \quad \beta_1 \leqq 0,$$

$$x = \gamma_1 q + \gamma_2 p, \quad \gamma_1 + \gamma_2 = 1, \quad \gamma_1, \gamma_2 \geqq 0.$$

Hence

$$x = \gamma_1\alpha_1 s + \gamma_1\alpha_2 v + \gamma_2\beta_1 r + \gamma_2\beta_2 u =$$

$$= (\gamma_1\alpha_1 + \gamma_2\beta_1)\left(\frac{\gamma_1\alpha_1}{\gamma_1\alpha_1+\gamma_2\beta_1} s + \frac{\gamma_2\beta_1}{\gamma_1\alpha_1+\gamma_2\beta_1} r\right) +$$

$$+ (\gamma_1\alpha_2 + \gamma_2\beta_2)\left(\frac{\gamma_1\alpha_2}{\gamma_1\alpha_2+\gamma_2\beta_2} v + \frac{\gamma_2\beta_2}{\gamma_1\alpha_2+\gamma_2\beta_2} u\right) = \delta_1 y + \delta_2 z,$$

where

$$\delta_1 = \gamma_1\alpha_1 + \gamma_2\beta_1, \quad \delta_2 = \gamma_1\alpha_2 + \gamma_2\beta_2,$$

$$y = \frac{\gamma_1\alpha_1}{\gamma_1\alpha_1+\gamma_2\beta_1} s + \frac{\gamma_2\beta_1}{\gamma_1\alpha_1+\gamma_2\beta_1} r,$$

$$z = \frac{\gamma_1\alpha_2}{\gamma_1\alpha_2+\gamma_2\beta_2} v + \frac{\gamma_2\beta_2}{\gamma_1\alpha_2+\gamma_2\beta_2} u.$$

This representation shows that $y \in \operatorname{conv}\{s, r\}$, $z \in \operatorname{conv}\{u, v\}$ (the coefficients are non-negative and sum up to 1). Finally, $\delta_1 + \delta_2 = 1$, $\delta_1 \leqq 0$ yield that $z \in \operatorname{conv}\{x, y\}$.

The following extension of this lemma is now proved by an easy induction over k.

Corollary. *Let* $p_0, p_1, \ldots, p_k, q_0, q_1, \ldots, q_k, r_0, r_1, \ldots, r_k \in E^{n-1}$. *Assume* $r_i \in \operatorname{conv}\{p_i, q_i\}$, $i = 0, 1, \ldots, k$. *Let* $p \in \operatorname{conv}\{p_0, p_1, \ldots, p_k\}$. *Then there exist* $q \in \operatorname{conv}\{q_0, q_1, \ldots, q_k\}$ *and* $r \in \operatorname{conv}\{r_0, r_1, \ldots, r_k\}$ *such that* $r \in \operatorname{conv}\{p, q\}$.

Now we pass on to prove the dual n-distributivity of $\mathfrak{L}(E^{n-1})$. Let

$$p\in\prod_{j=0}^{n}[X+\prod_{\substack{i=0\\ i\neq j}}^{n}Y_i],$$

where $X, Y_0, Y_1, \dots, Y_n\in\mathfrak{L}(E^{n-1})$. Then there exist sequences $\{p_{jm}\}_{m\in N}$, $j=0, 1, \dots$ $\dots, n$, each converging to p, such that

$$p_{jm}\in X\vee\prod_{\substack{i=0\\ i\neq j}}^{n}Y_i,\quad m\in N,\quad j=0, 1, \dots, n.$$

Now choose, for all $m\in N$ and $j=0, 1, \dots, n$,

$$x_{jm}\in X,\quad y_{jm}\in\prod_{\substack{i=0\\ i\neq j}}^{n}Y_i$$

such that p_{jm} is a convex linear combination of x_{jm} and y_{jm}. By Helly's theorem there exists an

$$y_m\in\prod_{i=0}^{n}Y_i$$

for all $m\in N$, and y_m can be chosen to be an element of conv $\{y_{0m}, y_{1m}, \dots, y_{nm}\}$. Thus, by the Corollary, there exist points $x_m\in$conv $\{x_{0m}, x_{1m}, \dots, x_{nm}\}$ and $p_m\in$ $\in$conv $\{p_{0m}, p_{1m}, \dots, p_{nm}\}$ with $p_m\in$conv $\{x_m, y_m\}$ for all $m\in N$. Obviously, $p_m\to p$ as $m\to\infty$, thus p is in the topological closure of $\{p_m\}_{m\in N}$ and each p_m is a member of $X\vee\prod_{i=0}^{n}Y_i$. Hence

$$p\in X+\prod_{i=0}^{n}Y_i.$$

The counterexamples at the end of Section 2 also show that $\mathfrak{L}(E^{n-1})\notin\Delta_{n-1}$, ∇_{n-1}.

5. Complemented modular lattices revisited. n-distributivity of complemented modular lattices was studied in [4]. Here we add a result describing those projective geometries in which "Carathéodory's theorem holds". As it is well-known by FRINK [2] there is a one-to-one correspondence between projective geometries and their subspace lattices, which are exactly the complete, complemented, modular, atomic lattices such that every atom is compact. It will be convenient to call *these lattices* projective geometries. We say that a projective geometry M satisfies the property (C_n) iff, for any atoms $p, p_1, \dots, p_m$, $m\geqq n+1$ of M with $p\leqq\bigvee_{i=1}^{m}p_i$, there exist $i_1, i_2, \dots, i_n\in\{1, 2, \dots, m\}$ such that $p\leqq\bigvee_{j=1}^{n}p_{i_j}$.

A lattice is called infinitely n-distributive iff it satisfies the identity

$$x \wedge \bigvee_{i \in I} Y_i = \bigvee_{\substack{K \subseteq I \\ |K| = n}} [x \wedge \bigvee_{i \in K} Y_i]$$

for arbitrary index set I. It is called completely n-distributive iff the identity

$$\bigwedge_{i \in I} \bigvee_{j \in J_i} x_{ij} = \bigvee_{\varphi} \bigwedge_{i \in I} \bigvee_{j \in \varphi(i)} x_{ij}$$

holds in it for arbitrary I and J_i, $i \in I$ and $|J_i| \geqq n$, where the $\bigvee_\varphi$ at the right hand side is to be formed for all choice functions $\varphi\colon I \to \bigcup_{i \in I} P_n(J_i)$ (with $\varphi(i) \in P_n(J_i)$), where $P_n(J_i)$ denotes the set of n element subsets of J_i, $i \in I$. Now we are ready to state the main result of this section.

Theorem 5.1. *Let L be a complete complemented modular lattice. Then the following conditions are equivalent:*

(i) *L is a projective geometry satisfying (C_n);*

(ii) *L is atomic and infinitely n-distributive;*

(iii) *L is completely n-distributive,*

(iv) *L is isomorphic to a direct product of irreducible projective geometries of length $\leqq n$.*

Corollary. *The dual of a projective geometry satisfying (C_n) also satisfies (C_n). The dual of a completely n-distributive complemented modular lattice is also completely n-distributive.*

Proof. (i)$\Rightarrow$(iv). If (i) holds, then, by FRINK [2], Theorem 7, Corollary, L is a direct product of irreducible projective geometries L_γ, $\gamma \in \Gamma$. We show that L_γ must be of length $\leqq n$ for all $\gamma \in L$. Indeed, in the contrary case L_γ contains an independent set of $n+1$ atoms: $p_0, p_1, \ldots, p_n$. By irreducibility, $p_0 \vee p_1 \geqq p_{01}$ for some atom $p_{01} \neq p_0, p_1$. We have also $p_0 \vee p_1 \vee p_2 \geqq p_{01} \vee p_2 \geqq p_{012}$ for some atom $p_{012} \neq p_{01}, p_2$. Clearly, $p_{012} \not\leqq p_0 \vee p_1$ (otherwise $p_0 \vee p_1 \geqq p_{012} \vee p_{01} \geqq p_2$, a contradiction). Similarly, for $\{i, j\} = \{0, 1\}$, $p_{012} \not\leqq p_i \vee p_2$ as otherwise $p_i \vee p_2 = p_i \vee p_{012} \vee p_2 = p_i \vee p_{01} \vee p_2 = p_j \vee p_{01} \vee p_2 \geqq p_j$. By induction, we find an atom $p_{01\ldots n} \leqq p_0 \vee p_1 \vee \ldots \vee p_n$ such that $p_{01\ldots n} \not\leqq p_0 \vee \ldots \vee p_{i-1} \vee p_{i+1} \vee \ldots \vee p_n$, $i = 0, 1, \ldots, n$. This contradicts (C_n).

(iv)$\Rightarrow$(iii). Irreducible projective geometries of length $\leqq n$ are completely n-distributive (in fact, any meet of joins equals one of the meets of n element subjoins), hence so are their direct products.

(iii)$\Rightarrow$(ii). It is easily seen that complete n-distributivity implies infinite n-distributivity. So we only have to show that L is atomic. It suffices to show that every element of L is a join of elements of height $\leqq n$. Let $x \in L$ be of height greater than n. Consider all independent sets $\{x_{\gamma 0}, x_{\gamma 1}, \ldots, x_{\gamma n}\}$, $\gamma \in \Gamma$ such that $\bigvee_{i=0}^{n} x_{\gamma i} = x$. As usual, H_n^Γ denotes the set of all mappings of the set Γ to $H_n=\{0, 1, \ldots, n\}$. By the complete n-distributive law,

$$x = \bigwedge_{\gamma\in\Gamma} \bigvee_{i=0}^{n} x_{\gamma i} = \bigvee_{m_1\in H_n^\Gamma} \cdots \bigvee_{m_n\in H_n^\Gamma} \bigwedge_{\gamma\in\Gamma} (x_{\gamma m_1(\gamma)} \vee \ldots \vee x_{\gamma m_n(\gamma)}).$$

We show that the elements

$$z_{m_1\ldots m_n} = \bigwedge_{\gamma\in\Gamma} \bigvee_{i=1}^{n} x_{\gamma m_i(\gamma)}$$

are of height $\leqq n$. Indeed, in the contrary case, some of the intervals $[0, z_{m_1\ldots m_n}]$ contains a chain of $n+1$ elements. Thus there is an independent set $\{x_1, x_2, \ldots, x_n\}$ such that $x_0' := \bigvee_{i=1}^{n} x_i < z_{m_1\ldots m_n}$ and $\bigwedge_{i=1}^{n} x_i = 0$. Let x_0 be a complement of x_0' in $[0, x]$. Then $\bigvee_{i=0}^{n} x_i = x$. Therefore, some of the joins $\bigvee_{i=0, i\neq j}^{n} x_i$ occurs in the $\wedge$-representation of $z_{m_1\ldots m_n}$. For $j=0$, this yields $x_0' \geqq z_{m_1\ldots m_n}$, a contradiction. If $j\neq 0$, then

$$x_0' = x_0' \wedge z_{m_1\ldots m_n} \leqq x_0' \wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} x_i = \bigvee_{\substack{i=0\\ i\neq 0, j}}^{n} x_i < \bigvee_{i=1}^{n} x_i = x_0'.$$

This contradiction yields (ii).

The implication (ii)$\Rightarrow$(i) being very easy, the proof is complete.

6. Modular lattices. In this section we prove Theorem 1.4. By a result of FAIGLE [1], every modular lattice M can be embedded into a modular lattice M' such that every element of M' is a join of compact completely join-irreducible elements. If we prove that M' is in $\mathrm{HSP}(\Delta_n \cap F)$, then the theorem follows. Let $\mathscr{P}$ be the set of all completely join-irreducible elements of M (these elements are all compact) and let $\mathscr{H}$ be the set of all finite subsets of $\mathscr{P}$. For any $H\in\mathscr{H}$, let M_H denote the set of all finite joins (in M') of elements of H. M_H is clearly a lattice relative to the ordering of M'. Let $\wedge^H$ and $\vee^H$ denote the operations in M_H (note that $\vee^H$ is the same as $\vee$). For any element $x\in M'$, and, for any $H\in\mathscr{H}$, let $x_H = \sup\{y \mid y \leqq x, y\in M_H\}$. Then

$$x\wedge y = \bigvee_{H\in\mathscr{H}} (x_H \wedge^H y_H)$$

and

$$x\vee y = \bigvee_{H\in\mathscr{H}} (x_H \vee^H y_H).$$

Indeed, observe that $x=\bigvee_H x_X$ and $H\subseteqq G\in\mathscr{H}$ implies $x_H \leqq x_G$. If $p\leqq x\wedge y$ for some $p\in\mathscr{P}$ then $x_H=y_H=p$ holds for $H=\{p\}$, whence $p\leqq p\wedge p = x_H \wedge^H y_H$. This proves the first equality. Now let $p\leqq x\vee y$. Then $p\leqq \bigvee_{H,K}(x_H\vee y_K) = \bigvee_H (x_H \vee y_H) = \bigvee_H (x_H \vee^H y_H)$, proving the second equality.

Assume that $p=q$ is an m-ary lattice identity holding in all finite n-distributive lattices. Then $p=q$ holds in all the lattices M_H. Let $x_1, x_2, \ldots, x_m \in M'$, and let p^H and q^H be the realizations of p and q in M. Then

$$p(x_1, x_2, \ldots, x_m) = \bigvee_{H \in \mathscr{H}} p^H((x_1)_H, (x_2)_H, \ldots, (x_m)_H) =$$

$$= \bigvee_{H \in \mathscr{H}} q^H((x_1)_H, (x_2)_H, \ldots, (x_m)_H) = q(x_1, x_2, \ldots, x_m).$$

References

[1] U. Faigle, Frink's theorem for modular lattices, Preprint.

[2] O. Frink, Complemented modular lattices and projective spaces of infinite dimension, *Trans. Amer. Math. Soc.*, **60** (1946), 452—467.

[3] H. Gerstmann, n-Distributivgesetze, *Acta Sci. Math.*, **46** (1983), 99—113.

[4] A. Huhn, Two notes on n-distributive lattices, *Colloq. Math. Soc. János Bolyai*, **14** *Lattice Theory*, North-Holland (Amsterdam, 1976), 137—147.

[5] V. L. Klee [editor], *Convexity, Proc. Sympos. Pure Math.*, Vol. **7**, Amer. Math. Soc. (Providence, R. I., 1963).