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Group theoretic results in Clifford semigroups

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Clifford semigroups or strong semilattices of groups are a class of inverse semigroups which are obviously very closely related to groups. This paper attempts to exploit this close relationship. Petrich's characterization of congruences on inverse semigroups is analyzed in this special case to obtain a description of homomorphisms and their images in terms of the groups involved. Next, the idea of classes and closure operations due to P. HALL, which has proved very useful in group theory, is extended. Some results are obtained, but there are many interesting open problems left. This is applied to nilpotency of groups and a number of interesting results are extended, in particular Fitting's Theorem, the Hirsch-Plotkin Theorem and the characterization of nilpotent groups in terms of subnormal subgroups. Finally some remarks on solubility are made. The techniques demonstrated here should lead to a very large number of results being transferred.

This paper describes a technique for applying group theoretic ideas and results to Clifford semigroups mainly by giving some examples of it in action.

I would like to thank Drs. KOWOL and MITSCH for a preprint of their paper [4] and for stimulating conversation and, later, correspondence.

I would also like to thank Dr. O'CARROLL for much help.

We refer to Howie's book [3] for background on the subject. In this paper we are exclusively concerned with Clifford semigroups and we give a definition now to establish notation.

Definition. A semigroup S is a *Clifford semigroup* or *strong semilattice of groups* if S is the disjoint union of a set of groups $\{S_\alpha\colon \alpha\in E\}$, where E is a meet semilattice and for all α, β in E such that $\alpha\geqq\beta$, there exists a homomorphism $\varphi_{\alpha,\beta}\colon S_\alpha\to S_\beta$ satisfying

$$\varphi_{\alpha,\beta}\varphi_{\beta,\gamma}=\varphi_{\alpha,\gamma} \quad \text{for all} \quad \alpha\geqq\beta\geqq\gamma \quad \text{in} \quad E.$$

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The homomorphisms $\{\varphi_{\alpha,\beta};\ \alpha \geqq \beta \text{ in } E\}$ are called the *linking homomorphisms*. For all α in E, $\varphi_{\alpha,\alpha}$ is the identity map on S_α. For s_1, s_2 in S, the product is defined by

$$s_1 s_2 = (s_1 \varphi_{\alpha,\alpha\beta})(s_2 \varphi_{\beta,\alpha\beta})$$

where $s_1 \in S_\alpha$, $s_2 \in S_\beta$, $\alpha\beta$ is the join in E and the product on the right is the product in the group $S_{\alpha\beta}$.

We denote the identity of S_α by e_α. Then $\{e_\alpha: \alpha \in E\}$ is a semilattice of idempotents isomorphic to E, and we will often denote it by $E(S)$ or even simply E. This will not cause any confusion. Note that e_α is central in S for all $\alpha \in E$.

It can be shown from Howie [3], and it is in any case well known, that Clifford semigroups form a variety of algebras, a subvariety of the variety of inverse semigroups. Petrich [7] has defined a concept of congruence pairs for inverse semigroups and related them to congruences. This enables a link to be made between congruences and a substructure which strongly resembles normal subgroups. This correspondence is analysed closely in the context of Clifford semigroups in section 1. In section 2, some applications are made of the concept of closure operations. In section 3, we deal with extensions of the idea of nilpotency from groups to Clifford semigroups, and finally we deal with solubility in the final section.

1. Congruences on Clifford semigroups

This material is a slight extension of the results of Petrich [7] as applied to Clifford semigroups. From now on, unless explicitly stated otherwise, all semigroups are assumed to be Clifford semigroups. Let S be a semigroup, with constituent groups $\{S_\alpha: \alpha \in E\}$, linking homomorphisms $\{\varphi_{\alpha,\beta}: \alpha \geqq \beta,\ \alpha, \beta \in E\}$ and semilattice of idempotents $\{e_\alpha: \alpha \in E\}$.

Definition 1.1. An inverse subsemigroup T of S is called *normal* if $a^{-1}Ta \subseteqq T$ for all $a \in S$ and *full* if $E \subseteqq T$.

This definition departs from standard practice, as usually normal subsemigroups are necessarily full. We do not require this.

Definition 1.2. A pair (ϱ, N) is called a *congruence pair* if N is a normal full subsemigroup and $ae \in N$, $e \varrho a^{-1}a$ implies $a \in N$, where $a \in S$, $e \in E$.

If we define

$$a \varkappa(\varrho, N) b \quad \text{if and only if} \quad a^{-1}a \varrho b^{-1}b,\ ab^{-1} \in N$$

then Petrich [7] shows that $\varkappa(\varrho, N)$ is a congruence on S and every congruence σ on S

is of this form, where

$$\varrho = \operatorname{tr}\sigma \quad \text{(the restriction of } \sigma \text{ to } E\times E),$$
$$N = \ker\sigma := \{s\sigma e\colon\ e\in E\}.$$

Our version is simpler than his because we take advantage of the fact that S is a Clifford semigroup. We now present some fairly straightforward results concerning the concepts that we have just defined. But first a useful notational device. If T is an inverse subsemigroup of S we write T_α for $T\cap S_\alpha$. Then $T=\bigcup_{\alpha\in E} T_\alpha$. In general some of the T_α may be empty. But T is full if and only if $T_\alpha\neq\emptyset$ for all $\alpha\in E$.

Lemma 1.3.

(i) *If N is a normal inverse subsemigroup of S then N_α is a normal subgroup of S_α for all $\alpha\in E$ such that $N_\alpha\neq\emptyset$.*

(ii) *Let N be an inverse subsemigroup of S. Then $NE\subseteqq N$ if and only if $N_\alpha\varphi_{\alpha,\beta}\subseteqq N_\beta$ for all $\alpha,\beta\in E$, $\alpha\geqq\beta$.*

(iii) *Let N be an inverse subsemigroup of S such that $NE\subseteqq N$. Then N is normal in S if and only if N_α is a normal subgroup of S_α for all $\alpha\in E$ such that $N_\alpha\neq\emptyset$.*

(iv) *Let N be a full inverse subsemigroup of S. Then N is normal in S if and only if N_α is normal in S_α for all $\alpha\in E$.*

(v) *The condition in Definition* 1.2 *is equivalent to: for all $\alpha,\beta\in E$ such that $e_\alpha\varrho e_\beta$ we have $N_{\alpha\beta}\varphi^{-1}_{\alpha,\alpha\beta}\subseteqq N_\alpha$.*

(vi) *If $NE\subseteqq N$ and $N_\alpha\neq\emptyset$, then* $\ker\varphi_{\alpha,\beta}\subseteqq N_\alpha$ *for all $\beta\leqq\alpha$, $\alpha,\beta\in E$.*

(vii) *Let ϱ be a congruence on E, N a normal full subsemigroup of S. Then (ϱ, N) is a congruence pair if and only if for all $\alpha,\beta\in E$ such that $e_\alpha\varrho e_\beta$ then $N_{\alpha\beta}\varphi^{-1}_{\alpha,\alpha\beta}\subseteqq N_\alpha$.*

These results can all be checked very easily and so no details of proof will be given. We now look at the minimum group congruence σ on S. Then σ is a congruence on S such that S/σ is a group and all group images of S can be factored through S/σ. See Howie [3] p. 139.

Lemma 1.4. *Let S be a Clifford semigroup with semilattice of idempotents E. Let $X\subseteqq E$ be a chain with the property that for all $\alpha\in E$ there exists $\beta\in X$ such that $\beta\leqq\alpha$. Then S/σ is the direct limit of the chain of groups*

$$\{S_\alpha, \varphi_{\alpha,\beta}\colon\ \alpha,\beta\in X\}.$$

Note that such a chain always exists. If E has a minimal element δ, then we can take $X=\{\delta\}$ and then $S/\sigma\cong S_\delta$. A special case will be used later.

Corollary 1.5. *Using the notation of Lemma* 1.4, *assume that $\varphi_{\alpha,\beta}$ is a monomorphism for all $\alpha,\beta\in X$. Then without loss of generality we may assume $S_\alpha\subseteqq S_\beta$ for all $\alpha\geqq\beta$ and $S/\sigma=\bigcup_{\alpha\in X} S_\alpha$.*

These results do not need proving as they seem well-known, and can in any case be checked quickly. To finish this section we consider homomorphic images of Clifford semigroups. We use ε to denote the identity congruence, i.e., $a\varepsilon b$ if and only if $a=b$. It is obvious from the definition that (ε, N) is a congruence-pair for all full normal subsemigroups N of S.

Lemma 1.6. *Let ϱ be a congruence on E. Then the least full normal subsemigroup $N(\varrho)$ such that $(\varrho, N(\varrho))$ is a congruence pair is defined by*

$$N(\varrho)_\alpha = \prod_{\alpha\varrho\beta} \ker \varphi_{\alpha,\alpha\beta}.$$

In particular if $\varphi_{\alpha,\alpha\beta}$ are monomorphisms for all $\alpha, \beta\in E$ such that $\alpha\varrho\beta$, then (ϱ, E) is a congruence pair.

Again this result is easy to prove, especially if we use Lemma 1.3.

Lemma 1.7. *Let N be a full normal inverse subsemigroup of S. Let $\varkappa=\varkappa(\varepsilon, N)$, and let $T=S/\varkappa$. Then $T_\alpha=S_\alpha/N_\alpha$ and $\theta_{\alpha,\beta}\colon T_\alpha\to T_\beta$ where $\alpha\geqq\beta$ is defined by $t\theta_{\alpha,\beta}=N_\beta s\varphi_{\alpha,\beta}$, where $t=N_\alpha s$,* i.e., *$\theta_{\alpha,\beta}$ is induced naturally by $\varphi_{\alpha,\beta}$.*

This follows easily from the definitions. We finally consider a general congruence pair.

Lemma 1.8. *Let (ϱ, N) be a congruence pair on S. Let $\varkappa=\varkappa(\varrho, N)$, $T=S/\varkappa$. Let $\lambda=\varkappa(\varrho, E)$ defined on T. Then*

$$T/\lambda \cong S/\varkappa(\varrho, N).$$

If $\{A_\gamma\colon \gamma\in C\}$ are the congruence classes of ϱ on E, then T/λ is obtained from T by replacing $\bigcup_{\alpha\in A_\gamma} T_\alpha$ by its maximal group homomorphic image T_γ, and for $\gamma, \delta\in C$, $\gamma\geqq\delta$, $\psi_{\gamma,\delta}$ is defined as the natural extension of the $\theta_{\alpha,\beta}$ for $\alpha\in A_\gamma$, $\beta\in A_\delta$.

Proof. We first note that, using the notation of Lemma 1.7, the homomorphism $\theta_{\alpha,\beta}\colon T_\alpha\to T_\beta$ is a monomorphism. Hence (ϱ, E) is a congruence pair on T by Lemma 1.6. Hence T_γ can be written as a union of a tower of groups as described in Lemma 1.4. This makes the definition of $\psi_{\gamma,\delta}$ easy to verify. All the rest is very easy to check.

2. Closure operations on classes

We use the ideas of classes of groups and closure operations as developed by P. Hall and apply them to Clifford semigroups. A good presentation of these can be found in Robinson [8] chapter 1, section 1. They have also been used in many other settings by many other people. In particular Cohn [1] uses them in the context of universal algebras.

The only condition imposed on a class $\mathfrak{X}$ of groups is that $\{e\}\in\mathfrak{X}$ and if $G\in\mathfrak{X}$ and $H\cong G$ then $H\in\mathfrak{X}$. A closure operation on classes of groups is a map A from classes of groups to classes of groups $A\colon \mathfrak{X}\to A\mathfrak{X}$ satisfying $A\mathfrak{X}\supseteq\mathfrak{X}$, $\mathfrak{X}\subseteq\mathfrak{Y}$ implies $A\mathfrak{X}\subseteq A\mathfrak{Y}$ and $AA\mathfrak{X}=A\mathfrak{X}$. A class $\mathfrak{X}$ is A-closed if $A\mathfrak{X}=\mathfrak{X}$. Any intersection of A-closed classes is A-closed. Hence to define A we only need to specify the A-closed classes. For then $A\mathfrak{X}=\bigcap\{\mathfrak{Y}\colon \mathfrak{Y}\supseteq\mathfrak{X},\ A\mathfrak{Y}=\mathfrak{Y}\}$. The concept of classes and closure operations can be transferred to any other algebraic structure, and, in particular, to Clifford semigroups.

Definition 2.1. For a class $\mathfrak{X}$ of groups, we define $\mathfrak{X}_S$ to be the class of Clifford semigroups given by

$$S\in\mathfrak{X}_S \quad \text{if and only if} \quad S_\alpha\in\mathfrak{X} \quad \text{for all} \quad \alpha\in E(S).$$

This gives the natural extension of the definition of a class of groups to a class of Clifford semigroups. We will see later that this extension of the definition is not always the most useful one. There is immediately a family of questions which can be posed.

Problem 2.2. Given a class $\mathfrak{X}$ of groups and a closure operation A on classes, determine whether $A\mathfrak{X}_S=(A\mathfrak{X})_S$. Alternatively if $A\mathfrak{X}=\mathfrak{X}$, is $A\mathfrak{X}_S=\mathfrak{X}_S$?

We will deal with a few cases of this problem, but there is a great deal more that can be done in this area. We first define the closure operations which we will be using, to cover both groups and Clifford semigroups.

The class $\mathfrak{X}$ is S closed if every substructure of an $\mathfrak{X}$ structure is itself an $\mathfrak{X}$-structure.

The class $\mathfrak{X}$ is Q closed (sometimes the symbol H is used) if every epimorphic image of an $\mathfrak{X}$ structure is itself an $\mathfrak{X}$ structure.

The class $\mathfrak{X}$ is R closed if given a structure Y such that a family of homomorphisms $\{\theta_i\colon i\in I\}$ exists with $Y\theta_i\in\mathfrak{X}$ for all $i\in I$ and $\bigcap_{i\in I}\ker\theta_i$ is trivial, then $Y\in\mathfrak{X}$. We say $\mathfrak{X}$ is residually closed.

The class $\mathfrak{X}$ is L closed if given a structure Y such that every finite subset $\{y_1, \dots, y_n\}$ of Y is contained in an $\mathfrak{X}$ substructure of Y, then $Y\in\mathfrak{X}$.

The class $\mathfrak{X}$ is $N(N_0)$ closed if every structure Y which can be expressed as a product of a (finite) number of normal substructures is again in $\mathfrak{X}$.

Lemma 2.3. *Let* $S\mathfrak{X}=\mathfrak{X}$. *Then* $S\mathfrak{X}_S=\mathfrak{X}_S$.

Proof. Let $T\in S\mathfrak{X}_S$. Then there exists $U\in\mathfrak{X}_S$ such that T is a Clifford subsemigroup of U. Hence for all $\alpha\in E(U)$, T_α is a subgroup of U_α or is empty. But $U_\alpha\in\mathfrak{X}=S\mathfrak{X}$. Hence $T_\alpha\in\mathfrak{X}$ or is empty. Thus $T\in\mathfrak{X}_S$. Thus $\mathfrak{X}_S=S\mathfrak{X}_S$.

Example 2.4. Let $\mathfrak{X}$ be the class of finite p-groups for some prime p. Let the semilattice E be the set of negative integers with the natural order inducing the semilattice structure. So $(-n)\cdot(-m)=\min\{-n, -m\}$. Let S_{-n} be the cyclic group of order p^n, $\varphi_{-n,-m}$ for $n\leqq m$ be the natural embedding. Then S, the Clifford semigroup so defined has as maximal group homomorphic image the group C_{p^∞}, the Prüfer group of type p^∞, which is certainly not a finite p-group. So in this case $\mathfrak{X}=Q\mathfrak{X}$ but $Q\mathfrak{X}_S\neq\mathfrak{X}_S$.

The problem with Q closure occurs because group homomorphic images of Clifford semigroups include direct limits. This leads to the following result.

Lemma 2.5. *Let $\mathfrak{X}$ be a class of groups closed under the operation of taking direct limits. Then $\mathfrak{X}_S$ is Q closed.*

Proof. Let $S\in Q\mathfrak{X}_S$. Then S is the homomorphic image of a semigroup $T\in\mathfrak{X}_S$. From Lemma 1.8, we see that the component groups of S are obtained from those of T by taking homomorphic images and direct limits. Since the component groups of T lie in $\mathfrak{X}$ and $\mathfrak{X}$ is closed under direct limits, and hence Q closed, it follows that $S\in\mathfrak{X}_S$. This finishes the proof.

We next look at L closure. First we prove a result used later.

Lemma 2.6. *Let $\{s_1, \ldots, s_n\}$ be a finite subset of a Clifford semigroup S. Then the inverse subsemigroup of S generated by $\{s_1, \ldots, s_n\}$ is contained in the union of a finite number of finitely generated groups forming a semigroup.*

Proof. Let $E=E(S)$, and let X be the finite subset of E defined by $\alpha\in X$ if and only if $s_i\in S_\alpha$ for some i, $1\leqq i\leqq n$. Then X generates a finite subsemilattice Y of E. For all $\beta\in Y$, we define

$$Z_\beta=\{s_i\varphi_{\alpha,\beta}\colon\ 1\leqq i\leqq n,\ \alpha\geqq\beta,\ \alpha\in Y,\ s_i\in S_\alpha\}.$$

Then Z_β is a finite subset of S_β and so generates a finitely generated subgroup G_β of S_β. It is routine to check that the inverse subsemigroup of S generated by $\{s_1, \ldots, s_n\}$ is contained in $\bigcup_{\beta\in Y} G_\beta$, and this is a semigroup, which is all we wished to show.

Lemma 2.7. *Let $\mathfrak{X}=L\mathfrak{X}$. Then $L\mathfrak{X}_S=\mathfrak{X}_S$.*

Proof. Let $S\in L\mathfrak{X}_S$. We need to show that $S_\alpha\in\mathfrak{X}$ for all $\alpha\in E=E(S)$. Let $\{s_1, \ldots, s_n\}$ be a finite subset of S_α. Then $\{s_1, \ldots, s_n\}\subseteqq T\in\mathfrak{X}_S$, T an inverse subsemigroup of S. In particular $T_\alpha\supseteqq\{s_1, \ldots, s_n\}$ and lies in $\mathfrak{X}$. Thus $S_\alpha\in L\mathfrak{X}=\mathfrak{X}$. Hence the result is true.

Lemma 2.8. *Let $\mathfrak{X}=Q\mathfrak{X}=S\mathfrak{X}=L\mathfrak{X}$. Then $\mathfrak{X}_S=Q\mathfrak{X}_S=S\mathfrak{X}_S=L\mathfrak{X}_S$.*

Proof. Following Lemma 2.3 and Lemma 2.7, we only need to show that

$\mathfrak{X}_S = Q\mathfrak{X}_S$. Let $S \in Q\mathfrak{X}_S$, S a homomorphic image of $T \in \mathfrak{X}_S$. From Lemma 1.8, each S_α is obtained from $\{T_\beta\colon \beta \in E(T)\}$ by taking homomorphic images and unions of towers. Let $\{G_\gamma\colon \gamma \in X\}$ be a tower of groups in $\mathfrak{X}$, $G = \bigcup_{\gamma \in X} G_\gamma$. Then any finite subset of G is contained in G_γ for some γ, and $G_\gamma \in \mathfrak{X}$. Hence $G \in L\mathfrak{X} = \mathfrak{X}$. Thus each $S_\alpha \in \mathfrak{X}$ and $S \in \mathfrak{X}_S$.

For any class $\mathfrak{X}$ we denote by $V\mathfrak{X}$ the least variety containing $\mathfrak{X}$. It is a standard result from universal algebra that $V\mathfrak{X} = \mathfrak{X}$ if and only if $\mathfrak{X} = S\mathfrak{X} = Q\mathfrak{X} = R\mathfrak{X}$. (Cohn [1] IV. 3). We now state

Lemma 2.9. *Let $\mathfrak{X}$ be a class of groups. Then $\mathfrak{X}_S$ is a variety if and only if $\mathfrak{X}$ is a variety.*

This is an easy consequence of known results (Petrich [6]) or can be proved directly without much trouble.

Corollary 2.10. *$\mathfrak{X}$ is Q, R, S closed if and only if $\mathfrak{X}_S$ is Q, R, S closed.*

3. Nilpotency and its generalizations

Let $\mathfrak{N}$ be the class of nilpotent groups, and let $\mathfrak{N}_c$ be the class of nilpotent groups of nilpotency class at most c. Then $\mathfrak{N}_c$ is a variety and $\mathfrak{N} = \bigcup_{c \geq 1} \mathfrak{N}_c$. The most obvious generalization of $\mathfrak{N}$ to Clifford semigroups is $\mathfrak{N}_S$, but this leads to problems as we now see.

Example 3.1. Let G_n be a nilpotent group of nilpotency class exactly n, in particular let G_n be the group of $(n+1)\times(n+1)$ unitriangular matrices over some field F. Then we can embed G_n in G_{n+1} by mapping $(a_{ij}) \in G_n \to (b_{ij}) \in G_{n+1}$, where for $j > i$, $a_{ij} = b_{ij+1}$, $b_{ii+1} = 0$. Let S be the Clifford semigroup whose semilattice of idempotents is isomorphic to the negative integers with the natural order. Compare Example 2.4. For each $-n \in E$, let $S_{-n} = G_n$ and $\varphi_{-n,\,-m}$ be the embedding obtained from the embeddings outlined above. Then $S \in \mathfrak{N}_S$, but S has as a homomorphic image $G = \bigcup_{n \geq 1} G_n$, the maximal group homomorphic image of S. And G is not nilpotent, since it contains subgroups of arbitrarily high nilpotency class.

Because of this example, we make the following definition.

Definition 3.2. The class of *nilpotent Clifford semigroups* is defined to be

$$\hat{\mathfrak{N}} = \bigcup_{c \geq 1} (\mathfrak{N}_c)_S.$$

Hence $S\in\hat{\mathfrak{N}}$ if and only if $S_\alpha\in\mathfrak{N}_c$ for all $\alpha\in E=E(S)$, and some $c=c(S)$. This coincides with LALLEMENT's definition [5]. As KOWOL and MITSCH dealt with finite semigroups, either definition would have served. In the infinite case this definition leads to a more satisfactory theory. Denote $(\mathfrak{N}_c)_S$ by $\hat{\mathfrak{N}}_c$.

Lemma 3.3. *$\hat{\mathfrak{N}}$ is S and Q closed.*

Proof. Let $T\in S\hat{\mathfrak{N}}$. Then there exists $U\in\hat{\mathfrak{N}}$ and T is a subsemigroup of U. So $U\in\hat{\mathfrak{N}}_c$ and $S\mathfrak{N}_c=\mathfrak{N}_c$. By Lemma 2.3 $S\hat{\mathfrak{N}}_c=\hat{\mathfrak{N}}_c$, hence $T\in\hat{\mathfrak{N}}_c\subseteqq\hat{\mathfrak{N}}$. The case of Q closure follows the same pattern, using Corollary 2.10 since $\mathfrak{N}_c$ is a variety.

We now introduce upper and lower central series for Clifford semigroups which extend the corresponding ideas for groups, as was done in Kowol and Mitsch [4].

Definition 3.4. Let S be a Clifford semigroup, N_i full normal subsemigroups of S for $0\leqq i\leqq r$.

(i) $Z(S)$, the *centre* of S is defined by $Z(S)=\{x\in S\colon xs=sx \text{ for all } s\in S\}$.

(ii) Let H, K be inverse subsemigroups of S. Define $[H, K]$ to be the inverse subsemigroup of S generated by

$$\{[h,k]=h^{-1}k^{-1}hk\colon h\in H, k\in K\}.$$

(iii) A sequence

$$E(S)=N_0\subseteqq N_1\subseteqq\ldots\subseteqq N_r=S$$

is called a central series of S if

$$N_i\subseteqq Z(S/\varkappa(\varepsilon, N_{i-1}))\theta_{i-1}^{-1}$$

for $1\leqq i\leqq r$, where θ_{i-1} is the natural homomorphism associated with $\varkappa(\varepsilon, N_{i-1})$.

(iv) The upper central series of S is defined inductively by

$$Z_0(S)=E(S),$$
$$Z_{i+1}(S)\theta_i=Z(S/\varkappa(\varepsilon, Z_i(S))),$$

for $i\geqq 0$, where θ_i is the natural homomorphism associated with $\varkappa(\varepsilon, Z_i(S))$ and $Z_{i+1}(S)$ is maximal such.

(v) The lower central series of S is defined inductively by

$$\gamma_1(S)=S,$$
$$\gamma_{i+1}(S)=[S,\gamma_i(S)],$$

for $i\geqq 1$.

We now list some easy consequences of this composite definition.

Lemma 3.5. *Let S be a Clifford semigroup.*

(i) *$Z_i(S)$ is a normal full subsemigroup of S for all $i\geqq 0$.*

(ii) *$\gamma_i(S)$ is a normal full subsemigroup of S for all $i\geqq 1$.*

(iii) $S/\varkappa(\varepsilon, N)$ *is commutative if and only if* $N \supseteqq \gamma_2(S)$, *where N is a normal full subsemigroup of S.*

(iv) $[s_1, s_2] \in E(S)$ *if and only if* $s_1 s_2 = s_2 s_1$.

Proof. This is all easy to prove or can be deduced easily from Section 3 of Kowol and Mitsch [4].

Lemma 3.6. *Let S be a Clifford semigroup. Then*

$$\gamma_i(S) = \bigcup_{\alpha \in E} \gamma_i(S_\alpha).$$

Proof. Obviously $\gamma_i(S_\alpha) \subseteqq \gamma_i(S)$ for all $\alpha \in E$. Conversely we prove by induction on i that $\gamma_i(S) \subseteqq \bigcup_{\alpha \in E} \gamma_i(S_\alpha)$. This is true trivially for $i=1$. So assume that this is true for i. Let $s \in S$, $t \in \gamma_i(S)$. Then $[s, t] = s^{-1}t^{-1}st = (s\varphi_{\alpha,\alpha\beta})^{-1}(t\varphi_{\beta,\alpha\beta})^{-1} \cdot (s\varphi_{\alpha,\alpha\beta})(t\varphi_{\beta,\alpha\beta})$, where $s \in S_\alpha$, $t \in S_\beta$. So $[s, t] \in [S_{\alpha\beta}, \gamma_i(S_{\alpha\beta})]$ using the induction hypothesis. This suffices to prove the result since now the generators of $\gamma_{i+1}(S)$ lie in $\bigcup_{\alpha \in E} \gamma_{i+1}(S_\alpha)$ and this is easily checked to be a normal full subsemigroup.

Lemma 3.7. *The upper and lower central series of a Clifford semigroup are central series.*

Proof. This is immediate from Definition 3.4 and Lemma 3.5.

Theorem 3.8. *Let S be a Clifford semigroup with a central series*

$$E(S) = N_0 \subseteqq N_1 \subseteqq \dots \subseteqq N_r = S. \tag{3.9.}$$

Then $Z_r(S) = S$, $\gamma_{r+1}(S) = S$ *and for all i,* $0 \leqq i \leqq r$, $N_i \subseteqq Z_i(S)$ *and* $N_{r-i} \supseteqq \gamma_{1+i}(S)$.

Proof. We only need to prove the two inequalities which we do by induction on i. Both are true trivially for $i=0$. Assume that both are true for i. Let $x \in N_{i+1}$. Then $N_{i+1}\theta_i \subseteqq Z(S/\varkappa(\varepsilon, N_i))$, where θ_i is the natural homomorphism associated with $\varkappa(\varepsilon, N_i)$. Let $s \in S$. Then $(xs)\theta_i = x\theta_i s\theta_i = s\theta_i x\theta_i$ since (3.9) is central, and so $xs\varkappa(\varepsilon, N_i)sx$. Since $Z_i(S) \supseteqq N_i$, it follows that $xs\varkappa(\varepsilon, Z_i(S))sx$ for all $s \in S$. Thus $x\varphi_i \in Z(S/\varkappa(\varepsilon, Z_i(S))) = Z_{i+1}(S)\varphi_i$, where φ_i is the natural homomorphism associated with $\varkappa(\varepsilon, Z_i(S))$. Hence $x \in Z_{i+1}(S)$. Thus $N_{i+1} \subseteqq Z_{i+1}(S)$.

Let $x \in N_{r-i}$, $s \in S$ now. Then $xs\varkappa(\varepsilon, N_{r-i-1})sx$ as before. So $x^{-1}s^{-1}xs \in N_{r-i-1}$. Thus $[N_{r-i}, S] \subseteqq N_{r-i-1}$. Hence $\gamma_{2+i}(S) = [\gamma_{1+i}(S), S] \subseteqq [N_{r-i}, S] \subseteqq N_{r-i-1}$ using the induction hypothesis. This finishes the induction step for both inequalities and hence the proof of the theorem.

Corollary 3.9. *A Clifford semigroup S is in* $\hat{\mathfrak{N}}$ *if and only if there exist c and d such that* $Z_c(S) = S$, $\gamma_{d+1}(S) = E(S)$ *and the least such c and d satisfy* $c = d$.

This is the *nilpotency class* of S and is the least c such that $S\in\hat{\mathfrak{N}}_c$.

This result follows directly from Lemmas 3.6, 3.7 and Theorem 3.8. Notice the close connection with the work of Kowol and Mitsch [4], Section 4. We now prove a selection of theorems about nilpotency and its generalizations in Clifford semigroups by transferring the results from group theory. As source book for the group theoretic results any standard text book will serve. We mention particularly Hall [2], an excellent account of the particular areas under consideration here, but not widely available, and also Robinson [8] and Scott [10].

Theorem 3.10. *Let S be a nilpotent Clifford semigroup. Then elements of coprime order commute.*

Proof. The order of $s\in S$ is its order in S_α, where $s\in S_\alpha$, i.e., the least integer $n>0$ such that $s^n\in E(S)$. Let $s_1, s_2\in S$. If $s_1\in S_\alpha$, $s_2\in S_\beta$, then $s_1s_2=$ $=s_1\varphi_{\alpha,\alpha\beta}s_2\varphi_{\beta,\alpha\beta}=s_2\varphi_{\beta,\alpha\beta}s_1\varphi_{\alpha,\alpha\beta}$, using the group theoretic result in $S_{\alpha\beta}$. Since the order of $s\varphi_{\gamma,\delta}$ divides the order of s, the result follows.

Theorem 3.11. *In a torsion nilpotent Clifford semigroup, the elements of order a power of p, a prime, form an inverse subsemigroup.*

Theorem 3.12. *In a nilpotent Clifford semigroup, the elements of finite order form an inverse subsemigroup, the torsion subsemigroup.*

These both follow immediately from Theorem 3.10, and the corresponding results from group theory. Most of the results from Section 4 of Kowol and Mitsch [4] can be obtained by transferring from group theory, and we will not repeat them here. The exception to this is Theorem 4.3 on the representation of an element of a nilpotent Clifford semigroup as a product of elements of prime power order.

Theorem 3.13. *Let S be a torsion nilpotent Clifford semigroup, and let $\{P_i\colon i\in I\}$ be the Sylow subsemigroups of S,* i.e., *$P_i=\{s\in S\colon$ order of s is a power of $p_i\}$, where $\{p_i\colon i\in I\}$ are a set of distinct primes. If $s\in S$, then $s=a_1\dots a_n$ is a uniquely defined representation of s, where $a_i\in S_\alpha\cap P_i$, α is defined by $s\in S_\alpha$, $1\leq i\leq n$, a finite subset of I.*

This follows directly from the group theoretic result. This seems to be the only uniqueness result of this kind, applicable in general. But under very special circumstances, there is a maximal version of the theorem.

Theorem 3.14. *Let S be a torsion nilpotent Clifford semigroup such that $E=E(S)$ is a lattice with the maximal condition, and such that all linking homomorphisms are monomorphisms. Let $\{P_i\colon i\in I\}$ be the Sylow subsemigroups of S, where $\{p_i\colon i\in I\}$ are a set of distinct primes. If $s\in S$, then $s=b_1\dots b_n$ is a uniquely defined* *representation of s, where* $b_i \in P_i \cap S_{\beta(i)}$, *and* $\beta(i)$ *is defined by* $\beta(i)$ *is maximal in* E *such that* $b_i \varphi_{\beta(i),\alpha} = a_i$, *using the notation of Theorem* 3.13.

Proof. Since E is a lattice with the maximum condition, $\beta(i)$ is unique. Since $\varphi_{\beta(i),\alpha}$ is a monomorphism b_i is uniquely defined, since a_i is unique given Theorem 3.13.

From the proof of Theorem 3.14, it is obvious how examples could be constructed to show that $\beta(i)$ has to be uniquely defined, and that $\varphi_{\beta(i),\alpha}$ has to be a monomorphism, to obtain a unique "maximal" representation.

The next results we will prove are the Clifford semigroup theoretic versions of famous group theoretic results on nilpotency. The first is Fitting's Theorem, the one about normal nilpotent subgroups.

Lemma 3.15. *Let* S *be a Clifford semigroup. Let* N *be a normal inverse subsemigroup of* S, T *an inverse subsemigroup of* S. *Then* $NT = TN$ *is an inverse subsemigroup of* S. *Also* $(NT)_\alpha = N_\alpha T_\alpha$, *if* $TE = T$, *for all* $\alpha \in E$. *If* T *is normal, then so is* NT.

Proof. Let $n_1 t_1, n_2 t_2 \in NT$, where $n_i \in N$, $t_i \in T$, $i = 1, 2$. Then $n_1 t_1 n_2 t_2 = n_1 t_1 t_1^{-1} t_1 n_2 t_2 = n_1 t_1 n_2 t_1^{-1} t_1 t_2 = n_1 n_3 t_1 t_2 \in NT$. So NT is a subsemigroup. Let $tn \in TN$. Then $tn = tt^{-1}tn = tnt^{-1}t = n't \in NT$. Hence $TN \subseteq NT$. Similarly $NT \subseteq TN$. Thus $NT = TN$ is an inverse subsemigroup as $(nt)^{-1} = t^{-1}n^{-1} \in TN = NT$. We now show that $(NT)_\alpha = N_\alpha T_\alpha$. Certainly $N_\alpha T_\alpha \subseteq (NT)_\alpha$. Let $nt \in (NT)_\alpha$. Then there exist $\beta \geqq \alpha, \gamma \geqq \alpha$ such that $\beta\gamma = \alpha$, $n \in N_\beta$, $t \in T_\gamma$ and $nt = n\varphi_{\beta,\alpha} t\varphi_{\gamma,\alpha}$. But $n\varphi_{\beta,\alpha} \in N_\alpha$, $t\varphi_{\gamma,\alpha} \in T_\alpha$. Hence $(NT)_\alpha \subseteq N_\alpha T_\alpha$. Thus $N_\alpha T_\alpha = (NT)_\alpha$. Finally let T be also normal and let $nt \in NT$, $s \in S$. Then $s^{-1}nts = s^{-1}ntss^{-1}s = s^{-1}nss^{-1}ts \in NT$, for all $s \in S$. Hence the whole lemma is proved.

This result extends directly Lemma 2.4 of Kowol and Mitsch [4]. We now come to Fitting's Theorem.

Theorem 3.16. *Let* S *be a Clifford semigroup. The product of two normal nilpotent subsemigroups of* S *is normal and nilpotent.*

Proof. Let N, M be normal and nilpotent subsemigroups of S. Then NM is a normal subsemigroup by Lemma 3.15. Also $(NM)_\alpha = N_\alpha M_\alpha$ for all $\alpha \in E$. Suppose $N \in \hat{\mathfrak{N}}_c$, $M \in \hat{\mathfrak{N}}_d$, then $N_\alpha \in \mathfrak{N}_c$, $M_\alpha \in \mathfrak{N}_d$ and by standard group theory, $N_\alpha M_\alpha \in \mathfrak{N}_{c+d}$. Hence $(NM)_\alpha \in \mathfrak{N}_{c+d}$ for all $\alpha \in E$, and $NM \in \hat{\mathfrak{N}}_{c+d}$.

Corollary 3.17. *Let* $N \in \hat{\mathfrak{N}}_c$, $M \in \hat{\mathfrak{N}}_d$ *be normal nilpotent subsemigroups of* S. *Then* $NM \in \hat{\mathfrak{N}}_{c+d}$.

Corollary 3.18. *Let* S *be a Clifford semigroup which satisfies the maximal condition on normal subsemigroups. Then* S *contains a unique maximal normal nilpotent* *subsemigroup containing all normal nilpotent subsemigroups, called the Fitting subsemigroup.*

The next result which we extend is the Hirsch—Plotkin Theorem.

Theorem 3.19. *Let S be a Clifford semigroup. Then the product of two normal locally nilpotent subsemigroups is a normal locally nilpotent subsemigroup. There is a unique maximal normal locally nilpotent subsemigroup, containing all normal locally nilpotent subsemigroups, the Hirsch—Plotkin radical of S.*

Proof. Because of Lemma 3.15 we only need to show, for the first part, that if $N, M \in L\hat{\mathfrak{N}}$ are normal, then $NM \in L\hat{\mathfrak{N}}$. Since $N \in L\hat{\mathfrak{N}}$, it follows that N_α, M_α are locally nilpotent groups which are normal in S_α. Let $\{n_1 m_1, \ldots, n_r m_r \colon n_i \in N, m_i \in M\}$ be a finite subset of NM. Let $Z = \{n_1, \ldots, n_r, m_1, \ldots, m_r\}$. By Lemma 2.6, T, the inverse subsemigroup generated by Z, is generated by a finite set of elements of the form $n_i \varphi_{\beta,\alpha}, m_j \varphi_{\gamma,\delta}$. T_α is generated as a group by a finite set of the form $\{n_i \varphi_{\beta,\alpha}, m_j \varphi_{\gamma,\alpha}\}$, which is a finite subset of $N_\alpha M_\alpha$, the product of two locally nilpotent normal subgroups of S_α. Hence $N_\alpha M_\alpha$ is locally nilpotent by the Hirsch—Plotkin Theorem and thus T_α is nilpotent. T is the union of a finite number of groups of the form T_α. Hence we can find c such that $T_\alpha \in \mathfrak{N}_c$ for all T_α, and so $T \in \hat{\mathfrak{N}}_c$. Since $\{n_1 m_1, \ldots \ldots, n_r m_r\} \subseteqq T$, we have shown that $NM \in L\hat{\mathfrak{N}}$.

The last part follows as in the group case. The product of any finite set of normal locally nilpotent subsemigroups is locally nilpotent by the first part. Consider the product H of all the normal locally nilpotent subsemigroups of S. It is normal and any finite subset of H is contained in the product of a finite number of normal locally nilpotent subsemigroups which is locally nilpotent, hence is contained in a nilpotent subsemigroup. Thus H is locally nilpotent. This finishes the proof.

The next result which we extend is a well-known one concerning minimal normal subgroups of locally nilpotent groups.

Theorem 3.20. *Let S be a locally nilpotent Clifford semigroup, N a minimal normal subsemigroup of S. Then there exists a unique* $\alpha \in E$ *such that* $N_\alpha \supset \{e_\alpha\}$ *and* $N_\alpha \subseteqq Z(S_\alpha)$, *and for all* $\beta \leqq \alpha$, *we have* $\ker \varphi_{\alpha,\beta} \supseteqq N_\alpha$.

Proof. By Lemma 1.3, it is easy to see that if there are two elements α, β of E such that $N_\alpha \supset \{e_\alpha\}$, $N_\beta \supset \{e_\beta\}$, then N is not minimal. If S is locally nilpotent, then so is S_α. So N_α is a normal subgroup of S_α such that for all $\beta \in E$, $\beta \leqq \alpha$, $N_\alpha \subseteqq \ker \varphi_{\alpha,\beta}$. It follows that N_α can be replaced by any normal subgroup of S_α contained in it, and we would still have a minimal normal subsemigroup. Then minimality of N forces N_α to be a minimal normal subgroup of S_α, hence by group theory $N_\alpha \subseteqq Z(S_\alpha)$.

The last results about nilpotency which we will present concern normalizers.

Definition 3.21. Let T be an inverse subsemigroup of a Clifford semigroup S. The *normalizer* $N_S(T)$ of T in S is the unique largest inverse subsemigroup of S in which T is normal.

A priori $N_S(T)$ may not always exist. We will show that it does.

Lemma 3.22. *Let T be an inverse subsemigroup of a Clifford semigroup S. Then $N_S(T)$ always exists and is defined by*

$$N_S(T) = \{x \in S:\ x^{-1}Tx \subseteqq T\}.$$

Proof. If U defined to be $\{x:\ x^{-1}Tx \subseteqq T\}$ is an inverse subsemigroup, then it must be $N_S(T)$. Now U is obviously closed under products. Let $x \in U$. Then $xTx^{-1} \supseteqq xx^{-1}Txx^{-1} = Txx^{-1}$. But $x^{-1}Tx \subseteqq T$. So if $t \in T_\beta$ and $x \in S_\alpha$ then $x^{-1}tx = x^{-1}\varphi_{\alpha,\alpha\beta}t\varphi_{\beta,\alpha\beta}x\varphi_{\alpha,\alpha\beta} \in T$. Hence $T_{\alpha\beta} \neq \varphi$. So $txx^{-1} = t\varphi_{\beta,\alpha\beta}e_{\alpha\beta} = te_{\alpha\beta} \in T$, since $e_{\alpha\beta} \in T_{\alpha\beta} \subseteqq T$.

Definition 3.23. An inverse subsemigroup T of a Clifford semigroup S is called *subnormal* if there exists a sequence of inverse subsemigroups

$$T = T_0 \subseteqq T_1 \subseteqq \dots \subseteqq T_n = S$$

such that T_i is normal in T_{i+1} for $0 \leqq i \leqq n-1$. The least length n of such a series is called the *index of subnormality.*

Theorem 3.24. *Let S be a nilpotent Clifford subsemigroup, T an inverse subsemigroup such that $TE \subseteqq T$. Then T is subnormal of index at most c where c is the nilpotency class of S.*

Proof. We show that if $\{Z_i:\ 0 \leqq i \leqq c\}$ is the upper central series of S, then TZ_i is normal in TZ_{i+1}, replacing TZ_c by S. Note that $Z_0 = E$, so $T = TE = TZ_0$. By Lemma 3.15 TZ_i is an inverse subsemigroup of S. Let $x \in Z_{i+1}, y \in TZ_i$. Then $x^{-1}yx = x^{-1}yy^{-1}yx = yy^{-1}x^{-1}yx = y[y,x] \in TZ_i$ since $y \in TZ_i$ and $[y,x] \in Z_i$ since $x \in Z_{i+1}$. Thus $Z_{i+1} \subseteqq N_S(TZ_i)$. This is enough to prove the result. If $i = c-1$, then $Z_c = S \subseteqq N_S(TZ_{c-1})$, $E \subseteqq N_S(TZ_{c-1})$ so $S = SE \subseteqq N_S(TZ_{c-1})$.

We could have used the group theoretic results and transferred them. But the details of the links to the group theory would be longer than the direct proof, which parallels very closely the group theory proof.

Theorem 3.25. *Let S be a Clifford semigroup with the property that all its full inverse subsemigroups are subnormal of index at most c. Then $S \in \hat{\mathfrak{N}}_d$ where d is a function of c.*

Proof. Let $\alpha \in E$ and consider U a subgroup of S_α. Let T be a full inverse subsemigroup of S such that $T_\alpha = U$, e.g. $T_\beta = \{e_\beta\}$ if $\beta \not\leqq \alpha$, $T_\beta = U\varphi_{\alpha,\beta}$ if $\beta \leqq \alpha$.

Then $T=T_0 \subseteqq T_1 \subseteqq \ldots \subseteqq T_c = S$ is a sequence such that T_i is normal in T_{i+1} for $0 \leqq i \leqq c-1$. In particular $T_{i,\alpha}$ is a normal subgroup of $T_{i+1,\alpha}$. Hence $T_{0,\alpha}=U$ is subnormal of index at most c in S_α. This is true for all subgroups of S_α. By Roseblade [9], S_α is nilpotent of class at most $f(c)=d$ say. Hence $S \in \hat{\mathfrak{N}}_d$.

Corollary 3.26. *Let S be a finite Clifford semigroup such that all its full inverse subsemigroups are subnormal. Then S is nilpotent.*

The result that gives as a sufficient condition for a finite group to be nilpotent that all its maximal subgroups are normal does not carry over in the most obvious way.

Example 3.27. Let E consist of three elements α, β and $\alpha\beta=\gamma$. With $S_\alpha \cong C_2 \cong S_\beta$ a cyclic group of order 2, S_γ the symmetric group on three symbols. Then $\varphi_{\alpha,\gamma}$: $S_\alpha \to \{e_\gamma, (12)\}$, $\varphi_{\beta,\gamma}$: $S_\beta \to \{e_\gamma, (13)\}$ defines $S=S_\alpha \cup S_\beta \cup S_\gamma$ as a Clifford semigroup. It is easy to check that the only maximal inverse subsemigroups are $E \cup S_\gamma \cup S_\alpha$ and $E \cup S_\gamma \cup S_\beta$, both normal. But S is not nilpotent.

We leave the reader to find some possible generalizations of this result.

4. Solubility

Let $\mathfrak{S}$ be the class of soluble groups, and $\mathfrak{S}_d$ the class of soluble groups of solubility class at most d. Then $\mathfrak{S}_d$ is a variety and $\mathfrak{S} = \bigcup_{d \geqq 1} \mathfrak{S}_d$. Example 3.1 shows that $\mathfrak{S}_S$ again leads to problems. The semigroup S of Example 3.1 is in $\mathfrak{S}_S$, but its maximal group homomorphic image G is not soluble, although it is a homomorphic image of S.

Definition 4.1. The class of *soluble* Clifford semigroups is defined to be

$$\hat{\mathfrak{S}} = \bigcup_{d \geqq 1} (\mathfrak{S}_d)_S.$$

Hence $S \in \hat{\mathfrak{S}}$ if and only if $S_\alpha \in \mathfrak{S}_d$ for all $\alpha \in E$ and some $d=d(S)$. Denote $(\mathfrak{S}_d)_S$ by $\hat{\mathfrak{S}}_d$. Lemma 3.3 extends very easily.

Lemma 4.2. *$\hat{\mathfrak{S}}$ is S and Q closed.*

Definition 4.3. Let S be a Clifford semigroup. The *derived series* of S is defined to be

$$\delta_0(S) = S, \quad \delta_{i+1}(S) = [\delta_i(S), \delta_i(S)].$$

A sequence

$$E(S) = N_r \subseteqq N_{r-1} \subseteqq \ldots \subseteqq N_0 = S$$

is called an *abelian series* of S if N_i is normal in N_{i-1} and $N_{i-1}/\varkappa(\varepsilon, N_i)$ is commutative for $r \geqq i \geqq 1$.

Lemma 4.4. *Let S be a Clifford semigroup. Then $\delta_i(S)$ is a full normal subsemigroup of S for all* $i \geqq 1$.

Lemma 4.5. *Let S be a Clifford semigroup. Then*

$$\delta_i(S) = \bigcup_{\alpha \in E} \delta_i(S_\alpha).$$

Lemma 4.6. *The derived series of S is an abelian series.*

These results all follow in much the same way as the corresponding results at the beginning of Section 3.

Theorem 4.7. *Let S be a Clifford semigroup with an abelian series*

$$E(S) = N_r \subseteqq N_{r-1} \subseteqq \dots \subseteqq N_0 = S.$$

Then $N_i \supseteqq \delta_i(S)$ *for all* $i \geqq 0$ *and* $\delta_r(S) = E(S)$.

Proof. We prove the result by induction. Obviously $S = N_0 \supseteqq \delta_0(S) = S$. Assume that $N_i \supseteqq \delta_i(S)$. Then $N_i/\varkappa(\varepsilon, N_{i+1})$ is commutative and so $[s_1, s_2] \in N_{i+1}$ for all $s_1, s_2 \in N_i$. Hence by Lemma 3.5 (iv) $[s_1, s_2] \in N_{i+1}$ for all $s_1, s_2 \in \delta_i(S) \subseteqq N_i$. Then $\delta_{i+1}(S) \subseteqq N_{i+1}$. This gives the result by induction.

Corollary 4.8. *A Clifford semigroup S is in $\hat{\mathfrak{S}}$ if and only if there exists d such that* $\delta_d(S) = E(S)$.

The least such d satisfying this is called the *solubility class* of S. It is the least d such that $S \in \hat{\mathfrak{S}}_d$.

Lemma 4.9. *Let S be a Clifford semigroup. Let N be a normal full subsemigroup. Then* $S/\varkappa(\varepsilon, N) \in \hat{\mathfrak{S}}_d$ *if and only if* $\delta_d(S) \subseteqq N$.

Proof. It is immediate that if θ is a homomorphism, then $[s_1, s_2]\theta = [s_1\theta, s_2\theta]$. Hence $\delta_i(S/\varkappa(\varepsilon, N)) = \delta_i(S)\varkappa(\varepsilon, N)/\varkappa(\varepsilon, N)$ by a simple induction argument. Then $S/\varkappa(\varepsilon, N) \in \hat{\mathfrak{S}}_d$ by Corollary 4.8 if and only if $\delta_d(S/\varkappa(\varepsilon, N)) = E(S/\varkappa(\varepsilon, N))$, i.e. $\delta_d(S)\varkappa(\varepsilon, N) = E(S/\varkappa(\varepsilon, N))$. This is just $\delta_d(S) \subseteqq N$.

Theorem 4.10. *Let S be a Clifford semigroup. Let N be a normal full subsemigroup such that* $N \in \hat{\mathfrak{S}}_d$ *and* $S/\varkappa(\varepsilon, N) \in \hat{\mathfrak{S}}_e$. *Then* $S \in \hat{\mathfrak{S}}_{d+e}$.

Proof. By Lemma 4.9, $S/\varkappa(\varepsilon, N)$ is in $\hat{\mathfrak{S}}_e$ implies $\delta_e(S) \subseteqq N$. By a simple induction argument $\delta_i(N) \supseteqq \delta_{e+i}(S)$. But $N \in \hat{\mathfrak{S}}_d$ implies $\delta_d(N) = E(N)$ as N is full. So $\delta_{e+d}(S) = E(S)$ and $S \in \hat{\mathfrak{S}}_{d+e}$.

Theorem 4.11. *Let* $N \in \hat{\mathfrak{S}}_c$, $M \in \hat{\mathfrak{S}}_d$ *be normal soluble subsemigroups of S, a Clifford semigroup. Then* $NM \in \hat{\mathfrak{S}}_{c+d}$.

Proof. The proof follows closely that of Theorem 3.16. It would be instructive to develop a proof involving a more general version of Theorem 4.10 and paralleling the group theoretic proof.

Theorem 4.12. *Let S be a Clifford semigroup which satisfies the maximal condition on normal subsemigroups. Then S contains a unique maximal normal soluble subsemigroup containing all normal soluble subsemigroups.*

We present the locally soluble version of Theorem 3.17.

Theorem 4.13. *Let S be a locally soluble Clifford semigroup, N a minimal normal subsemigroup of S. Then there exists a unique* $\alpha \in E$ *such that* $N_\alpha \supset \{e_\alpha\}$, *N is commutative and for all* $\beta \leqq \alpha$, *we have* $\ker \varphi_{\alpha,\beta} \supseteqq N_\alpha$.

Proof. A minimal normal subgroup of a locally soluble group is abelian by a standard result from group theory. The same technique as in the proof of Theorem 3.20 now proves the result.

We will leave the extension of results from group theory here. There is obviously an almost inexhaustible supply of results which could be transferred, and there are also some traps for the unwary. Before finishing a few comments might be in order. Finite soluble group theory has a beautiful set of results in the formation theory of Gaschütz. The right extension of this to finite Clifford semigroups should be an interesting exercise with pleasing results. The other point concerns nilpotent versus soluble groups. The laws of $\mathfrak{N}_c$ can be defined without reference to inverses. Using this Lallement [5] showed that regular nilpotent semigroups were Clifford semigroups in $\hat{\mathfrak{N}}_c$. This might be expected because idempotents should be central in a nilpotent semigroup. The same could be done for solubility. There the natural expectation is for idempotents to commute. So it should be a theory naturally based in general inverse semigroups. This is what we hope to attempt soon.

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