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Integral manifolds, stability and decomposition of singularly perturbed systems in Banach space

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1. Introduction. This paper is dealing with the study of infinite dimensional singularly perturbed systems near an integral manifold.

Consider the system

$$\dot{x} = f(t, x, y, \varepsilon)$$
$$(1.1)$$
$$\varepsilon\dot{y} = Ay + \varepsilon g(t, x, y, \varepsilon)$$

where x and y are elements of Banach spaces X and Y with norms $\|\cdot\|$, A is a constant linear bounded operator in Y, and

$$f\colon R\times X\times Y\times[0, \varepsilon_0] \to X, \quad g\colon R\times X\times Y\times[0, \varepsilon_0] \to Y$$

are continuous nonlinear operator functions. Using the method of integral manifolds [1, 2] we shall study the stability problem for (1.1) and the problem of decomposition of (1.1) by transforming it to the form

$$\dot{u} = F(t, u, \varepsilon), \tag{1.2}$$

$$\varepsilon\dot{v} = Av + \varepsilon G(t, u, v, \varepsilon). \tag{1.3}$$

Then we shall apply this method for investigation of linear singularly perturbed systems.

2. Slow manifold. We first recall the definition of an integral manifold for the equation $\dot{x} = X(t, x)$ where x is an element of a Banach space. A set S is said to be an integral manifold if for $(t_0, x_0)\in S$, the solution $(t, x(t))$, $x(t_0) = x_0$ is in S for $t\in R$. If $(t, x(t))\in S$ only at a finite interval, then we shall say that S is a local integral manifold.

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Let $B_r=\{y\in Y,\ \|y\|\leqq r\}$, $I_{\varepsilon_0}=[0,\varepsilon_0]$, $\Omega=R\times X\times B_r\times I_{\varepsilon_0}$. Assume that f and g are bounded and satisfy the Lipschitz condition in x, y on Ω:

$$\|f(t,x,y,\varepsilon)\|\leqq M,\quad \|g(t,x,y,\varepsilon)\|\leqq M, \tag{2.1}$$

$$\begin{aligned}\|f(t,x,y,\varepsilon)-f(t,\bar{x},\bar{y},\varepsilon)\|&\leqq l(\|x-\bar{x}\|+\|y-\bar{y}\|),\\ \|g(t,x,y,\varepsilon)-g(t,\bar{x},\bar{y},\varepsilon)\|&\leqq l(\|x-\bar{x}\|+\|y-\bar{y}\|),\end{aligned} \tag{2.2}$$

where M and l are positive constants.

Assume that the spectrum $\sigma(A)$ of the linear bounded operator A satisfies the inequality $\operatorname{Re}\sigma(A)\leqq -2\alpha<0$. Then there exists a positive number K such that

$$\|e^{At}\|\leqq Ke^{-\alpha t},\quad t\geqq 0. \tag{2.3}$$

We shall say that the integral manifold of system (1.1) is a slow manifold if it can be represented of form $y=h(t,x,\varepsilon)$, where h is a continuous operator-function. If ε_0 is sufficiently small then for each $\varepsilon\in(0,\varepsilon_0)$ the system (1.1) has an integral manifold (slow manifold) represented of form $y=\varepsilon h(t,x,\varepsilon)$ (see [1], p. 438). Here h is a continuous and bounded operator-function defined on $\Omega_1=R\times X\times I_{\varepsilon_0}$ and satisfies the Lipschitz condition in x:

$$\|h(t,x,\varepsilon)-h(t,\bar{x},\varepsilon)\|\leqq \Delta\|x-\bar{x}\|,\quad \Delta>0. \tag{2.4}$$

Moreover, if f and g are continuously differentiable on Ω to k order and their derivatives are bounded and Lipschitzian in x, y then h is continuously differentiable on Ω_1 to k and its derivatives are bounded and Lipschitzian in x. In this case the operator-function h can be represented as asymptotic expansion $\varepsilon h=\varepsilon h_1(t,x)+\dots$ $\dots+\varepsilon^k h_k(t,x)+h_{k+1}(t,x,\varepsilon)$ where $h_{k+1}=O(\varepsilon^{k+1})$. The coefficients h_i of this expansion can be found from the equation

$$\varepsilon\frac{\partial h}{\partial t}+\varepsilon\frac{\partial h}{\partial x}f(t,x,\varepsilon h,\varepsilon)=Ah+g(t,x,\varepsilon h,\varepsilon). \tag{2.5}$$

For finite dimensional systems this method of approximating slow manifolds was essentially used in [3]. The method of approximation used in [4] can be generalized to infinite dimensional problems in an obvious way.

The flow on a slow manifold is governed by the reduced equation (1.2), where $F(t,u,\varepsilon)=f(t,u,\varepsilon h(t,u,\varepsilon),\varepsilon)$.

It is well-known for finite dimensional spaces X that the condition $f(t,0,0,\varepsilon)=0$, $g(t,0,0,\varepsilon)=0$ implies $h(t,0,\varepsilon)=0$ and if the zero solution of (1.2) is stable (asymptotically stable, unstable) then the zero solution of (1.1) is stable (asymptotically stable, unstable). We shall prove below this statement for infinite dimensional X.

3. Integral manifold for auxiliary system. Let us suppose that f and g are continuously differentiable on Ω and their derivatives are bounded and Lipschitzian in x, y and introduce new variables u, z and x_1 by the formulae $z=y-\varepsilon h(t, x, \varepsilon)$, $x_1=x-u$ where u satisfies (1.2). Consider the following auxiliary differential system

$$\begin{aligned} \dot{u} &= F(t, u, \varepsilon) \\ \dot{x}_1 &= f_1(t, u, x_1, z, \varepsilon) \\ \varepsilon\dot{z} &= Az+\varepsilon Z(t, u, x_1, z, \varepsilon), \end{aligned} \tag{3.1}$$

where $f_1=f\big(t, u+x_1, z+\varepsilon h(t, u+x_1, \varepsilon), \varepsilon\big)-F(t, u, \varepsilon)$,

$$Z = g\big(t, u+x_1, z+\varepsilon h(t, u+x_1, \varepsilon), \varepsilon\big)-g\big(t, u+x_1, \varepsilon h(t, u+x_1, \varepsilon), \varepsilon\big)-$$

$$-\varepsilon\frac{\partial h}{\partial x}(t, u+x_1, \varepsilon)\big[f\big(t, u+x_1, z+\varepsilon h(t, u+x_1, \varepsilon), \varepsilon\big)-f\big(t, u+x_1, \varepsilon h(t, u+x_1, \varepsilon), \varepsilon\big)\big].$$

By means of our assumptions it is easy to show that there exists a constant $N>0$ such that f_1 and Z satisfy the following inequalities

$$\|f_1(t, u, x_1, z, \varepsilon)\| \leqq N(\|x_1\|+\|z\|), \tag{3.2}$$

$$\|Z(t, u, x_1, z, \varepsilon)\| \leqq N\|z\|, \tag{3.3}$$

$$\|f_1(t, u, x_1, z, \varepsilon)-f_1(t, u, \bar{x}_1, \bar{z}, \varepsilon)\| \leqq N(\|x_1-\bar{x}_1\|+\|z-\bar{z}\|), \tag{3.4}$$

$$\|Z(t, u, x_1, z, \varepsilon)-Z(t, u, \bar{x}_1, \bar{z}, \varepsilon)\| \leqq N(\|x_1-\bar{x}_1\|+\|z-\bar{z}\|), \tag{3.5}$$

$$\begin{aligned} &\|f_1(t, u, x_1, z, \varepsilon)-f_1(t, \bar{u}, \bar{x}_1, \bar{z}, \varepsilon)\| \leqq \\ &\leqq N[1+\|x_1\|+\|z\|][\|z-\bar{z}\|+(1+\|x_1\|)\|x_1-\bar{x}_1\|+(\|x_1\|+\|z\|)\|u-\bar{u}\|], \end{aligned} \tag{3.6}$$

$$\|Z(t, u, x_1, z, \varepsilon)-Z(t, \bar{u}, \bar{x}_1, \bar{z}, \varepsilon)\| \leqq N[\|z-\bar{z}\|+\|z\|(\|u-\bar{u}\|+\|x_1-\bar{x}_1\|)], \tag{3.7}$$

where $t\in R$, $u, \bar{u}\in X$, $x_1, \bar{x}_1\in X$, $z, \bar{z}\in B_{r_1}$, $0<r_1\leqq r$.

We shall show that the system (3.1) has an integral manifold represented of form $x_1=\varepsilon H(t, u, z, \varepsilon)$, where H is an operator-function defined and continuous on $\Omega_2=R\times X\times B_\varrho\times I_{\varepsilon_1}$, $0<\varrho<\frac{r_1}{K}$, $0<\varepsilon\leqq\varepsilon_0$, and H satisfies the inequalities:

$$\|H(t, u, z, \varepsilon)\| \leqq a\|z\|, \tag{3.8}$$

$$\|H(t, u, z, \varepsilon)-H(t, u, \bar{z}, \varepsilon)\| \leqq b\|z-\bar{z}\|, \tag{3.9}$$

$$\|H(t, u, z, \varepsilon)-H(t, \bar{u}, z, \varepsilon)\| \leqq c\|z\|\cdot\|u-\bar{u}\|, \tag{3.10}$$

with $a, b, c>0$ for $t\in R$, $u, \bar{u}\in X$, $z, \bar{z}\in B_\varrho$, $\varepsilon\in I_{\varepsilon_1}$.

The flow on this manifold is governed by reduced equations (1.2), (1.3), where $F=f\big(t, u, \varepsilon h(t, u, \varepsilon), \varepsilon\big)$, $G=Z\big(t, u, \varepsilon H(t, u, v, \varepsilon), v, \varepsilon\big)$.

Moreover, every solution of (1.1) with $\|y(t_0)-\varepsilon h(t_0, x(t_0), \varepsilon)\| \leqq \varrho$ can be represented of form

$$x = u+\varepsilon H(t, u, v, \varepsilon),$$

(3.11)

$$y = v+\varepsilon h(t, x, \varepsilon) = v+\varepsilon h\big(t, u+\varepsilon H(t, u, v, \varepsilon), \varepsilon\big),$$

where u, v is the corresponding solution of (1.2), (1.3).

Our proof of this statements is modelled on KELLEY [5].

Let S be the set of operator-functions εH: $\Omega_2 \to X$ such that H is continuous and satisfies (3.8)—(3.10). Let d be a metric on S defined by

$$d(\varepsilon H, \varepsilon \bar{H}) = \sup\left\{\frac{1}{\|z\|}\varepsilon\|H(t, u, z, \varepsilon)-\bar{H}(t, u, z, \varepsilon)\|, t\in R, u\in X, z\in B_\varrho\right\}$$

for each $\varepsilon\in(0, \varepsilon_1]$, εH, $\varepsilon\bar{H}\in S$ and note that S is a complete metric space with respect to d.

For each $\varepsilon H\in S$, we consider the system

$$\dot{u} = F(t, u, \varepsilon), \tag{3.12}$$

$$\varepsilon\dot{z} = Az+\varepsilon Z\big(t, u, \varepsilon H(t, u, z, \varepsilon), z, \varepsilon\big), \tag{3.13}$$

with solutions denoted by $u=\Phi(t, t_0, u_0, \varepsilon)$, $z=\Psi(t, t_0, u_0, z_0, \varepsilon|H)$ where $\Phi(t_0, t_0, u_0, \varepsilon)=u_0$, $\Psi(t_0, t_0, u_0, z_0, \varepsilon|H)=z_0$. The operator-functions $F(t, u, \varepsilon)$, $Z(t, u, \varepsilon H(t, u, z, \varepsilon), z, \varepsilon)$ are uniformly bounded on their domains, hence, any solution of (3.12), (3.13) is defined for all t.

As usually, (see [1, 2, 5]) the equality $x_1=\varepsilon H(t, u, z, \varepsilon)$ describes an integral manifold for (3.1) if and only if the operator-function εH is a solution of the equation

$$\varepsilon H(\tau, u, z, \varepsilon) = -\int_\tau^\infty f_1\big(t, \Phi(t, \tau, u, \varepsilon), \varepsilon H(t, \Phi(t, \tau, u, \varepsilon), \Psi(t, \tau, u, z, \varepsilon|H), \varepsilon\big), \Psi(t, \tau, u, z, \varepsilon|H), \varepsilon\big)\,dt. \tag{3.14}$$

Let $\varphi(t)=\Phi(t, \tau, u, \varepsilon)$, $\psi(t)=\Psi(t, \tau, u, z, \varepsilon|H)$ then by the "variation of constants" formula

$$\psi(t) = e^{(1/\varepsilon)A(t-\tau)}z+\int_\tau^t e^{(1/\varepsilon)A(t-s)}Z\big(s, \varphi(s), \varepsilon H\big(s, \varphi(s), \psi(s), \varepsilon\big)\,ds.$$

By (2.3), (3.3) and (3.8) there holds for all $-\infty<\tau\leqq t<\infty$, $\|z\|\leqq\varrho$, $\varepsilon\in(0, \varepsilon_1]$:

$$\|\psi(t)\| \leqq Ke^{-(\alpha/\varepsilon)(t-\tau)}\|z\|+\int_\tau^t Ke^{-(\alpha/\varepsilon)(t-s)}N\|\psi(s)\|\,ds.$$

Therefore, by Gronwall's Lemma, we obtain

$$\|\psi(t)\| \leqq Ke^{-(\alpha_1/\varepsilon)(t-\tau)}\|z\|, \quad -\infty<\tau\leqq t<\infty, \tag{3.15}$$

where $\alpha_1=\alpha-\varepsilon KN>\gamma>0$ for sufficiently small ε_1.

Now define an operator T on S by setting

$$T(H)(\tau, u, z, \varepsilon) = -\int_{\tau}^{\infty} f_1(t, \varphi(t), \varepsilon H(t, \varphi(t), \psi(t), \varepsilon), \psi(t), \varepsilon)\, dt. \tag{3.16}$$

The improper integral here converges by virtue of (3.2), (3.8) and (3.15). It is clear that $T(H)$ as defined in (3.16) is continuous on Ω_2. Also, by (3.2), (3.8) and (3.15) we obtain

$$\|T(H)(\tau, u, z, \varepsilon)\| \leqq \int_{\tau}^{\infty} N(1+\varepsilon a)Ke^{-(\alpha_1/\varepsilon)(t-\tau)}\|z\|\, dt = \varepsilon\frac{NK}{\alpha_1}(1+\varepsilon a)\|z\|,$$

and therefore $T(H)$ satisfies the boundedness condition required by (3.8) if $\varepsilon_1\dfrac{NK}{\alpha_1}<1$ and $a\geqq\dfrac{NK}{\alpha_1}\Big/\left(1-\varepsilon\dfrac{NK}{\alpha_1}\right)$.

To prove that $T(H)$ satisfies the conditions, required by (3.9), (3.10) we reason as follows. Let $u\in X$, $z, \bar{z}\in B_\varrho$, $\psi_1=\Psi(t, \tau, u, \bar{z}, \varepsilon|H)$. Then, by (3.5), (3.9), (2.3) and by the "variations of constants" formula we have

$$\|\psi(t)-\psi_1(t)\| \leqq Ke^{-(\alpha/\varepsilon)(t-\tau)}\|z-\bar{z}\| + \int_{\tau}^{t} Ke^{-(\alpha/\varepsilon)(t-s)}N(1+\varepsilon b)\|\psi(s)-\psi_1(s)\|\, ds.$$

Therefore, by Gronwall's Lemma, we obtain

$$\|\psi(t)-\psi_1(t)\| \leqq Ke^{-(\alpha_2/\varepsilon)(t-\tau)}\|z-\bar{z}\|, \quad -\infty<\tau\leqq t<\infty, \tag{3.17}$$

$$\alpha_2 = \alpha-\varepsilon KN(1+\varepsilon b).$$

Then, by (3.4), (3.9) and (3.17)

$$\|T(H)(\tau, u, z, \varepsilon)-T(H)(\tau, u, \bar{z}, \varepsilon)\| \leqq \int_{\tau}^{\infty} N(1+\varepsilon b)\|\psi(t)-\psi_1(t)\|\, dt \leqq$$

$$\leqq \varepsilon\frac{KN}{\alpha_2}(1+\varepsilon b)\|z-\bar{z}\|$$

It is clear that for sufficiently small ε_1 a constant b can be choosen such that $\alpha_2>\gamma$ and $\dfrac{K^2N}{\alpha_2}(1+\varepsilon_1 b)\leqq b$. From this inequality it follows that $T(H)$ satisfies the Lipschitz condition required by (3.9).

In exactly the same way by the inequality

$$\|\Phi(t, \tau, u, \varepsilon)-\Phi(t, \tau, \bar{u}, \varepsilon)\| \leqq e^{l(1+\varepsilon\Delta)(t-\tau)}\|u-\bar{u}\|, \quad -\infty<\tau\leqq t<\infty$$

and (3.10), (3.7) and (3.6) it is easy to show that for some $c>0$ and sufficiently small ε_1 the operator-function $T(H)$ satisfies the condition (3.10). Now, let $\varepsilon H, \varepsilon\bar{H}\in S$, $\psi_2(t)=\Psi(t, \tau, u, z, \varepsilon|\bar{H})$. Then by (3.4) and (3.9)

$$\|T(H)(\tau, u, z, \varepsilon)-T(\bar{H})(\tau, u, z, \varepsilon)\| \leqq \tag{3.18}$$

$$\leqq \int_{\tau}^{\infty} N\big[(1+\varepsilon b)\|\psi(t)-\psi_2(t)\|+\varepsilon\big\|H(t, \varphi(t), \psi_2(t), \varepsilon)-\bar{H}(t, \varphi(t), \psi_2(t), \varepsilon)\big\|\big]\,dt \leqq$$

$$\leqq \int_{\tau}^{\infty} N[(1+\varepsilon b)\|\psi(t)-\psi_2(t)\|+Ke^{-(\gamma/\varepsilon)(t-\tau)}\|z\|\, d(\varepsilon H, \varepsilon\bar{H})]\,dt.$$

Using (3.5) and (3.9) we find that

$$\|\psi(t)-\psi_2(t)\| \leqq$$

$$\leqq \int_{\tau}^{t} Ke^{-(\alpha/\varepsilon)(t-\tau)}N[(1+\varepsilon b)\|\psi(s)-\psi_2(s)\|+Ke^{-(\gamma/\varepsilon)(t-s)}\|z\|\, d(\varepsilon H, \varepsilon\bar{H})]\,dt.$$

Substitution of this into (3.18) yields

$$\frac{1}{\|z\|}\|T(H)(\tau, u, z, \varepsilon)-T(\bar{H})(\tau, u, z, \varepsilon)\| \leqq \varepsilon\frac{KN}{\gamma}\left[(1+\varepsilon b)\frac{KN}{\alpha_2-\gamma}+1\right]d(\varepsilon H, \varepsilon\bar{H}).$$

From this last inequality it easily follows that T is a contraction mapping if ε_1 is sufficiently small.

Thus, T is a contraction mapping of S into itself and so, by the known Banach Contraction Principle, T must have a unique fixed point $\varepsilon H\in S$. The operator-function εH is a solution of (3.14) and, therefore, the equality $x_1=\varepsilon H(t, u, z, \varepsilon)$ represents an integral manifold for (3.1). The flow on this manifold is governed by (1.2), (1.3) where

$$F=f(t, u, \varepsilon h(t, u, \varepsilon), \varepsilon), \quad G=Z(t, u, \varepsilon H(t, u, v, \varepsilon), v, \varepsilon).$$

4. Decomposition and stability. Our next object is to obtain the representation (3.11). Let $x=x(t)$, $y=y(t)$ be a solution of (1.1) with $x(t_0)=x_0$, $y(t_0)=y_0$, $\|y_0-\varepsilon h(t_0, x_0, \varepsilon)\|\leqq\varrho$. We shall show that there exists a solution $u=u(t)$, $u(t_0)=u_0$, $v=v(t)$, $v(t_0)=v_0$ of (1.2), (1.3) such that

$$\begin{aligned} x(t) &= u(t)+\varepsilon H(t, u(t), v(t), \varepsilon),\\ y(t) &= v(t)+\varepsilon h(t, x(t), \varepsilon). \end{aligned} \tag{4.1}$$

It is sufficient to show that (4.1) holds for $t=t_0$. Substitution $t=t_0$ into (4.1) yields

$$x_0 = u_0+\varepsilon H(t_0, u_0, v_0, \varepsilon), \quad y_0 = v_0+\varepsilon h(t_0, x_0, \varepsilon)$$

and, therefore, $v_0=y_0-\varepsilon h(t_0, x_0, \varepsilon)$. For u_0 we obtain the equation

$$x_0 = u_0+\varepsilon H\big(t_0, u_0, y_0-\varepsilon h(t_0, x_0, \varepsilon), \varepsilon\big). \tag{4.2}$$

This last equation can be represented of form

$$u_0 = P(u_0, \varepsilon) = x_0-\varepsilon H\big(t_0, u_0, y_0-\varepsilon h(t_0, x_0, \varepsilon), \varepsilon\big).$$

From (3.10) it is easy to obtain that for each $\varepsilon\in(0, \varepsilon_1]$ and fixed x_0, y_0 such that $\|y_0-\varepsilon h(t_0, x_0, \varepsilon)\|\leqq\varrho<\dfrac{1}{\varepsilon_1 c}$, P is a contraction mapping of X into itself and so, by the Banach Contraction Mapping Principle, P must have a unique fixed point $u_0\in X$ which is the required solution of (4.2).

Now, we consider the stability problem for (1.1). Using (4.1) we obtain that every solution $x=x(t)$, $y=y(t)$ with $\|y_0-\varepsilon h(t_0, x_0, \varepsilon)\|\leqq\varrho$ can be represented as

$$\begin{aligned} x(t) &= u(t)+\varphi_1(t),\\ y(t) &= \varepsilon h\big(t, u(t), \varepsilon\big)+\varphi_2(t), \end{aligned} \tag{4.3}$$

where $(u(t), \varepsilon h(t, u(t), \varepsilon))$ is a solution lying in the manifold $y=\varepsilon h(t, x, \varepsilon)$; $\varphi_1= =\varepsilon H(t, u(t), v(t), \varepsilon)$, $\varphi_2=v(t)+\varepsilon h\big(t, u(t)+\varepsilon H(t, u(t), v(t), \varepsilon), \varepsilon\big)-\varepsilon h(t, u(t), \varepsilon)$. This and (2.4), (3.8) and (3.15) allow us to write

$$\begin{aligned} \|\varphi_1(t)\| &\leqq \varepsilon a K e^{-(\gamma/\varepsilon)(t-t_0)}\|v_0\|,\\ \|\varphi_2(t)\| &\leqq (1+\varepsilon^2 a\Delta)K e^{-(\gamma/\varepsilon)(t-t_0)}\|v_0\|,\\ \varepsilon\in(0, \varepsilon_1], &\quad t\geqq t_0, \quad v_0 = y_0-\varepsilon h(t_0, x_0, \varepsilon). \end{aligned} \tag{4.4}$$

Assume that $f(t, 0, 0, \varepsilon)=0$, $g(t, 0, 0, \varepsilon)=0$; then $h(t, 0, \varepsilon)=0$ and $F(t, 0, \varepsilon)=0$. By (4.3) and (4.4) we obtain

$$\|x(t)\| \leqq \|u(t)\|+\varepsilon a K e^{-(\gamma/\varepsilon)(t-t_0)}\|v_0\|.$$

$$\|y(t)\| \leqq \varepsilon\Delta\|u(t)\|+(1+\varepsilon^2 a\Delta)K e^{-(\gamma/\varepsilon)(t-t_0)}\|v_0\|, \quad t\geqq t_0.$$

From this last inequalities it easily follows that if the zero solution of (1.2) is stable (asymptotically stable) then the zero solution of (1.1) is stable (asymptotically stable). It is obvious that the instability of the zero solution of (1.2) implies the instability of the zero solution of (1.1).

Now, we can summarize our results in the following

Theorem 4.1. *Let f and g in* (1.1) *be continuous, bounded and satisfy* (2.1), (2.2) *on $R\times X\times B_r\times I_{\varepsilon_0}$; let us assume that the spectrum of the linear bounded operator A satisfies* $\operatorname{Re}\sigma(A)\leqq-2\varkappa<0$. *Then there exist numbers ε_1 and ϱ_1 such that the following assertations are true:*

(i) *For each* $\varepsilon \in (0, \varepsilon_1]$, $\varrho \in (0, \varrho_1)$ *and* t_0 *there exists for* (3.1) *an integral manifold represented by an equation of form* $x_1 = \varepsilon H(t, u, z, \varepsilon)$ *where* H *is an operator-function defined and continuous on* $R \times X \times B_\varrho \times I_{\varepsilon_1}$ *and, moreover,* H *satisfies* (3.8)—(3.10).

(ii) *Every solution* $x = x(t)$, $y = y(t)$ *of* (1.1) *with* $x(t_0) = x_0, y(t_0) = y_0$, $\|y_0 - \varepsilon h(t_0, x_0, \varepsilon)\| \leq \varrho$ *can be represented of form* (3.11), *where* $u = u(t)$, $u(t_0) = u_0$ *is a solution of* (1.2), u_0 *is a solution of* (4.2); $v = v(t)$ *is a solution of* (1.3) *with* $u = u(t)$, $v(t_0) = v_0 = y_0 - \varepsilon h(t_0, x_0, \varepsilon)$.

(iii) *If* $f(t, 0, 0, \varepsilon) = 0$, $g(t, 0, 0, \varepsilon) = 0$ *and the zero solution of* (1.2) *is stable (asymptotically stable, unstable), then the zero solution of* (1.1) *is stable (asymptotically stable, unstable).*

Note, that in the proof of this theorem we did not use the boundedness of A. So, Theorem 4.1 can be extended onto the system (1.1) with an unbounded operator A, if A is the generator of a strongly continuous linear semigroup $S(t)$ such that $\|S(t)\| \leq Ke^{-\alpha t}$, $t \geq 0$.

It should be observed that similar problems for systems with unbounded operators were studied in [2, 6].

The next result shows that, in principle, the operator-function H can be approximated to any degree of accuracy with respect to ε. Let $D(\varepsilon H) = \varepsilon \frac{\partial H}{\partial t} + \varepsilon \frac{\partial H}{\partial u} F(t, u, \varepsilon) + \frac{\partial H}{\partial v}(Av + \varepsilon Z(t, u, \varepsilon H, v, \varepsilon)) - f_1(t, u, \varepsilon H, v, \varepsilon)$. *If* $D(\varepsilon \bar{H}) = O(\varepsilon^{k+1})$ *then* $\|H - \bar{H}\| = O(\varepsilon^k)$.

The idea of the proof of this statement is very simple. Let us introduce a new variable $x_2 = x_1 - \varepsilon \bar{H}(t, u, z, \varepsilon)$; then for u, x_2, z we obtain the following system

$$\dot{u} = f(t, u, \varepsilon),$$

$$\dot{x}_2 = f_2(t, u, x_2, z, \varepsilon),$$

$$\varepsilon \dot{z} = Az + \varepsilon Z(t, u, x_2 + \varepsilon \bar{H}, z, \varepsilon),$$

where $f_1 = f_1(t, u, x_2 + \varepsilon \bar{H}, z, \varepsilon) - f_1(t, u, \varepsilon \bar{H}, z, \varepsilon) - \varepsilon \frac{\partial \bar{H}}{\partial z}[Z(t, u, x_2 + \varepsilon \bar{H}, z, \varepsilon) - Z(t, u, \varepsilon \bar{H}, z, \varepsilon)]$, $\varepsilon \bar{H} = \varepsilon \bar{H}(t, u, z, \varepsilon)$. This last system has an integral manifold $x_2 = \varepsilon H_{k+1}(t, u, z, \varepsilon)$ such that $H_{k+1} = O(\varepsilon^k)$. It means that the system (3.1) has the integral manifold $x_1 = \varepsilon H(t, u, z, \varepsilon) = \varepsilon \bar{H}(t, u, z, \varepsilon) + O(\varepsilon^{k+1})$.

In many problems, H can be found as asymptotic expansion

$$\varepsilon H = \varepsilon H_1(t, u, v) + \ldots + \varepsilon^k H_k(t, u, v) + O(\varepsilon^{k+1})$$

from the equation $D(\varepsilon H) = 0$. Note, that u_0 can be found as asymptotic expansion

$$u_0 = u_0(\varepsilon) = u_0^0 + \varepsilon u_0^1 + \ldots + \varepsilon^k u_0^k + O(\varepsilon^{k+1})$$

from (4.2). It is easy to see that $u_0^0 = x_0$, $u_0^1 = -H_1(t_0, x_0, y_0)$.

5. Linear systems. Consider the following system

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + f_1, \tag{5.1}$$
$$\varepsilon\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + f_2,$$

where x_i, $f_i = f_i(t, \varepsilon)$ vary in the Banach space X_i, and $A_{ij} = A_{ij}(t, \varepsilon)$ are operator-functions $A_{ij}\colon X_j \to X_i$ $(i, j = 1, 2)$. Assume A_{ij} and f_i to have high order continuous and bounded derivatives with respect to t and ε, for $t \in R$, $\varepsilon \in [0, \varepsilon_0]$. Therefore, they can be represented as asymptotic expansions

$$A_{ij} = A_{ij}^{(0)}(t) + \varepsilon A_{ij}^{(1)}(t) + \ldots + \varepsilon^k A_{ij}^{(k)}(t) + O(\varepsilon^{k+1}),$$
$$f_i = f_i^{(0)}(t) + \varepsilon f_i^{(1)}(t) + \ldots + \varepsilon^k f_i^{(k)}(t) + O(\varepsilon^{k+1})$$

with smooth and bounded coefficients.

Let us suppose that the family $A_{22}^{(0)}(t)$, $t \in R$, is compact, the spectrum $\sigma(A_{22}^{(0)})$ of $A_{22}^{(0)}(t)$ satisfies the inequality

$$\operatorname{Re}\sigma(A_{22}^{(0)}) \leqq -2\alpha < 0, \quad t \in R \tag{5.2}$$

and there exists bounded operator $[A_{22}^{(0)}]^{-1}$. Under such assumptions there exists a transformation

$$x_1 = u + \varepsilon H(t, \varepsilon)v,$$
$$x_2 = v + L(t, \varepsilon)x_1 + l(t, \varepsilon) = (I + \varepsilon LH)v + Lu + l(t, \varepsilon),$$

analogous to (3.11) for the linear case. The new variables u, v satisfy the equations

$$\dot{u} = (A_{11} + A_{12}L)u + f_1 + A_{12}l, \tag{5.3}$$

$$\varepsilon\dot{v} = (A_{22} - \varepsilon LA_{12})v. \tag{5.4}$$

The operator-functions L, H and the function l can be found from the equations

$$\varepsilon\dot{L} + \varepsilon L(A_{11} + A_{12}L) = A_{21} + A_{22}L, \tag{5.5}$$

$$\varepsilon\dot{H} + H(A_{22} - \varepsilon LA_{12}) = \varepsilon(A_{11} + A_{12}L)H + A_{12}, \tag{5.6}$$

$$\varepsilon\dot{l} + \varepsilon Lf_1 = (A_{22} - \varepsilon LA_{12})l + f_2 \tag{5.7}$$

as asymptotic expansions $L = L^{(0)}(t) + \varepsilon L^{(1)}(t) + \ldots + \varepsilon^k L^{(k)}(t) + O(\varepsilon^{k+1})$,

$$H = H^{(0)}(t) + \varepsilon H^{(1)}(t) + \ldots + \varepsilon^{k-1} H^{(k-1)}(t) + O(\varepsilon^k),$$
$$l = l^{(0)}(t) + \varepsilon l^{(1)}(t) + \ldots + \varepsilon^k l^{(k)}(t) + O(\varepsilon^{k+1}).$$

It is a straightforward computation to obtain expressions for $L^{(i)}$, $H^{(i)}$, $l^{(i)}$ from (5.5)—(5.7).

Note that L is a bounded solution of the Riccati equation (5.5) on R and, therefore, satisfies the integral equation

$$L(t, \varepsilon) = \frac{1}{\varepsilon} \int_{-\infty}^{t} U(t, s, \varepsilon)\big[A_{21}(s, \varepsilon) - \varepsilon L(s, \varepsilon)\big(A_{11}(s, \varepsilon) + A_{12}(s, \varepsilon)L(s, \varepsilon)\big)\big]\, ds,$$

where U is the evolutional operator of the equation $\varepsilon\dot{x}_2 = A_{22}x_2$. Using (5.2) and the compactness of $A_{22}^{(0)}(t)$ we obtain

$$\|U(t, s, \varepsilon)\| \leqq Ke^{-(\alpha/\varepsilon)(t-s)}, \quad -\infty < s \leqq t < \infty. \tag{5.8}$$

For H and l we have the exact expressions

$$H = -\frac{1}{\varepsilon} \int_t^{\infty} V(t, s, \varepsilon) A_{12}(s, \varepsilon) W(s, t, \varepsilon)\, ds,$$

$$l = \frac{1}{\varepsilon} \int_{-\infty}^{t} W(t, s, \varepsilon)[f_2(s, \varepsilon) - \varepsilon L(s, \varepsilon) f_1(s, \varepsilon)]\, ds,$$

where V is the evolutional operator of the equation $\dot{x}_1 = (A_{11} + A_{12}L)x_1$ and W is the one of the equation $\varepsilon\dot{x}_2 = (A_{22} - \varepsilon LA_{12})x_2$. The improper integrals here converge by virtue of (5.8). As earlier, the stability of (5.3) is equivalent to the stability of (5.1).

In conclusion it should be noted that the stability and decomposition problems for finite dimensional systems were considered in [7].

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