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Non-atomic measure spaces and Fredholm composition operators

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1. Introduction. Let $(X, \mathcal{S}, \lambda)$ be a sigma-finite measure space and let T be a measurable nonsingular ($\lambda T^{-1}(E)=0$ whenever $\lambda(E)=0$ for $E \in \mathcal{S}$) transformation from X into itself. Then the composition transformation C_T on $L^2(\lambda)$ is defined as $C_T f = f \circ T$ for every $f \in L^2(\lambda)$. If the range of C_T is in $L^2(\lambda)$ and C_T is bounded, then we call C_T the composition operator induced by T. It has been proved that a nonsingular measurable transformation T induces a composition operator C_T if and only if there exists a constant $M>0$ such that $\lambda T^{-1}(E) \leqq M\lambda(E)$ for every $E \in \mathcal{S}$. Hence the induced measure λT^{-1} is absolutely continuous with respect to the measure λ. Let f_0 denote the Radon—Nikodym derivative of the measure λT^{-1} with respect to λ.

The main purpose of this paper is to study Fredholm, essentially unitary and essentially normal composition operators on $L^2(\lambda)$ when the underlying measure space is non atomic. In case X is the unit interval of the real line and λ is the Lebesgue measure on the Borel subsets X it turns out that the composition operator C_T on $L^2(\lambda)$ is Fredholm if and only if C_T is invertible [2]. We prove here that the above result is true for a general non-atomic measure space. We also prove that the set of essentially unitary composition operators on $L^2(\lambda)$ coincides with the set of unitary composition operators on $L^2(\lambda)$ and the set of essentially isometric composition operators coincide with the set of isometric composition operators on $L^2(\lambda)$. It is also proved that when C_T has dense range, C_T is essentially normal if and only if C_T is normal. Note that a measure space $(X, \mathcal{S}, \lambda)$ is said to be non-atomic if for every nonnull $E \in \mathcal{S}$, there exists a nonnull $F \in \mathcal{S}$ such that $F \subset E$ and $\lambda(F) < \lambda(E)$.

Definitions. Let $B(H)$ denote the Banach algebra of all operators on a Hilbert space H and $C(H)$ denote the ideal of compact operators on H. Let π be the natural homomorphism from $B(H)$ to the Calkin algebra $B(H)/C(H)$. An

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operator $A \in B(H)$ is said to be Fredholm, essentially unitary, essentially normal, an essential isometry or an essential coisometry according as $\pi(A)$ is invertible, unitary, normal, an isometry or a coisometry, respectively. It has been proved that A is Fredholm if and only if A has closed range, and the kernel of A and the kernel of A^* are finite dimensional. A is called quasiunitary if A^*A-I and AA^*-I are finite rank operators [2].

2. Fredholm composition operators. If $C_T \in B(L^2(\lambda))$, then we know that $C_T^* C_T = M_{f_0}$ [3]. So, $\ker C_T = \ker C_T^* C_T = \ker M_{f_0} = L^2(X_0)$, where $X_0 = \{x : f_0(x) = 0\}$. The following theorem computes the kernel of C_T^* which is useful in proving the main theorem of this section.

Theorem 2.1. *Let* $C_T \in B(L^2(\lambda))$. *Then*

$$\ker C_T^* = \Big\{f\colon f \in L^2(\lambda) \text{ and } \int_{T^{-1}(E)} f\, d\lambda = 0 \text{ for all } E \in \mathcal{S}\Big\}.$$

Proof. Let $f \in L^2(\lambda)$. Then $f \in \ker C_T^*$ if and only if the inner product $\langle f, g\rangle = 0$ for every $g \in (\ker C_T^*)^{\perp} = \overline{\operatorname{Ran} C_T}$. Since the span of the characteristic function $\{X_{T^{-1}(E)}\colon \lambda T^{-1}(E) < \infty \text{ and } E \in \mathcal{S}\}$ is dense in $\overline{\operatorname{Ran} C_T}$, we conclude that $f \in \ker C_T^*$ if and only if $\int_{T^{-1}(E)} f\, d\lambda = 0$ for every $E \in \mathcal{S}$. Hence the proof is completed.

Definition. If $(X, \mathcal{S}, \lambda)$ is a measure space, then the sigma-algebra $T^{-1}(\mathcal{S}) = \{T^{-1}(E)\colon E \in \mathcal{S}\}$ is said to be essentially all of $\mathcal{S}$ if for every $E \in \mathcal{S}$ there exists $T^{-1}(F) \in T^{-1}(\mathcal{S})$ such that $\lambda(E \triangle T^{-1}(F)) = \lambda\{(E \setminus T^{-1}(F)) \cup (T^{-1}(F) \setminus E)\} = 0$.

It has been proved by WHITLEY [6] and SINGH and KUMAR [4] that C_T has dense range if and only if $T^{-1}(\mathcal{S})$ is essentially all of $\mathcal{S}$. This we can conclude from the above theorem also.

Corollary 2.1. *No characteristic function belongs to* $\ker C_T^*$. *In fact, no positive function belongs to* $\ker C_T^*$.

Corollary 2.3. *If* $C_T \in B(L^2(\lambda))$, *then* $\ker C_T \subset \ker C_T^*$ *implies that* C_T *is an injection.*

Theorem 2.4. *Let* C_T *be a normal composition operator on* $L^2(\lambda)$. *Then* C_T *is Fredholm if and only if* C_T *is invertible.*

Proof. Since every normal composition operator on $L^2(\lambda)$ is an injection [4], the result follows.

The above theorem is not true in general as evident from the following example:

Example 2.5. Let l^2 denote the Hilbert space of all square summable sequences of complex numbers. Define the operator $A\colon l^2 \to l^2$ by

$$(Ax)(n) = \begin{cases} 0 & \text{if } n = 1 \\ x_n & \text{if } n > 1 \end{cases}$$

for $x=\{x_n\colon n\in\mathbf{N}\}$ in l^2. Then $A=A^*$ and hence A is normal. Also $\dim\ker A = \dim\ker A^* = 1$ and the range of A is closed. Hence A is a normal Fredholm operator. But clearly, A is not invertible.

From now on we assume that the measure space $(X, \mathscr{S}, \lambda)$ is non-atomic. The following theorem shows that the set of Fredholm composition operators and the set of invertible composition operators on $L^2(\lambda)$ coincide.

Theorem 2.6. *Let $C_T \in B(L^2(\lambda))$. Then C_T is Fredholm if and only if C_T is invertible.*

Proof. Suppose C_T is Fredholm. Then $\ker C_T$ and $\ker C_T^*$ are finite dimensional and C_T has closed range. But $\ker C_T = L^2(X_0)$, where $X_0=\{x\colon f_0(x)=0\}$ and λ is non-atomic implies that $\ker C_T$ is $\{0\}$. Hence to prove that C_T is invertible it is enough to prove that C_T has dense range. Suppose C_T does not have dense range. Then there exists a measurable set G in $\mathscr{S}$ such that G is not in $T^{-1}(\mathscr{S})$. We can find a measurable set E such that $T^{-1}(E)\supset G$. Let $T^{-1}(E)=G\cup F$. Then F is a nonnull measurable set and F does not belong to $T^{-1}(\mathscr{S})$. If we partition E into countable disjoint measurable sets, then at least one set among those partitions, say E^1, will be such that $T^{-1}(E^1)$ contains nonnull measurable subsets G^1 of G and F^1 of F where G^1 and F^1 are not in $T^{-1}(\mathscr{S})$ and $\lambda(E^1)<1$. Again partition E^1. Then we get at least one E^2 such that $\lambda(E^2)<1/2$ and $T^{-1}(E^2)$ containing nonnull parts of G and F which are not in $T^{-1}(\mathscr{S})$. Repeat this process. If at each stage of partition, there is exactly one measurable set E^n such that $T^{-1}(E^n)$ contains nonnull parts G^n of G and F^n of F such that G^n and F^n are not in $T^{-1}(E)$, then E^n can be made to approach a null set, since $\lambda(E^n)<1/n$. This will imply that G and F are in $T^{-1}(\mathscr{S})$ which is a contradiction. Hence we can get a disjoint sequence $\{E_n\colon n\in\mathbf{N}\}$ of measurable subsets of E such that for every n, $T^{-1}(E_n)=G_n\cup F_n$ where $G_n\subset G$ and $F_n\subset F$ and G_n and F_n are not in $T^{-1}(\mathscr{S})$. Now consider the sequence $\{G_n\colon n\in\mathbf{N}\}$. This is a disjoint sequence and G_n does not belong to $T^{-1}(\mathscr{S})$ for every $n\in\mathbf{N}$. Also $G_1\cup G_2\cup\ldots\cup G_k$ does not belong to $T^{-1}(\mathscr{S})$ for every $k\in\mathbf{N}$. For, if $G_1\cup G_2\in T^{-1}(\mathscr{S})$, then there is a measurable set $K\subset E_1\cup E_2$ such that $T^{-1}(K)=G_1\cup G_2$. Hence $T^{-1}(K\cap E_1)=G_1$ which implies that $G_1\in T^{-1}(\mathscr{S})$ which is a contradiction.

Now, since G_1 does not belong to $T^{-1}(\mathscr{S})$, there is a function f_1 in $L^2(\lambda)$ such that $\int_{T^{-1}(E)} f_1\,d\lambda=0$ for every $E\in\mathscr{S}$ and $\int_{G_1} f_1\,d\lambda\neq 0$. For, if not, then $\ker C_T^*\subset (X_{G_1})^{\perp}$ and hence $(X_{G_1})^{\perp\perp}\subset \operatorname{Ran} C_T$. This implies that $X_{G_1}\in\operatorname{Ran} C_T$ which is a contradiction since G_1 does not belong to $T^{-1}(\mathcal{S})$. Again, there is a function f_2 in $L^2(\lambda)$ such that $\int_{T^{-1}(E)} f_2 d\lambda=0$ for every $E\in\mathcal{S}$ and $\int_{G_1} f_2 d\lambda=0$ but $\int_{G_2} f_2 d\lambda\neq 0$. For, if not, then $(\ker C_T^*)\cap(X_{G_1})^\perp\subset(X_{G_2})^\perp$. Hence $(X_{G_2})^{\perp\perp}\subset\text{span}\,\{\text{Ran}\,C_T, X_{G_1}\}$. This implies that $X_{G_2}=f+\alpha x_{G_1}$ for some f in $\text{Ran}\,C_T$ and $\alpha\in\mathbf{C}$. Hence $f=X_{G_2}-$ $-\alpha X_{G_1}$ and this will imply that f is not measurable with respect to the sigma algebra $T^{-1}(\mathcal{S})$ which is a contradiction. Proceeding like this we will get a sequence $\{f_n\colon n\in\mathbf{N}\}\subset\ker C_T^*$ such that $\int_{G_k} f_n d\lambda$ is not equal to zero for $k=n$ and is zero for $k<n$. Hence, no two functions in $\{f_n\colon n\in\mathbf{N}\}$ are linearly dependent and hence $\dim\ker C_T^*=\infty$ which is a contradiction. Hence the theorem is proved.

3. Essentially unitary and essentially normal composition operators. First we shall characterise essentially isometric and essentially coisometric composition operators on $L^2(\lambda)$.

Theorem 3.1. *Let $C_T\in B(L^2(\lambda))$. Then C_T is an essential isometry if and only if C_T is an isometry.*

Proof. Let C_T be an essential isometry. Then $\pi(C_T)^*\pi(C_T)=\pi(I)$ which implies that $C_T^*C_T-I$ is compact. But $C_T^*C_T-I=M_{f_0}-I$ is compact on $L^2(\lambda)$ if and only if $f_0=1$ a.e. [5]. This implies that C_T is an isometry and hence the proof is completed.

Theorem 3.2. *Let $C_T\in B(L^2(\lambda))$. Then C_T is an essential coisometry if and only if it is a coisometry.*

Proof. Let C_T be an essential coisometry. Now, it is clear that C_T is an essential coisometry if and only if $C_TC_T^*-I$ is compact. But

$$C_TC_T^*-I=\begin{cases} M_{f_0\circ T^{-1}} & \text{on } \overline{\text{Ran}\,C_T} \\ -I & \text{on } \ker C_T^*. \end{cases}$$

Since $\overline{\text{ran}\,C_T}$ and $\ker C_T^*$ are invariant under $C_TC_T^*-I$, $C_TC_T^*-I$ is compact if and only if $M_{f_0\circ T^{-1}}$ is compact on $\overline{\text{ran}\,C_T}$ and $-I$ is compact on $\ker C_T^*$. But $-I$ is compact on $\ker C_T^*$ if and only if $\ker C_T^*$ is finite dimensional which further implies that $\ker C_T^*=\{0\}$. Hence C_T has dense range and $M_{f_0\circ T^{-1}}$ is compact on $\overline{\text{ran}\,C_T}=L^2(\lambda)$ if and only if $f_0\circ T=1$ a.e. This implies that C_T is a coisometry and hence the theorem is proved.

Theorem 3.3. *Let $C_T\in B(L^2(\lambda))$. Then C_T is essentially unitary if and only if C_T is unitary.*

Proof. C_T is essentially unitary if and only if C_T is essentially an isometry and essentially a coisometry. Hence the theorem follows from Theorems 3.1 and 3.2.

Corollary 3.4. *Let $C_T \in B(L^2(\lambda))$. Then C_T is quasiunitary if and only if C_T is unitary.*

Theorem 3.5. *Let C_T be a composition operator on $L^2(\lambda)$ with dense range. Then C_T is essentially normal if and only if C_T is normal.*

Proof. C_T is essentially normal if and only if $C_T^*C_T - C_TC_T^*$ is compact. But when C_T has dense range, $C_T^*C_T - C_TC_T^* = M_{f_0 - f_0 \circ T}$ and hence $M_{f_0 - f_0 \circ T}$ is compact on $L^2(\lambda)$ implies that $f_0 = f_0 \circ T$ a.e. and this further implies that C_T is normal. Thus the theorem is proved.

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