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An extension of the Lindeberg—Trotter operator-theoretic approach to limit theorems for dependent random variables

II. Approximation theorems with *O*-rates, applications to martingale difference arrays

PAUL L. BUTZER, HERIBERT KIRSCHFINK and DIETMAR SCHULZ

This is Part II of the paper [10]. The contents of Part I, particularly the notations and preliminary results, are assumed to be known. References are in alphabetical order in each part, a few of the basic papers of Part I being recalled here. The sections are numbered consecutively.

Whereas Part I is concerned with convergence assertions without as well as with o-rates for dependent r.v.'s, all established with the help of the conditional Trotter operator first defined there, the purpose of Part II is to deal with O-error estimates, not only for convergence in distribution but also for the uniform convergence of distribution functions. The specializations to martingales carried out in Section 8 enable one to compare the results with those of other authors. Firstly some modifications and corrections are made to Part I.

3. A generalization of the Trotter-operator for dependent r.v.'s. — A revisit. Let us recall the definition of the generalized Trotter operator in terms of the conditional expectation given in Section 3.

Definition 1. Let $X\in\mathfrak{L}(\Omega, \mathfrak{A}, P)$ and $\mathfrak{G}$ be an arbitrary sub-σ-algebra of $\mathfrak{A}$. The conditional Trotter operator $V_{X|\mathfrak{G}}\colon C_B\to C_B\times(\mathfrak{Z}(\Omega, \mathfrak{G}))$ of X relative to $\mathfrak{G}$ is defined for $f\in C_B$ by

$$V_{X|\mathfrak{G}}f(y):=\inf_{x\in A_\alpha(y,f)} E[f(X+x)|\mathfrak{G}]\quad (y\in\mathbf{R})$$

for an $\alpha>0$ with $\alpha\in Q$ (=set of rationals), where $A_\alpha(y,f):=\{x\in Q; f(x)>f(y), y\in B_{\alpha x}\}$, $B_{\alpha x}:=\{y\in\mathbf{R}^1,\ |x-y|<\alpha\}$. Again, $(V_{X|\mathfrak{G}}f)(y,\omega):=(V_{X|\mathfrak{G}}f(y))(\omega)$.

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In comparison with Definition 1 of [10] the present version has been modified by the introduction of the infimum. This will assure that assertions to be derived with this operator theoretical approach are valid almost surely in $\omega\in\Omega$ not only for each fixed $y\in\mathbf{R}$ but uniformly in $y\in\mathbf{R}$. The space $\mathbf{R}^1$ endowed with the usual topology has a countable base, namely $\mathfrak{B}:=\{B_{\alpha x};\ \alpha, x\in Q,\ x>0\}$. Such spaces, namely complete, separable metric spaces are called "Polish spaces"; they are in particular Borel spaces. Now it is well known that each finite Borel-measure μ on a Polish space is a regular measure (see e.g. [16, p. 373]). This ensures the existence of a regular distribution $P_{X|\mathfrak{G}}$ which is in particular $\mathfrak{G}$-measurable for each fixed $B\in\mathfrak{B}$ as well as a measure on $\mathfrak{B}$ for each fixed $\omega\in\Omega$. Therefore the integral representation of the conditional expectation (2.12) of [10] holds. In view of these considerations, the above infimum is taken only countably often for all $y\in\mathbf{R}$, so uniformly for all $y\in\mathbf{R}$. The condition "$f(x)>f(y)$" assures that the conditional Trotter operator will coincide with the classical one in case $\mathfrak{A}(X)$ is independent of $\mathfrak{G}$.

Under this modification Lemma 2 and Corollary 1 of [10] is readily seen to be valid, Lemma 3 is superfluous, and Lemmas 4 (the case $n=2$ of La. 5) and 5 are to be replaced by

Lemma 5. *Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of r.v.'s from $\mathfrak{L}(\Omega, \mathfrak{A}, P)$, $(\mathfrak{G}_n)_{n\in\mathbf{N}}$ a sequence of sub-σ-algebras from $\mathfrak{A}$ for which $\mathfrak{G}_0:=\{\Omega, \emptyset\}\subset\mathfrak{G}_1\subset\mathfrak{G}_2\subset\ldots\subset\mathfrak{G}_n\subset\ldots$.*

a) *For each $f\in C_B$ one has*

$$V_{X_1|\mathfrak{G}_1}(V_{X_2|\mathfrak{G}_2}(\ldots V_{X_n|\mathfrak{G}_n}f(\ )\ldots))(y,\omega)=$$

$$=(V_{X_1|\mathfrak{G}_1}V_{X_2|\mathfrak{G}_2}\ldots V_{X_n|\mathfrak{G}_n}f)(y,\omega)=(V_{S_n|\mathfrak{G}_1}f)(y,\omega)\quad \text{a.s.}\quad (y\in\mathbf{R};\ n\in\mathbf{N}).$$

If, in particular $\mathfrak{G}_1=\mathfrak{G}_0$, then

$$(V_{X_1|\mathfrak{G}_1}\ldots V_{X_n|\mathfrak{G}_n}f)(y,\omega)=V_{S_n}f(y,\omega)\quad a.s.\quad (y\in\mathbf{R};\ n\in\mathbf{N}).$$

b) *If $(Z_n)_{n\in\mathbf{N}}$ is a further sequence from $\mathfrak{L}(\Omega, \mathfrak{A}, P)$, it being assumed that the Z_n are independent amongst themselves as well as of the X_n, then for each $f\in C_B$,*

$$\|V_{S_n|\mathfrak{G}_1}f-V_{\sum_{k=1}^{n}Z_k}f\|\leqq\sum_{k=1}^{n}\|V_{X_k|\mathfrak{G}_k}f-V_{Z_k}f\|\quad (n\in\mathbf{N}).$$

If, in particular, $\mathfrak{G}_k=\mathfrak{G}_0$, all $k\in\mathbf{N}$, then

$$\|V_{S_n}f-V_{\sum_{k=1}^{n}Z_k}f\|\leqq\sum_{k=1}^{n}\|V_{X_k}f-V_{Z_k}f\|\quad (n\in\mathbf{N}).$$

Proof. a) First take $n=2$. Noting (2.7) and (2.11) of [10], the latter being valid only for $\mathfrak{G}\subset\mathfrak{G}'$, one readily has

$$(V_{X_1|\mathfrak{G}_1}V_{X_2|\mathfrak{G}_2}f)(y,\omega)=\big(V_{X_1|\mathfrak{G}_1}\big\{\inf_{\bar{x}\in A_\alpha(\cdot,f)}E[f(X_2+\bar{x}(\cdot))|\mathfrak{G}_2]\big\}\big)(y,\omega)=$$
$$=\inf_{x\in A_\alpha(y,V_{X_2|\mathfrak{G}_2}f)}\{E[f(X_1+X_2+x)|\mathfrak{G}_1](\omega)\}=V_{X_1+X_2|\mathfrak{G}_1}f(y,\omega).$$

The general result follows by induction, the particular case by Lemma 2e) of [10].

b) The proof follows immediately by Corollary 1 and Lemmas 1 and 2e) of [10], as well as by the following result: Let $U_1,\dots,U_n,\dots$ and $V_1,\dots,V_n,\dots$ be contraction endomorphisms of C_B such that U_iU_j is only defined for $i\leqq j$, but the V_i may commute amongst themselves, and $U_iV_j=V_jU_i$ for any i,j. Then for each $f\in C_B$

$$\|U_1\dots U_nf-V_1\dots V_nf\|\leqq\sum_{k=1}^{n}\|U_kf-V_kf\|\quad(n\in\mathrm{N}).$$

6. O-approximation theorems for convergence in distribution

6.1. General theorems. In the proofs of the O-estimates of Section 6 the hypotheses of the corresponding o-convergence theorems of Section 5 may either be weakened or partially dropped entirely. Thus Lindeberg conditions are not needed either for the sequences $(X_k)_{k\in\mathrm{N}}$ or $(Z_k)_{k\in\mathrm{N}}$; the conditional moments of the r.v.'s X_k relative to $\mathfrak{G}_k$ need only coincide with the moments of Z_k up to the order $r-1$ (compare (6.2)).

Theorem 7. *Let $(X_k)_{k\in\mathrm{N}}$ be a sequence of (possibly) dependent r.v.'s from $\mathfrak{L}(\Omega,\mathfrak{A},P)$, let $(\mathfrak{G}_k)_{k\in\mathrm{N}}$ be a sequence of sub-σ-algebras of $\mathfrak{A}$ with $\mathfrak{G}_0:=\{\Omega,\emptyset\}\subset\subset\mathfrak{G}_1\subset\mathfrak{G}_2\subset\dots\subset\mathfrak{G}_n\subset\dots$, and Z be a φ-decomposable r.v. with decomposition components Z_k, $k\in\mathrm{N}$. Assume that for an $r\in\mathrm{N}\setminus\{1\}$*

$$E[|X_k|^r|\mathfrak{G}_k]\leqq M_{k,r}\quad a.s.\quad(k\in\mathrm{N})\tag{6.1}$$

for some constant $M_{k,r}>0$ as well as $E[|Z_k|^r]<\infty$. If furthermore

$$E[X_k^j|\mathfrak{G}_k]=E[Z_k^j]\quad a.s.\quad(k,j\in\mathrm{N};\ 1\leqq j\leqq r-1),\tag{6.2}$$

then there holds for $f\in C_B$

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1}f-V_Zf\|\leqq 2c_{2,r}\omega_r\left(\left[\frac{\varphi(n)^r}{(r-1)!}M(n)\right]^{1/r};f;C_B\right),\tag{6.3}$$

where

$$M(n):=\sum_{k=1}^{n}(M_{k,r}+E[|Z_k|^r]),\tag{6.4}$$

$c_{2,r}$ being the constant of (2.1) in [10].

Proof. In view of (2.7) and (2.8) one has for $f\in C_B$ and any $g\in C_B^r$,

$$\Big|\inf_{x\in A_\alpha(y,f)} E[f(\varphi(n)S_n+x)\,|\,\mathfrak{G}_1]-E[f(Z+y)]\Big| \le \tag{6.5}$$
$$\le 2\|f-g\|+\Big|\inf_{x\in A_\alpha(y,g)} E[g(\varphi(n)S_n+x)\,|\,\mathfrak{G}_1]-E[g(Z+y)]\Big|.$$

Further, in view of Lemma 5b,

$$\|V_{\varphi(n)S_n|\mathfrak{G}_1}g-V_Z g\| \le \sum_{k=1}^{n} \|V_{\varphi(n)X_k|\mathfrak{G}_k}g-V_{\varphi(n)Z_k}g\|. \tag{6.6}$$

Furthermore, there holds the estimate

$$\Big|\inf_{x\in A_\alpha(y,g)} \{E[g(\varphi(n)X_k+x)\,|\,\mathfrak{G}_k]\}-E[g(\varphi(n)Z_k+y)]\Big| \le \tag{6.7}$$
$$\le \sup_{x\in A_\alpha(y,g)} \{\big|E[g(\varphi(n)X_k+x)\,|\,\mathfrak{G}_k]-E[g(\varphi(n)Z_k+x)]\big|\}.$$

On account of the integral representation (2.12), and Taylor's formula applied to $g(u+x)$ twice,

$$\big|E[g(\varphi(n)X_k+x)\,|\,\mathfrak{G}_k]-E[g(\varphi(n)Z_k+x)]\big|=$$
$$=\Big|\int_{\mathbb{R}} g(u+x)\,dF_{\varphi(n)X_k}(u\,|\,\mathfrak{G}_k)(\omega)-\int_{\mathbb{R}} g(u+x)\,dF_{\varphi(n)Z_k}(u)\Big| \le$$
$$\le \Big|\int_{\mathbb{R}}\Big\{\sum_{j=0}^{r-1}\frac{u^j}{j!}g^{(j)}(x)\Big\}\,d\big(F_{\varphi(n)X_k}(u\,|\,\mathfrak{G}_k)-F_{\varphi(n)Z_k}(u)\big)\Big|+$$
$$+\Big|\int_{\mathbb{R}}\frac{1}{(r-2)!}\Big[\int_0^1 (1-t)^{r-2}\{g^{(r-1)}(x+tu)-g^{(r-1)}(x)\}u^{r-1}\,dt\Big]\,dF_{\varphi(n)X_k}(u|\mathfrak{G}_k)\Big|+$$
$$+\Big|\int_{\mathbb{R}}\frac{1}{(r-2)!}\Big[\int_0^1 (1-t)^{r-2}\{g^{(r-1)}(x+tu)-g^{(r-1)}(x)\}u^{r-1}\,dt\Big]\,dF_{\varphi(n)Z_k}(u)\Big|.$$

Since $g\in C_B^r$, $g^{(r-1)}\in \mathrm{Lip}\,(1;1;C_B)$ with Lipschitz constant $L_g=\|g^{(r)}\|$. So for $0<t\le 1$, $\big|\{g^{(r-1)}(x+tu)-g^{(r-1)}(x)\}u^{r-1}\big|\le\|g^{(r)}\|\,|u|^r$. In view of (6.2) and (6.1) there holds

$$\sum_{k=1}^{n}\big|E[g(\varphi(n)X_k+x)\,|\,\mathfrak{G}_k]-E[g(\varphi(n)Z_k+x)]\big|\le \tag{6.8}$$
$$\le \sum_{k=1}^{n}\sum_{j=0}^{r-1}\Big|\frac{1}{j!}g^{(j)}(x)\Big\{\int_{\mathbb{R}} u^j\,d[F_{\varphi(n)X_k}(u\,|\,\mathfrak{G}_k)-F_{\varphi(n)Z_k}(u)]\Big\}\Big|+$$
$$+\Big|\frac{\|g^{(r)}\|}{(r-1)!}\sum_{k=1}^{n}\int_{\mathbb{R}}|u|^r\,d[F_{\varphi(n)X_k}(u\,|\,\mathfrak{G}_k)+F_{\varphi(n)Z_k}(u)]\Big|\le$$
$$\le \|g^{(r)}\|\,\Big|\frac{\varphi(n)^r}{(r-1)!}\sum_{k=1}^{n}(M_{k,r}+E[|Z_k|^r])\Big| = \|g^{(r)}\|\,\frac{\varphi(n)^r}{(r-1)!}M(n).$$

All in all, noting (6.5), (6.6) and (6.7),

$$\|V_{\varphi(n)S_n}f(y)-V_Zf(y)\| \leqq 2 \inf_{g\in C_B^r}\left\{\|f-g\|+\|g^{(r)}\|_{C_B}\frac{\varphi(n)^r}{(r-1)!}M(n)\right\} \leqq$$

$$\leqq 2K\left(\frac{\varphi(n)^r}{(r-1)!}M(n);\, f;\, C_B;\, C_B^r\right).$$

This gives (6.3) in view of (2.1).

The proof of the following result is immediate, noting La. 2e.

Theorem 8. a) *If the hypotheses in Theorem 7 are satisfied, and if in particular* $\mathfrak{G}_1=\mathfrak{G}_0$, *then*

$$\|V_{\varphi(n)S_n}f-V_Zf\| \leqq 2c_{2,r}\omega_r\left(\left[\frac{\varphi(n)^r}{(r-1)!}M(n)\right]^{1/r};\, f;\, C_B\right). \tag{6.9}$$

b) *If further* $f\in \mathrm{Lip}\,(\alpha; r, C_B)$, $0<\alpha\leqq r$, *then the left side of* (6.9) *has the bound*

$$2c_{2,r}\, L_f\,\varphi(n)^\alpha M(n)^{\alpha/r}. \tag{6.10}$$

Remark 1. The basic condition (6.2) of Thm. 7, which together with the assumed monotonicity of the $\mathfrak{G}_k$ is the only condition upon which the dependency structure of the r.v.'s X_k in question is subjected, could be replaced by the much weaker order condition

$$\sum_{=1}^{n}|E[X_k^j|\mathfrak{G}_k]-E[Z_k^j]| = O\left(\frac{\varphi(n)^r}{r!}M(n)\right) \quad \text{a.s.} \quad (1\leqq j\leqq r-1;\ n\to\infty). \tag{6.2*}$$

This will also insure the estimate (6.8) as does condition (6.2).

A comparable weaker version is given in [9] or [5] in the case of a weak invariance principle for dependent random functions. A further paper [6] deals in more detail with conditions like (6.2)*, called pseudo-moment conditions (with orders).

Remark 2. Concerning the proofs of Theorem 1, and analogously of Thms. 2—6 of [10], it should be mentioned that they have to be modified and corrected by taking the definition of the conditional Trotter operator in the form given here and by using likewise the arguments involving inequalities (6.6) and (6.7) of the proof of Thm. 7. This will assure results comparable to Thms. 7 and 8 for "little-o-rates" when assuming Lindeberg conditions provided $(\mathfrak{G}_n)_{n\in\mathbf{N}}$ is additionally assumed to be a monotone non-decreasing sequence. In fact, assertion (2.11) needed here is only valid if $\mathfrak{G}\subset\mathfrak{G}'$. In regard to GOVINDARAJULU, cited in [10], the authors cannot follow the proof of his main Theorem 3.1, in particular the step involving the norms on p. 1016, since the conditional expectations occurring there only hold for each fixed $y\in\mathbf{R}$ a.s. in $\omega\in\Omega$.

6.2. The CLT and WLLN with O-rates. The following statements dealing with the CLT are applications of Theorems 7 and 8, the usual specialisations being carried out.

Theorem 9. *Let $(X_k)_{k\in N}$, $(\mathfrak{G}_k)_{k\in N}$ be given as in Theorem 7, and let X^* be a standard normally distributed r.v. Set $\sigma_k^2=\mathrm{Var}\,[X_k]$, $k\in N$ and $s_n^2=\Sigma_{k=1}^n \sigma_k^2$, and assume that $E[|X_k|^r|\mathfrak{G}_k]\leqq M_{k,r}$ a.s. for some constant $M_{k,r}>0$ as well as that*

$$(6.11)\qquad E[X_k^j\,|\,\mathfrak{G}_k]=\sigma_k^j E[X^{*j}] \quad a.s. \quad (k, j\in N,\ 1\leqq j\leqq r).$$

a) *Under these hypotheses one has for any $f\in C_B$*

$$(6.12)\qquad \|V_{s_n^{-1}S_n|\mathfrak{G}_1}f-V_{X^*}f\|_{C_B}\leqq 2c_{2,r}\omega_r\left(\left[\frac{s_n^{-r}}{(r-1)!}\,\overline{M}(n)\right]^{1/r};\ f;\ C_B\right),$$

where $\overline{M}(n):=\Sigma_{k=1}^n (M_{k,r}+\sigma_k^r E[|X^*|^r])$.

b) *If $f\in \mathrm{Lip}\,(\alpha; r; C_B)$, $0<\alpha\leqq r$, and $\mathfrak{G}_1=\mathfrak{G}_0$, then*

$$(6.13)\qquad \|V_{s_n^{-1}S_n}f-V_{X^*}f\|_{C_B}\leqq 2c_{2,r}L_f s_n^{-\alpha}\overline{M}(n)^{\alpha/r}.$$

Concerning the proof, just as in that of Theorem 2 condition (2.4) is satisfied with $Z_k=\sigma_k X^*$, $\varphi(n)=s_n^{-1}$. Since $[|X^*|^r]<\infty$, $r\in N$, and so assertion (6.12) follows from (6.3), assertion (6.13) follows from (6.10).

In the case of the following WLLN with "O"-rates the basic moment condition, in this case (6.2) for $r>2$, must, for the same reasons as in Theorem 6, be weakened (see (6.15)), whereas for $r=2$ (6.10) reduces to the non-trivial requirement (6.17).

Theorem 10. *Let $(X_k)_{k\in N}$, $(\mathfrak{G}_k)_{k\in N}$, Z_k with $P(Z_k=0)=1$, $k\in N$ be defined as in Theorem 7.*

a) *If for some $r\in N\setminus\{1\}$*

$$(6.14)\qquad E[|X_k|^r\,|\,\mathfrak{G}_k]\leqq M_{k,r} \quad a.s. \quad (k\in N)$$

and if there exist constants c_j such that

$$(6.15)\quad \varphi(n)^j\sum_{k=1}^n |E[X_k^j\,|\,\mathfrak{G}_k]|\leqq c_j\varphi(n)^r\sum_{k=1}^n M_{k,r} \quad a.s. \quad (0\leqq j\leqq r-1;\ n\in N),$$

then for $f\in \mathrm{Lip}\,(r; r; C_B)$,

$$(6.16)\qquad \|V_{\varphi(n)S_n|\mathfrak{G}_1}f-f\|_{C_B}\leqq\left(c_f+\frac{L_f}{(r-1)!}\right)\varphi(n)^r\sum_{k=1}^n M_{k,r}$$

with $c_f:=\Sigma_{j=0}^{r-1}c_j\|f^{(j)}\|_{C_B}/j!$.

b) *If* $r=2$, *one has for* $f\in C_B^2$, *provided* $E[X_K|\mathfrak{G}_k]=0$ *a.s. and* $E[X_k^2|\mathfrak{G}_k]<M_{k,r}$ *a.s.*, $k\in\mathbf{N}$,

(6.17) $$\|V_{n^{-1}S_n}f-f\|_{C_B}\leqq 2c_{2,2}L_f n^{-2}\sum_{k=1}^{n}M_{k,2}.$$

Proof. Condition (2.4) is satisfied for independent r.v.'s Z_k with $P_{Z_k}=P_{X_0}$ for all $k\in\mathbf{N}$. Since $E[|X_0|^j]=0$ for any $j\geqq 1$, a Taylor expansion up to the order $r-1$ yields, similarly as in the proof of Theorem 7,

(6.18) $$|V_{\varphi(n)X_k|\mathfrak{G}_k}f(y,\omega)-V_{\varphi(n)Z_k}f(y)|\leqq$$
$$\leqq\sum_{j=1}^{r-1}\frac{\varphi(n)^j}{(r-1)!}\|f^{(j)}\|_{C_B}E[|X_k|^j|\mathfrak{G}_k]+L_f\frac{\varphi(n)^r}{(r-1)!}M_{k,r}.$$

Assertion (6.16) now follows by using condition (6.15) in formula (6.18). Part b) is the particular case of Theorem 8b) for $r=\alpha=2$ and $\varphi(n)=n^{-1}$, noting that $P_Z=P_{X_0}$.

7. O-approximation theorems for convergence in distribution. Just as in the case of martingales (cf. [5], [6]) it is possible to transfer our results concerned with rates for the weak convergence of the distributions $P_{\varphi(n)S_n}$ to P_Z to those for strong convergence. This is possible by applying a result contained implicitly in ZOLOTAREV [18], formulated explicitly in e.g. [5]. Using this result one can deduce from Theorem 7 the following theorem, noting that conditions (7.1) and (7.2) yield, for $f\in C_B^r$,

$$\|V_{\varphi(n)S_n}f-V_Zf\|_{C_B}=O(n\varphi(n)^r)\quad(n\to\infty).$$

Theorem 11. *Let* $(X_k)_{k\in\mathbf{N}}$, $(\mathfrak{G}_k)_{k\in\mathbf{P}}$, Z_k, $k\in\mathbf{N}$, *be defined as in Theorem 7, let the limiting r.v.* $Z\in\mathfrak{L}(\Omega,\mathfrak{A},P)$, *with distribution function* F_Z, *satisfy condition*

$$|F_Z(x_1)-F_Z(x_2)|\leqq M_Z|x_1-x_2|\quad(x_1,x_2\in\mathbf{R})$$

for some constant $M_Z>0$, *and assume that for* $r\in\mathbf{N}\setminus\{1\}$

(7.1, 2) $$E[|X_k|^r|\mathfrak{G}_k]<M_r\quad a.s.,\quad E[|Z_k|^r]<M_r^*\quad(k\in\mathbf{N}),$$

M_r, M_r^* *being positive constants, independent of k. If further* (6.2) *holds, then*

(7.3) $$\sup_{x\in\mathbf{R}}|F_{\varphi(n)S_n}(x)-F_Z(x)|=O\big(\varphi(n)^{r/(r+1)}n^{1/(r+1)}\big)\quad(n\to\infty).$$

If one applies Theorem 11 to the r.v. $Z:=X^*$, one obtains the following Berry—Esséen type estimates for dependent r.v.'s.

Theorem 12. *Let the assumptions of Theorem 9 be satisfied. If there exist two positive constants m, M such that* $m<\sigma_k^2<M$, *one obtains*

(7.4) $$\sup_{x\in\mathbf{R}}|F_{s_n^{-1}S_n}(x)-F_{X^*}(x)|=O(s_n^{r/(r+1)}n^{1/(r+1)})\quad(n\to\infty).$$

If the r.v.'s X_k and Z_k, $k \in \mathbf{N}$ are identically distributed, and $\sigma_k^2=1$, all $k \in \mathbf{N}$, then for $r=3$

$$\sup_{x\in\mathbf{R}} |F_{n^{-1/2}S_n}(x)-F_{X^*}(x)| = O(n^{-1/8}) \quad (n\to\infty). \tag{7.5}$$

Setting $\varphi(n)=s_n^{-1}$ one can show, just as in the proof of Theorem 9b, that (7.4) follows from (7.3). Estimate (7.5) is a result of (7.4) since $s_n^{-1}=n^{-1/2}$ for $\sigma_k^2=1$.

8. Applications to martingale difference arrays. Whereas the dependency structure of the r.v.'s in question has so far been very general, it will be concretized in this section. The particular type of dependency to be considered will be that defined by a martingale difference array (MDA). A MDA is a double indexed array $(X_{nk})_{1\le k\le k_n}$, $n\in\mathbf{N}$ of r.v.'s from $\mathfrak{L}(\Omega, \mathfrak{A}, P)$ that is connected with a scheme $(\mathfrak{F}_{nk})_{0\le k\le k_n}$, $n\in\mathbf{N}$ of sub-σ-algebras of $\mathfrak{A}$ in such a form that the following three conditions are satisfied:

i) the sequence $(\mathfrak{F}_{nk})_{0\le k\le k_n}$ is monotone non-decreasing in k for each $n\in\mathbf{N}$,
ii) X_{nk} is measurable with respect to $\mathfrak{F}_{nk}$ for $1\le k\le k_n$,
iii) $E[X_{nk}|\mathfrak{F}_{n,k-1}]=0$ a.s. for $1\le k\le k_n$, $n\in\mathbf{N}$.

The general convergence theorem of this paper, Theorem 1, may be applied to MDA, as well as that supplied with o-rates, namely Theorem 4. But in order to avoid repetitions in the formulations we shall just consider the applications of Theorem 7 and 12 to yield

Theorem 13. *Let $(X_{nk})_{1\le k\le k_n}$, $n\in\mathbf{N}$, be a MDA, $(\mathfrak{F}_{nk})_{0\le k\le k_n}$, $n\in\mathbf{N}$, the associated array of sub-σ-algebras of $\mathfrak{A}$ (non-decreasing in k per definition) with $\mathfrak{F}_{n0}=\{\emptyset, \Omega\}$ for all $n\in\mathbf{N}$, and let Z be a φ-decomposable r.v. with decomposition components Z_{nk}, $1\le k\le k_n$.*

Assume further that for an $r\in\mathbf{N}\setminus\{1\}$

$$E[|X_{nk}|^r|\mathfrak{F}_{n,k-1}] < M_{nk,r} \quad a.s. \quad (1\le k\le k_n;\ n\in\mathbf{N}) \tag{8.1}$$

for some constant $M_{nk,r}>0$, as well as

$$E[|Z_{nk}|^r] < \infty \quad (1\le k\le k_n;\ n\in\mathbf{N}) \tag{8.2}$$

together with

$$E[X_{nk}^j|\mathfrak{F}_{n,k-1}] = E[Z_{nk}^j] \quad a.s. \quad (1\le k\le k_n;\ n\in\mathbf{N};\ 1\le j\le r-1;\ j\in\mathbf{N}). \tag{8.3}$$

Then for any $f\in \mathrm{Lip}(\alpha; r; C_B)$, $0<\alpha\le r$, there holds for $T_{nk_n}:=\varphi(k_n)\Sigma_{k=1}^{k_n}X_{nk}$ the estimate

$$\|V_{T_{nk_n}}f-V_Zf\| \le 2c_{2,r}L_f\varphi(n_k)^\alpha \sum_{k=1}^{k_n}(M_{nk,r}+E[|Z_{nk}|^r]) \tag{8.4}$$

with $c_{2,r}$ and L_f from Theorem 8.

If for each $n\in\mathbf{N}$ the r.v.'s X_{nk} and Z_{nk} are in particular identically distributed for all $1\leqq k\leqq k_n$, and if there holds

$$E[|X_{nk}|^r\,|\,\mathfrak{F}_{n,k-1}] < M_{n,r} \quad a.s. \quad (1\leqq k\leqq k_n,\ n\in\mathbf{N})$$

where $M_{n,r}$ is a positive constant independent of k, as well as (8.2) *and* (8.3), *then for $f\in C_B^r$*

$$\|V_{T_{nk_n}}f-V_Zf\| \leqq L_f\frac{\varphi(k_n)^r}{(r-1)!}k_n(M_{n,r}+E[|Z_{nk}|^r]). \tag{8.5}$$

Proof. Assertion (8.4) is a direct application of Theorem 8b), replacing the X_k by X_{nk} and the $\mathfrak{G}_k$ by $\mathfrak{F}_{n,k-1}$, noting that $\mathfrak{G}_1=\mathfrak{F}_{n,0}=\{\emptyset,\Omega\}$, and that the distribution P_Z of the limit r.v. Z can, for each natural k_n, be representated as $P_Z=P_{\varphi(k_n)\sum_{k=1}^{k_n}Z_{nk}}$, whereby the independent decomposition components Z_k of (2.4) have here been written in the preciser form Z_{nk}. Inequality (8.5) follows by (6.8) in the proof of Theorem 7.

Now to the application of Theorem 12 to MDA; it is the CLT with rates for MDA.

Theorem 14. *Let $(X_{nk})_{1\leqq k\leqq k_n}$, $n\in\mathbf{N}$ and $(\mathfrak{F}_{nk})_{0\leqq k\leqq k_n}$, $n\in\mathbf{N}$ be defined as in Theorem 13. Let $m_n'<\sigma_{nk}^2:=\mathrm{Var}\,(X_{nk})<M_n'$, $1\leqq k\leqq k_n$, $n\in\mathbf{N}$. Assume further that for $r\in\mathbf{N}\setminus\{1\}$*

$$E[|X_{nk}|^r\,|\,\mathfrak{F}_{n,k-1}] < M_{n,r} \quad a.s. \quad (1\leqq k\leqq k_n,\ n\in\mathbf{N}),$$

$M_{n,r}$ being positive constants, independent of k, as well as

$$E[X_{nk}^j\,|\,\mathfrak{F}_{n,k-1}] = \sigma_{nk}^jE[X^{*j}] \quad a.s. \quad (1\leqq k\leqq k_n,\ n\in\mathbf{N};\ 1\leqq j\leqq r-1;\ j\in\mathbf{N}).$$

Then one has for $s_{n,k_n}:=(\sum_{k=1}^{k_n}\sigma_{nk}^2)^{1/2}$,

$$\sup_{x\in\mathbf{R}}|F_{s_{n,k_n}^{-1}\sum_{k=1}^{k_n}X_{nk}}(x)-F_{X^*}(x)| = O(s_{n,k_n}^{r/(r+1)}k_n^{1/(r+1)}) \quad (n\to\infty). \tag{8.6}$$

If, in addition, for each $n\in\mathbf{N}$ the r.v.'s X_{nk} are identically distributed for all $1\leqq k\leqq k_n$, and $\sigma_{nk}^2=1$ for $1\leqq k\leqq k_n$, $n\in\mathbf{N}$, then for $r=3$

$$\sup_{x\in\mathbf{R}}|F_{k_n^{-1/2}\sum_{k=1}^{k_n}X_{nk}}(x)-F_{X^*}(x)| = O(k_n^{-1/8}) \quad (n\to\infty). \tag{8.7}$$

The proof of this theorem consists in a consequent application of Theorem 12, using the special case of MDA with $\mathfrak{G}_k=\mathfrak{F}_{n,k-1}$.

If one would take $k_n=n$ in Theorem 14, then the rate in (8.7) reduces to $O(n^{-1/8})$, one which was also attained by Heyde and Brown [14], Chow and Teicher [11 p. 314] as well as by Erickson, Quine and Weber [12]. Improvements of this rate were achieved by Hall and Heyde [13 p. 84], namely with $O(n^{-1/4}\log n)$, by Mukerjee [17] with $O(n^{-1/4})$, Kato [15] with $O(n^{-1/2}(\log n)^3)$ as well as by Bolt-

HAUSEN [3] with $O(n^{-1/2}\log n)$, whereby the better rates of convergence by Kato and Bolthausen are restricted to uniformly bounded r.v.'s. If one just assumes the boundedness of the third absolute moments of the r.v.'s X_{nk} as well as the "near constancy" of the partial sums of the conditional variances, here expressed in the form

$$s_{n,k_n}^{-1} \sum_{k=1}^{k_n} E[X_{nk}^2 | \mathfrak{F}_{n,k-1}] \to 1 \quad \text{in probability} \quad (n \to \infty), \tag{8.8}$$

then the rate $O(n^{-1/4})$ is the best that has been obtained so far. It should be noted that condition (8.6) for $j=2$ implies (8.8); however, an assertion comparable to Theorem 14 could also be deduced by means of the conditional Trotter operator under the weaker assumption (8.8).

It must further be mentioned that the rates of Theorem 13, deduced from Theorem 7, dealing with rates for dependent r.v.'s, are just as good as those obtained in [4], [1], [2], [8], [9] and [5] for independent r.v.'s, MDS or MDA by means of the strongly modified Dvoretzky-method of proof mentioned in the introduction of [10]. But the conditional Trotter operator introduced in Section 3 allows one to prove the fundamental limit theorems equipped with rates in a *unified* way not only for various types of dependent r.v.'s but also for independent r.v.'s. It should be added that the definition and proofs involving the conditional Trotter operator and its properties also make use of set functions. So this operator theoretic approach stands in contrast to the more intricate "measure-theoretic" approach dealt with in most papers concerned with stochastic processes, in particular Markov processes.

It may be observed that the conditional Lindeberg—Trotter operator approach even makes it possible to deal with general limit theorems for Markov processes equiped with rates, see [7]. Similar results would be possible for inverse martingales or other dependency structure types.

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LEHRSTUHL A FÜR MATHEMATIK
AACHEN UNIVERSITY OF TECHNOLOGY
5100 AACHEN, FEDERAL REPUBLIC OF GERMANY