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Structure-filters in equality-free model theory

P. ECSEDI-TÓTH

Using a natural definition of (finite) meets of structures and that of the lattice ordering induced by the meet, we introduce the concept of filters on the similarity class of structures. Our main problem here is to answer the question whether the theories of such filters are characterizable by purely syntactical means. Restricting our considerations mostly to equality-free first order languages, we provide an affirmative solution to this problem.

1. Finite meet of structures has been introduced as a simple set theoretic construction in [3], where we have proved the following

Theorem 1.1 ([3], Theorem 2.14). *Let T be an equality-free first order theory. Then the two assertions below are equivalent:*

(i) *T has a set of universal equality-free Horn axioms;*

(ii) *T is preserved under finite meets (cf. Definition* 3.6, *below).*

It was shown, too, that this theorem fails for theories containing equality; more precisely, (ii) does not entail (i) if the equality is present, while the converse implication (i)⇒(ii) holds in general.

Our starting point in the present work is that, disregarding some set theoretic difficulties, the class of all similar structures forms a weak partial meet-semilattice. It is well-known, that the lattice ordering is uniquely determined in weak partial meet-semilattices. By means of the lattice ordering, filters are definable in the traditional way, and so the following natural questions arise:

(1) Which sentences (theories) are preserved under the lattice ordering induced by the meet?

(2) Which sentences (theories) have a class of models that forms a filter in the weak partial meet-semilattice of structures?

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We shall give here a complete answer to question (1) (cf. Corollary 4.4, Theorems 4.5, 4.6, 4.7), and a partial one to question (2) (cf. Theorems 5.3 and 5.5), in the sense, that we restrict our attention to equality-free languages, only.

It would be natural, too, to introduce and investigate the duals of these concepts; i.e. the join of structures and ideals of structures. These notions, however cannot be treated analogously to the meets and filters. For example, the meet of structures can be defined without any restrictions on the universes of structures (cf. Definition 3.1, below), nevertheless, a similar definition of the join would involve either the assumption that the universes of all structures in question are the same, or the permission for partial structures (in which functions may be partial). Beyond this definitional difficulty, some of our results do not have analogous dual forms. Thus, it seemed better to deal with these dual question in a separate paper.

2. Some of our assertions refer explicitly to proper classes, and so, in order to avoid set theoretic difficulties, the choice of the underlying set theory is important; in fact, our considerations could be carried out e.g. in the Bernays—Gödel set theory. We shall, however, present the material informally; the formal set theoretic development would be rather tedious.

By a similarity type t we shall mean an ordered quintuple $t=\langle\mathcal{R}, \mathcal{F}, \mathcal{C}, t_{\mathcal{R}}, t_{\mathcal{F}}\rangle$, where $\mathcal{R}$, $\mathcal{F}$, $\mathcal{C}$ are pairwise disjoint sets, $\mathcal{C}\neq\emptyset$, $t_{\mathcal{R}}\colon \mathcal{R}\to\omega-\{0\}$, $t_{\mathcal{F}}\colon \mathcal{F}\to\omega-\{0\}$.

By a structure of type t, we mean an ordered quadruplet

$$\mathfrak{A} = \langle|\mathfrak{A}|, \langle R_r^{(\mathfrak{A})}\rangle_{r\in\mathcal{R}}, \langle F_f^{(\mathfrak{A})}\rangle_{f\in\mathcal{F}}, \langle C_c^{(\mathfrak{A})}\rangle_{c\in\mathcal{C}}\rangle,$$

where $|\mathfrak{A}|$ is a nonvoid set, the universe of $\mathfrak{A}$; for each $r\in\mathcal{R}, f\in\mathcal{F}$ and $c\in\mathcal{C}$, $R_r^{(\mathfrak{A})}$ is a $t_{\mathcal{R}}(r)$-ary relation, $F_f^{(\mathfrak{A})}$ is a $t_{\mathcal{F}}(f)$-ary function and $C_c^{(\mathfrak{A})}$ is a constant on the set $|\mathfrak{A}|$, respectively.

From now on, we shall keep an arbitrary similarity type t be fixed. The class of all structures of type t will be denoted by $\mathfrak{M}^t$; we shall denote the elements of $\mathfrak{M}^t$ by German capitals, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, maybe with indices.

We shall use the standard notions and notations of [2]. Additionally, we need some supplementary facts, collected together in the rest of this section.

First, we mention the equality-free version of the well-known Łoś—Tarski preservation theorem (cf. [2], Theorem 3.2.2, p. 124).

Theorem 2.1 ([3], Lemma 2.10). *Let T be an equality-free first order theory. Then, the following two assertions are equivalent:*

(i) *T is preserved under substructures;*

(ii) *T has a set of universal equality-free axioms.*

Analogously, one can prove without major difficulty the dual form of this theorem.

Theorem 2.2. *Let T be an arbitrary equality-free first order theory. Then, the two assertions below are equivalent:*

(i) *T is preserved under extensions;*

(ii) *T has a set of existential equality-free axioms.*

The following concept has been introduced also in [3].

Definition 2.3 ([3], 2.4). Let X be an arbitrary set and consider the absolutely free algebra $\mathfrak{Fr}(X\cup\mathscr{C})$ of type t generated by the set $X\cup\mathscr{C}$ (cf. [5], Definition 0.4.19(i), Remarks 0.4.20, pp. 130—131). Let $\mathfrak{A}\in\mathfrak{M}^t$. It is well-known, that for arbitrary $h\colon X\cup\mathscr{C}\to|\mathfrak{A}|$, such that for all $c\in\mathscr{C}$, $h(c)=C_c^{(\mathfrak{A})}$ holds, there exists a unique homomorphism $\bar{h}$ from $\mathfrak{Fr}(X\cup\mathscr{C})$ into $\mathfrak{A}$ for which $h\subseteq\bar{h}$ (cf. [5], Definition 0.4.23, Theorem 0.4.24, Theorem 0.4.27(i), pp. 131—132). We define the *free structure* $\mathfrak{Fr}_h\mathfrak{A}$ induced by h over $\mathfrak{A}$ as follows:

(i) let $|\mathfrak{Fr}_h\mathfrak{A}|=|\mathfrak{Fr}(X\cup\mathscr{C})|$;

(ii) for every $r\in\mathscr{R}$, $t_{\mathscr{R}}(r)=n+1$ and for arbitrary elements $a_0, \dots, a_n\in|\mathfrak{Fr}_h\mathfrak{A}|$, let

$$\langle a_0, \dots, a_n\rangle\in R_r^{(\mathfrak{Fr}_h\mathfrak{A})} \Leftrightarrow \langle\bar{h}(a_0), \dots, \bar{h}(a_n)\rangle\in R_r^{(\mathfrak{A})},$$

where $\bar{h}$ is the unique extension of h to a homomorphism from $\mathfrak{Fr}(X\cup\mathscr{C})$ into $\mathfrak{A}$;

(iii) for every $f\in\mathscr{F}$, such that $t_{\mathscr{F}}(f)=n+1$ and for arbitrary $a_0, \dots, a_n\in|\mathfrak{Fr}_h\mathfrak{A}|$, let

$$F_f^{(\mathfrak{Fr}_h\mathfrak{A})}(a_0, \dots, a_n) = F_f^{(\mathfrak{Fr}(X\cup\mathscr{C}))}(a_0, \dots, a_n);$$

(iv) finally, for all $c\in\mathscr{C}$, let

$$C_c^{(\mathfrak{Fr}_h\mathfrak{A})} = C_c^{(\mathfrak{Fr}(X\cup\mathscr{C}))}.$$

It was shown in [3], that $\mathfrak{Fr}_h\mathfrak{A}$ is correctly defined and is of type t, provided $\mathfrak{A}\in\mathfrak{M}^t$. We shall need the following

Theorem 2.4 ([3], Lemma 2.5). *Let $\mathfrak{A}\in\mathfrak{M}^t$ and X be a set, $h\colon X\cup\mathscr{C}\to|\mathfrak{A}|$ such that $h(c)=C_c^{(\mathfrak{A})}$ for all $c\in\mathscr{C}$. If h is onto, then $\mathfrak{A}$ and $\mathfrak{Fr}_h\mathfrak{A}$ are elementarily equivalent for equality-free sentences.*

The next assertion is, on the one hand, a particular case of a well-known result of Shoenfield (cf. [2], Theorem 3.1.16, p. 118) in two respects: firstly, it concerns equality-free languages only, and secondly, it is restricted to the lowest levels of the quantifier hierarchy. On the other hand, however, it is a generalization of the mentioned result, since it is about theories instead of single sentences. Our proof, presented here, is purely model theoretic in character and differs from the one given in [2], p. 118.

Theorem 2.5. *Let T be an equality-free first order theory. Then, the following assertions are equivalent:*

(i) *T has both a set of universal equality-free and a set of existential equality-free axioms;*

(ii) *T is preserved under both substructures and extensions;*

(iii) *T has a set of quantifier-free (i.e. $\Pi_0=\Sigma_0=\Delta_0$) equality-free axioms.*

Proof. (i) and (ii) are equivalent by Theorem 2.1, and Theorem 2.2. Also, (iii) implies (i) trivially, since every quantifier-free (and equality-free) sentence can be considered as a universal, as well as an existential (equality-free) sentence. To complete the proof, we show that (ii) entails (iii).

First, we prove the following fact.

(3) Let $\mathfrak{A}, \mathfrak{B}\in\mathfrak{M}^t$, and assume that for any quantifier-free and equality-free sentence ψ, $\mathfrak{A}\models\psi\Leftrightarrow\mathfrak{B}\models\psi$. Then $\mathfrak{A}\models T\Leftrightarrow\mathfrak{B}\models T$.

Let X be an arbitrary set with cardinality large enough such that the onto mappings h: $X\cup\mathscr{C}\to|\mathfrak{A}|$ and g: $X\cup\mathscr{C}\to|\mathfrak{B}|$ exist. Consider the free structures $\mathfrak{Fr}_h\mathfrak{A}$ and $\mathfrak{Fr}_g\mathfrak{B}$ and let us denote by $\mathfrak{A}'$ and $\mathfrak{B}'$ those substructures of $\mathfrak{Fr}_h\mathfrak{A}$ and $\mathfrak{Fr}_g\mathfrak{B}$ which are generated by the set of constants, respectively. (By assumption, there exist constants in $\mathfrak{Fr}_h\mathfrak{A}$ and $\mathfrak{Fr}_g\mathfrak{B}$, so $\mathfrak{A}'$ and $\mathfrak{B}'$ exist.)

We claim that $\mathfrak{A}'=\mathfrak{B}'$.

Indeed, by Definition 2.3, we see that

$$|\mathfrak{Fr}_h\mathfrak{A}| = |\mathfrak{Fr}_g\mathfrak{B}|, \tag{4}$$

and for every $c\in\mathscr{C}$ and $f\in\mathscr{F}$,

$$C_c^{(\mathfrak{Fr}_h\mathfrak{A})} = C_c^{(\mathfrak{Fr}_g\mathfrak{B})}, \tag{5}$$

$$F_f^{(\mathfrak{Fr}_h\mathfrak{A})} = F_f^{(\mathfrak{Fr}_g\mathfrak{B})}. \tag{6}$$

From (4), (5) and (6), it follows that $|\mathfrak{A}'|=|\mathfrak{B}'|$ and $C_c^{(\mathfrak{A}')}=C_c^{(\mathfrak{B}')}$, $F_f^{(\mathfrak{A}')}=F_f^{(\mathfrak{B}')}$, for every $c\in\mathscr{C}$, $f\in\mathscr{F}$.

Finally, let $a_0, \ldots, a_n\in|\mathfrak{A}'|$ and $r\in\mathscr{R}$, such that $t_{\mathfrak{a}}(r)=n+1$. By the definition of $\mathfrak{A}'$, there are closed terms $\tau_0, \ldots, \tau_n$ such that

$$\tau_0^{(\mathfrak{A}')} = a_0, \ldots, \tau_n^{(\mathfrak{A}')} = a_n,$$

(where $\tau_i^{(\mathfrak{A}')}$ denotes the "value of τ_i in $\mathfrak{A}'$", cf. [2], 1.3.13, p. 27). Hence

$$\langle a_0, \ldots, a_n\rangle\in R_r^{(\mathfrak{A}')} \Leftrightarrow \langle\tau_0^{(\mathfrak{A}')}, \ldots, \tau_n^{(\mathfrak{A}')}\rangle\in R_r^{(\mathfrak{A}')} \Leftrightarrow \mathfrak{A}'\models r(\tau_0, \ldots, \tau_n).$$

Since $\tau_0, \ldots, \tau_n$ are closed, $\mathfrak{A}'\models r(\tau_0, \ldots, \tau_n)$ implies that $\mathfrak{Fr}_h\mathfrak{A}\models r(\tau_0, \ldots, \tau_n)$. By Theorem 2.4, $\mathfrak{A}\models r(\tau_0, \ldots, \tau_n)$. According to the assumption of (3), $\mathfrak{B}\models r(\tau_0, \ldots, \tau_n)$, from which $\mathfrak{Fr}_g\mathfrak{B}\models r(\tau_0, \ldots, \tau_n)$ and $\mathfrak{B}'\models r(\tau_0, \ldots, \tau_n)$ follow, again by Theorem 2.4 and by the closedness of $\tau_0, \ldots, \tau_n$.

This, however, means that $\langle \tau_0^{(\mathfrak{B}')}, \ldots, \tau_n^{(\mathfrak{B}')} \rangle \in R_r^{(\mathfrak{B}')}$ and so, using the fact that for all i $(0 \leq i \leq n)$, $\tau_i^{(\mathfrak{B}')} = a_i$, which follows from (5) and (6), we obtain: $\langle a_0, \ldots, a_n \rangle \in R_r^{(\mathfrak{B}')}$. Hence, $R_r^{(\mathfrak{A}')} \subset R_r^{(\mathfrak{B}')}$. The converse implication $R_r^{(\mathfrak{B}')} \subset R_r^{(\mathfrak{A}')}$ can be established similarly. Thus $R_r^{(\mathfrak{A}')} = R_r^{(\mathfrak{B}')}$ and, r being chosen arbitrarily, we have $\mathfrak{A}' = \mathfrak{B}'$.

If $\mathfrak{A} \models T$, then by Theorem 2.4, $\mathfrak{Fr}_h \mathfrak{A} \models T$, and since T is preserved under substructures, $\mathfrak{A}' \models T$. So, $\mathfrak{B}' \models T$. But T is preserved under extensions, too, hence $\mathfrak{Fr}_g \mathfrak{B} \models T$, whence we obtain $\mathfrak{B} \models T$; by Theorem 2.4. The converse implication $\mathfrak{B} \models T \Rightarrow \mathfrak{A} \models T$ can be seen in an analogous way.

Thus (3) is proved.

If T is inconsistent, then the set $\{r(c, c, \ldots, c), \neg r(c, c, \ldots, c)\}$, where $r \in \mathscr{R}$, $t_{\mathscr{R}}(r) = n+1$ and $c \in \mathscr{C}$ are arbitrary, is an axiom system for T in the required form. (In fact, speaking on equality-free languages, we may assume that $\mathscr{R} \neq \emptyset$, for otherwise no formula exists; on the other hand, $\mathscr{C} \neq \emptyset$ by assumption.)

Let us suppose that T is consistent and set $T_0 = \{\varphi | \varphi$ is a quantifier-free equality-free sentence and $T \models \varphi\}$. Then, $T \models T_0$ and so T_0 is consistent.

Let $\mathfrak{C} \models T_0$ be arbitrary. We claim that there is a structure $\mathfrak{D}$, such that $\mathfrak{D} \models T$, and for every quantifier-free equality-free sentence ψ, $\mathfrak{C} \models \psi \Leftrightarrow \mathfrak{D} \models \psi$.

Indeed, let $\Sigma = \{\varphi | \varphi$ is a quantifier-free equality-free sentence and $\mathfrak{C} \models \varphi\}$. Then $\Sigma \cup T$ is consistent. For if $\Sigma \cup T$ were inconsistent, then we could find a finite subset $\{\sigma_0, \ldots, \sigma_m\} \subset \Sigma$ such that $T \models \neg(\sigma_0 \wedge \ldots \wedge \sigma_m)$. But the sentence $\neg(\sigma_0 \wedge \ldots \wedge \sigma_m)$ is itself a quantifier-free equality-free sentence and so it is in T_0, hence $\mathfrak{C} \models \neg(\sigma_0 \wedge \ldots \wedge \sigma_m)$. Nevertheless, $\mathfrak{C} \models \sigma_0 \wedge \ldots \wedge \sigma_m$, by the definition of Σ. This contradiction indicates that $\Sigma \cup T$ is consistent.

Let $\mathfrak{D}$ be a model of $\Sigma \cup T$ and let ψ be an arbitrary equality-free quantifier-free sentence. If $\mathfrak{C} \models \psi$, then $\psi \in \Sigma$ and so $\mathfrak{D} \models \psi$. If $\mathfrak{C} \not\models \psi$, then $\mathfrak{C} \models \neg\psi$ and so $\neg\psi \in \Sigma$, hence $\mathfrak{D} \models \neg\psi$, i.e. $\mathfrak{D} \not\models \psi$.

Thus, $\mathfrak{C}$ and $\mathfrak{D}$ satisfy the condition of (3), and $\mathfrak{C} \models T$ follows from $\mathfrak{D} \models T$, by (3). □

3. Definition 3.1 ([3], 1.2). Let $t = \langle \mathscr{R}, \mathscr{F}, \mathscr{C}; t_{\mathscr{R}}, t_{\mathscr{F}} \rangle$ be a similarity type and let

$$\mathfrak{A}_i = \langle |\mathfrak{A}_i|, \langle R_r^{(\mathfrak{A}_i)} \rangle_{r \in \mathscr{R}}, \langle F_f^{(\mathfrak{A}_i)} \rangle_{f \in \mathscr{F}}, \langle C_c^{(\mathfrak{A}_i)} \rangle_{c \in \mathscr{C}} \rangle$$

be structures of type t for $i < n+1$, where $n \in \omega$. We define the set theoretic meet of $\mathfrak{A}_i$, $i < n+1$ as follows:

$$\bigcap_{i<n+1} \mathfrak{A}_i = \langle \bigcap_{i<n+1} |\mathfrak{A}_i|, \langle \bigcap_{i<n+1} R_r^{(\mathfrak{A}_i)} \rangle_{r \in \mathscr{R}}, \langle \bigcap_{i<n+1} F_f^{(\mathfrak{A}_i)} \rangle_{f \in \mathscr{F}}, \bigcap_{i<n+1} \langle C_c^{(\mathfrak{A}_i)} \rangle_{c \in \mathscr{C}} \rangle$$

where the meets on the right hand side of the equation are meant in the sense of set theory (i.e. the meet of functions is taken as the meet of sets of pairs representing those functions; the meet of sequences of constants is defined again as the meet of ordered sets).

If $\bigcap_{i<n+1} \mathfrak{A}_i \in \mathfrak{M}^t$, then it is called the *model theoretic meet* (from now on, simply, the meet) of $\mathfrak{A}_i$, $i<n+1$. We shall use the infix notation $\mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_n$ for the meet of $\mathfrak{A}_i$, $i<n+1$.

Clearly, $\bigcap_{i<n+1} \mathfrak{A}_i$ always exists as a tuple. The meet of the structures $\mathfrak{A}_i$, $i<n+1$, however, is a partial operation: it may well happen, that the meet of $\mathfrak{A}_i$, $i<n+1$ does not exist even if $\bigcap_{i<n+1} |\mathfrak{A}_i| \neq \emptyset$. We shall use synonymously the following two expressions:

$$\text{“}\bigcap_{i<n+1} \mathfrak{A}_i \in \mathfrak{M}^t\text{” and “}\mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_n \text{ exists”}.$$

The meet, if exists, is very close to the set theoretic meet. In particular, it possesses the following familiar properties.

Lemma 3.2. *Let* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathfrak{M}^t$ *be arbitrary.*

(i) $\mathfrak{A} \cap \mathfrak{A} = \mathfrak{A}$, *hence* $\mathfrak{A} \cap \mathfrak{A} \in \mathfrak{M}^t$.

(ii) *If* $\mathfrak{A} \cap \mathfrak{B} \in \mathfrak{M}^t$, *then* $\mathfrak{B} \cap \mathfrak{A} \in \mathfrak{M}^t$, *and* $\mathfrak{A} \cap \mathfrak{B} = \mathfrak{B} \cap \mathfrak{A}$.

(iii) *If* $\mathfrak{A} \cap \mathfrak{B} \in \mathfrak{M}^t$ *and* $\mathfrak{B} \cap \mathfrak{C} \in \mathfrak{M}^t$, *then* ($\alpha$) *and* ($\beta$) *below are equivalent and any of them implies* $(\mathfrak{A} \cap \mathfrak{B}) \cap \mathfrak{C} = \mathfrak{A} \cap (\mathfrak{B} \cap \mathfrak{C})$:

(α) $(\mathfrak{A} \cap \mathfrak{B}) \cap \mathfrak{L} \in \mathfrak{M}^t$,

(β) $\mathfrak{A} \cap (\mathfrak{B} \cap \mathfrak{L}) \in \mathfrak{M}^t$.

Proof. (i) and (ii) are trivial.

(iii): Assume that $\mathfrak{A} \cap \mathfrak{B} \in \mathfrak{M}^t$, $\mathfrak{B} \cap \mathfrak{C} \in \mathfrak{M}^t$. If ($\alpha$) is true, i.e. $(\mathfrak{A} \cap \mathfrak{B}) \cap \mathfrak{C} \in \mathfrak{M}^t$, then consider $\mathfrak{A} \cap (\mathfrak{B} \cap \mathfrak{C})$. By the associativity of the set theoretic meet, which is immediate by Definition 3.1, we have $(\mathfrak{A} \cap \mathfrak{B}) \cap \mathfrak{C} = \mathfrak{A} \cap (\mathfrak{B} \cap \mathfrak{C})$, hence ($\beta$) is true and $(\mathfrak{A} \cap \mathfrak{B}) \cap \mathfrak{C} = \mathfrak{A} \cap (\mathfrak{B} \cap \mathfrak{C})$ holds. The converse can be established similarly. □

An immediate consequence of this lemma is the following

Theorem 3.3. *Let t be a fixed similarity type. Then, the class of all structures of type t forms a weak partial meet-semilattice.*

Definition 3.4. Let us define the binary relation $\leqq$ on $\mathfrak{M}^t$ by the item: for any $\mathfrak{A}, \mathfrak{B} \in \mathfrak{M}^t$, $\mathfrak{A} \leqq \mathfrak{B}$ iff $\mathfrak{A} \cap \mathfrak{B}$ exists and $\mathfrak{A} \cap \mathfrak{B} = \mathfrak{A}$.

If $\mathfrak{A} \leqq \mathfrak{B}$, then we say that "$\mathfrak{A}$ is a *weak substructure* of $\mathfrak{B}$", or equivalently, that "$\mathfrak{B}$ is a *weak extension* of $\mathfrak{A}$".

The next assertion collects some useful facts about the relation $\leqq$. The proof is an easy verification or can be readily obtained from the general theory of lattices [4].

Lemma 3.5. *Let* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathfrak{M}^t$.

(i) $\leqq$ *is a partial ordering on* $\mathfrak{M}^t$.

(ii) *If* $\mathfrak{A} \cap \mathfrak{B}$ *exists, then* *(α)* $\mathfrak{A} \cap \mathfrak{B} \leqq \mathfrak{A}$ *and* $\mathfrak{A} \cap \mathfrak{B} \leqq \mathfrak{B}$;

(β) $\mathfrak{C} \leqq \mathfrak{A}$ *and* $\mathfrak{C} \leqq \mathfrak{B}$ *imply that* $\mathfrak{C} \leqq \mathfrak{A} \cap \mathfrak{B}$.

(iii) *If* $\mathfrak{A} \subset \mathfrak{B}$ *then* $\mathfrak{A} \leqq \mathfrak{B}$ *(where* $\subset$ *stands for the traditional concept of substructures). The converse implication is not true in general.*

(iv) $\mathfrak{A} \leqq \mathfrak{B}$ *iff* $|\mathfrak{A}| \subset |\mathfrak{B}|$ *and the identity mapping* $\imath\colon |\mathfrak{A}| \to |\mathfrak{B}|$, *defined by* $\imath(a) = a$, *is a homomorphism in the model theoretic sense.*

The clause (iii) of this lemma justifies the adjective "weak" in the naming of weak substructures.

Definition 3.6. Let T be an arbitrary first order theory. We say that

(i) T is *preserved under weak substructures* (resp. *under weak extensions*) iff for all $\mathfrak{A}, \mathfrak{B} \in \mathfrak{M}^t$, if $\mathfrak{A} \models T$ and $\mathfrak{B} \leqq \mathfrak{A}$ (resp. $\mathfrak{A} \leqq \mathfrak{B}$), then $\mathfrak{B} \models T$;

(ii) T is *preserved under finite meets* iff for all $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n \in \mathfrak{M}^t$, if $\mathfrak{A}_0 \models T$, $\mathfrak{A}_1 \models T, \ldots, \mathfrak{A}_n \models T$ and $\mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_n$ exists, then $\mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_n \models T$.

The next assertion is a slight strengthening of Lemma 3.5(ii), (iii), and is true for arbitrary first order languages.

Theorem 3.7. *Let T be an arbitrary first order theory.*

(i) *If T is preserved under weak substructures, then T is preserved under finite meets.*

(ii) *If T is preserved under finite meets, then T is preserved under traditional substructures.*

(iii) *None of these implications in* (i) *and* (ii) *can be reversed in general.*

Proof. (i): Let us suppose that T is preserved under weak substructures; let $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n \in \mathfrak{M}^t$, and assume that for all $i < n+1$, $\mathfrak{A}_i \models T$ and the meet $\mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_n$ exists. By Lemma 3.5(ii) it is easily seen that $\mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_n \leqq \mathfrak{A}_0$ and so, $\mathfrak{A}_0 \cap \mathfrak{A}_1 \cap \ldots \cap \mathfrak{A}_n \models T$, because T is preserved under weak substructures.

(ii): Let T be such that T is preserved under finite meets. Let $\mathfrak{A} \models T$, $\mathfrak{B} \subset \mathfrak{A}$. We define the structure $\mathfrak{A}'$ as follows. First set $|\mathfrak{A}'| = |\mathfrak{B}| \cup ((|\mathfrak{A}| - |\mathfrak{B}|) \times \{|\mathfrak{A}|\})$; then define $h\colon |\mathfrak{A}| \to |\mathfrak{A}'|$ by the item

$$h(a) = \begin{cases} a & \text{iff } a \in |\mathfrak{B}| \\ \langle a, |\mathfrak{A}|\rangle & \text{iff } a \in |\mathfrak{A}| - |\mathfrak{B}|. \end{cases}$$

Obviously, h is one-to-one and is onto. For all $c \in \mathscr{C}$, let $C_c^{(\mathfrak{A}')} = h(C_c^{(\mathfrak{A})})$. For every $f \in \mathscr{F}$, $t_{\mathscr{F}}(f) = n+1$ and for arbitrary elements $a_0', \ldots, a_n' \in |\mathfrak{A}'|$, let

$$F_f^{(\mathfrak{A}')}(a_0', \ldots, a_n') = h\big(F_f^{(\mathfrak{A})}(h^{-1}(a_0'), \ldots, h^{-1}(a_n'))\big).$$

Finally, for every $r \in \mathscr{R}$, $t_{\mathscr{R}}(r) = n+1$ and elements $a_0', \ldots, a_n' \in |\mathfrak{A}'|$; let

$$\langle a_0', \ldots, a_n'\rangle \in R_r^{(\mathfrak{A}')} \Leftrightarrow \langle h^{-1}(a_0'), \ldots, h^{-1}(a_n')\rangle \in R_r^{(\mathfrak{A})}.$$

Then $\mathfrak{A}'$ is correctly defined and $\mathfrak{A}' \in \mathfrak{M}^t$, provided $\mathfrak{A} \in \mathfrak{M}^t$. Moreover, $\mathfrak{A}'$ is isomorphic to $\mathfrak{A}$ by h. Thus $\mathfrak{A}' \models T$. By the construction, $\mathfrak{A} \cap \mathfrak{A}' = \mathfrak{B}$ and so, T being preserved under finite meets, $\mathfrak{B} \models T$.

(iii): Let us consider the (equality-free) theories

$$T_1 = \{(\forall x) r(x)\}, \quad T_2 = \{(\forall x)(r(x) \vee q(x))\},$$

where r and q are distinct unary relation symbols of an appropriate particular similarity type t.

By Theorem 2.1, T_2 is preserved under traditional substructures but, according to Theorem 1.1, is not preserved under finite meets.

Similarly, Theorem 1.1 shows, that T_1 is preserved under finite meets. Nevertheless, T_1 is not preserved under weak substructures as the following counterexample indicates. (This follows also from Theorem 4.7, below.)

Obviously, T_1 is consistent; let $\mathfrak{A}$ be a model of T_1. Let us define the structure $\mathfrak{C}$ as follows. First set $|\mathfrak{C}| = |\mathfrak{A}|$. Then, for every $f \in \mathscr{F}$, and $c \in \mathscr{C}$, put $F_f^{(\mathfrak{C})} = F_f^{(\mathfrak{A})}$ and $C_c^{(\mathfrak{C})} = C_c^{(\mathfrak{A})}$. Finally, for every $r \in \mathscr{R}$, let $R_r^{(\mathfrak{C})} = \emptyset$.

Trivially, $\mathfrak{C} \not\models T_1$ and $\mathfrak{C} \cap \mathfrak{A} = \mathfrak{C}$, i.e. $\mathfrak{C} < \mathfrak{A}$. □

The "dual" of this theorem is simply a reformulation of Lemma 3.5(iii) in terms of preservation properties.

Theorem 3.8. *Let T be an arbitrary first order theory. If T is preserved under weak extensions, then T is preserved under (traditional) extensions. The converse fails in general.*

Proof. Trivial by Lemma 3.5(iii). □

Corollary 3.9. *Let T be an arbitrary first order theory.*

(i) *If T is preserved under weak substructures or under finite meets, then T has a set of universal axioms. If, in addition, T is equality-free, then it has a universal axiom system which is equality-free.*

(ii) *If T is preserved under weak extensions, then T has a set of existential axioms, which are equality-free, provided T is such.*

Proof. (i): In contrary to the assertion, let us suppose that T has no universal axioms. Then T is consistent. By the well-known Łoś—Tarski preservation theorem ([2], Theorem 3.2.2, p. 124), we can find a model $\mathfrak{A}$ of T and a substructure $\mathfrak{B}$ of $\mathfrak{A}$, such that $\mathfrak{B} \not\models T$. By Theorem 3.7, T is preserved under neither weak substructures nor finite meets. If T is equality-free, then using Theorem 2.1 in place of the Łoś—Tarski theorem, the same argument applies.

(ii): Similar. □

4. This section is devoted to answering the question (1).

Definition 4.1. Let us suppose that φ is an arbitrary first order formula. By predicate logic, φ is equivalent to a formula ψ of the form

(7)

$$\psi=(Q_1x_1\dots Q_nx_n)\bigwedge_{i=1}^{m}(\neg\varphi_{i1}\vee\dots\vee\neg\varphi_{ij_i}\vee\psi_{i1}\vee\dots\vee\psi_{is_i}\vee\neg\varepsilon_{i1}\vee\dots\vee\neg\varepsilon_{ik_i}\vee\eta_{i1}\vee\dots\vee\eta_{il_i}),$$

where $n, m\in\omega$; for all i, $1\leqq i\leqq m$, $j_i, s_i, k_i, l_i\in\omega$, $\varphi_{i1}, \dots, \varphi_{ij_i}, \psi_{i1}, \dots, \psi_{is_i}$ are proper atomic formulae of the form $r(\tau_0, \dots, \tau_v)$ for some $r\in\mathscr{R}$, $t_{\mathscr{R}}(r)=v+1$ and terms $\tau_0, \dots, \tau_v$; and $\varepsilon_{i1}, \dots, \varepsilon_{ik_i}, \eta_{i1}, \dots, \eta_{il_i}$ are equations of the form $\tau_0\equiv\tau_1$, for some terms τ_0, τ_1; and finally, for all z, $1\leqq z\leqq n$, $Q_z\in\{\forall, \exists\}$.

We say that ψ (of the form (7)) is an *equationally-augmented negative* (resp. *positive*) formula, an EAN-formula (resp. EAP-formula), for short, iff for all i, $1\leqq i\leqq m$, $s_i=0$ (resp. $j_i=0$).

Lemma 4.2. *Let T be an arbitrary first order theory. If T has a set of existential* EAP-*axioms, then T is preserved under weak extensions.*

Proof. It will suffice to prove, that every existential EAP-sentence φ is preserved under weak extensions. We shall proceed by induction.

First we observe some trivial facts. Let $\mathfrak{A}, \mathfrak{B}\in\mathfrak{M}^t$, and $\mathfrak{B}\leqq\mathfrak{A}$. We shall denote the set of variables by V.

(8) If $k\colon V\to|\mathfrak{B}|$, then $k\colon V\to|\mathfrak{A}|$; that is, every assignment relative to $\mathfrak{B}$ can as well be regarded as an assignment relative to $\mathfrak{A}$.

(9) For every $r\in\mathscr{R}$, $R_r^{(\mathfrak{B})}\subset R_r^{(\mathfrak{A})}$, by Lemma 3.5(iv).

(10) If τ is a term in the variables $x_1, \dots, x_n$, then for all $k\colon V\to|\mathfrak{B}|$, $\tau^{(\mathfrak{B})}[k]=$ $=\tau^{(\mathfrak{A})}[k]$, by (8) and by Lemma 3.5(iv). (Here $\tau^{(\mathfrak{B})}[k]$ (resp. $\tau^{(\mathfrak{A})}[k]$) stands for "the value of τ in $\mathfrak{B}$ (resp. in $\mathfrak{A}$) at k"; cf. [2], 1.3.13, p. 27).

Now, let us suppose, that $\varphi=(\exists x_1\dots\exists x_n)\psi$, where ψ is an atomic formula in the variables $x_1, \dots, x_n$, and let $\mathfrak{B}\models\varphi$. Then there is an assignment $k\colon V\to|\mathfrak{B}|$, such that

(11) $$\mathfrak{B}\models\psi[k].$$

Recalling that ψ is in one of the following three forms: $\tau_0\equiv\tau_1$, $\neg(\tau_0\equiv\tau_1)$ and $r(\tau_0, \dots, \tau_v)$, we see that, in any case, $\mathfrak{A}\models\psi[k]$ is immediate from (11) by (8), (9) and (10).

The induction trivially passes over all the remaining cases, hence the assertion is proved. □

The converse of this lemma holds, too.

Theorem 4.3. *Let T be an arbitrary first order theory. If T is preserved under weak extensions, then T has a set of existential* EAP *axioms.*

Proof. If T is inconsistent, then the set $\{(\exists x)\neg(x\equiv x)\}$ is an axiom system for T in the required form. Hence we may assume, that T is consistent. Let $\Gamma=\{\varphi|\varphi$ is an existential EAP sentence and $T\models\varphi\}$. Then, obviously, $T\models\Gamma$ and Γ is consistent. We shall prove that $\Gamma\models T$.

Let $\mathfrak{A}\models\Gamma$. First we show, that there is a structure $\mathfrak{B}$ such that $\mathfrak{B}\models T$, and every existential EAP sentence holding in $\mathfrak{B}$ holds in $\mathfrak{A}$. To see this, let $\Sigma=\{\neg\varphi|\varphi$ is an existential EAP sentence and $\mathfrak{A}\models\neg\varphi\}$. We claim that $\Sigma\cup T$ is consistent. Indeed, if $\Sigma\cup T$ were inconsistent, then we could find a finite subset $\{\neg\sigma_0, \ldots, \neg\sigma_m\}\subset\Sigma$ such that $T\models\neg(\neg\sigma_0\wedge\ldots\wedge\neg\sigma_m)$. But $\neg(\neg\sigma_0\wedge\ldots\wedge\neg\sigma_m)$ is equivalent to an existential EAP sentence, say σ, and thus $T\models\sigma$ implies that $\sigma\in\Gamma$, hence $\mathfrak{A}\models\sigma$, that is $\mathfrak{A}\models\neg(\neg\sigma_0\wedge\ldots\wedge\neg\sigma_m)$. This, however, contradicts to the assumption that $\mathfrak{A}\models\neg\sigma_0, \ldots, \mathfrak{A}\models\neg\sigma_m$. So $\Sigma\cup T$ is consistent. Let $\mathfrak{B}$ be an arbitrary model of $\Sigma\cup T$ and suppose that χ is an existential EAP sentence which is true in $\mathfrak{B}$. Assume that $\mathfrak{A}\not\models\chi$, i.e. $\mathfrak{A}\models\neg\chi$. Then $\neg\chi\in\Sigma$ which entails that $\mathfrak{B}\models\neg\chi$, a contradiction. Thus $\mathfrak{A}\models\chi$.

Next we show that if $\mathfrak{B}$ is such that $\mathfrak{B}\models T$ and every existential EAP sentence holding in $\mathfrak{B}$ holds in $\mathfrak{A}$, then there are structures $\mathfrak{A}'$, $\mathfrak{B}'$ for which we have $\mathfrak{A}\prec\mathfrak{A}'$, $\mathfrak{B}'\leqq\mathfrak{A}'$ and $\mathfrak{B}'$ is isomorphic to $\mathfrak{B}$. (Here $\prec$ stands for the traditionally defined concept "elementary submodel", cf. [2], p. 107.)

Let c_a and d_b be new constant symbols for every $a\in|\mathfrak{A}|$ and $b\in|\mathfrak{B}|$, respectively, thus forming the diagram languages of $\mathfrak{A}$ and $\mathfrak{B}$ (cf. [2], p. 108). Make sure that $\{c_a|a\in|\mathfrak{A}|\}\cap\{d_b|b\in|\mathfrak{B}|\}=\emptyset$. Let $\Gamma_{\mathfrak{A}}$ be the elementary diagram of $\mathfrak{A}$ (cf. [2], p. 108). Let $\Delta_{\mathfrak{B}}^{+\mathrm{ea}}$ be the set of all positive atomic sentences and all negated equations in the diagram language of $\mathfrak{B}$ which hold in the diagram expansion $(\mathfrak{B}, b)_{b\in|\mathfrak{B}|}$ (cf. [2], p. 108). (That is, $\Delta_{\mathfrak{B}}^{+\mathrm{ea}}$ is a proper subset of the diagram $\Delta_{\mathfrak{B}}$ of $\mathfrak{B}$, cf. [2], p. 68, obtained from $\Delta_{\mathfrak{B}}$ by omitting all elements of the form $\neg r(\tau_0, \ldots, \tau_m)$.)

We claim that $\Gamma_{\mathfrak{A}}\cup\Delta_{\mathfrak{B}}^{+\mathrm{ea}}$ is consistent. Let us suppose the contrary: $\Gamma_{\mathfrak{A}}\cup\Delta_{\mathfrak{B}}^{+\mathrm{ea}}$ is inconsistent. Then we can find a finite subset $\{\delta_0, \ldots, \delta_m\}\subset\Delta_{\mathfrak{B}}^{+\mathrm{ea}}$, such that $\Gamma_{\mathfrak{A}}\models\neg(\delta_0\wedge\ldots\wedge\delta_m)$. Since the elements of $\{d_b|b\in|\mathfrak{B}|\}$ do not appear in $\Gamma_{\mathfrak{A}}$, we can treat them in $\neg(\delta_0\wedge\ldots\wedge\delta_m)$ as free variables. It follows from the universal Closure Theorem of predicate logic, that for an appropriately large $n\in\omega$,

$$\Gamma_{\mathfrak{A}}\models(\forall x_1\ldots\forall x_n)\neg(\delta_0(x_1, \ldots, x_n)\wedge\ldots\wedge\delta_m(x_1, \ldots, x_n)).$$

In particular, $(\mathfrak{A}, a)_{a\in|\mathfrak{A}|}\models(\forall x_1\ldots\forall x_n)\neg(\delta_0(x_1, \ldots, x_n)\wedge\ldots\wedge\delta_m(x_1, \ldots, x_n))$, and so

$$\mathfrak{A}\models(\forall x_1\ldots\forall x_n)\neg(\delta_0(x_1, \ldots, x_n)\wedge\ldots\wedge\delta_m(x_1, \ldots, x_n)) \tag{12}$$

because no elements of $\{c_a|a\in|\mathfrak{A}|\}$ appear in the sentence

$$\chi=(\forall x_1\ldots\forall x_n)\neg(\delta_0(x_1, \ldots, x_n)\wedge\ldots\wedge\delta_m(x_1, \ldots, x_n)).$$

On the other hand, however, $(\mathfrak{B}, b)_{b\in|\mathfrak{B}|}\models\delta_0\wedge\ldots\wedge\delta_m$, and so

$$\mathfrak{B}\models(\exists x_1\ldots\exists x_n)(\delta_0(x_1,\ldots,x_n)\wedge\ldots\wedge\delta_m(x_1,\ldots,x_n)).$$

But the sentence $(\exists x_1\ldots\exists x_n)(\delta_0(x_1,\ldots,x_n)\wedge\ldots\wedge\delta_m(x_1,\ldots,x_n))$ is an existential EAP sentence, hence, by assumption

$$\mathfrak{A}\models(\exists x_1\ldots\exists x_n)(\delta_0(x_1,\ldots,x_n)\wedge\ldots\wedge\delta_m(x_1,\ldots,x_n)),$$

which contradicts to (12). Thus $\Gamma_{\mathfrak{A}}\cup\Delta_{\mathfrak{B}}^{+ea}$ is consistent, indeed.

Let $(\mathfrak{A}', a', b')_{a\in|\mathfrak{A}|,b\in|\mathfrak{B}|}$ be a model of $\Gamma_{\mathfrak{A}}\cup\Delta_{\mathfrak{B}}^{+ea}$ (where a' and b' denote the interpretations of the new constant symbols c_a and d_b for every $a\in|\mathfrak{A}|$ and $b\in|\mathfrak{B}|$, respectively). We may assume that for all $a\in|\mathfrak{A}|$, $a'=a$; i.e. $|\mathfrak{A}|\subset|\mathfrak{A}'|$. Then $\mathfrak{A}\prec\mathfrak{A}'$, because $(\mathfrak{A}', a, b')_{a\in|\mathfrak{A}|,b\in|\mathfrak{B}|}\models\Gamma_{\mathfrak{A}}$. Let us define the mapping $g\colon|\mathfrak{B}|\to|\mathfrak{A}'|$ by the equation $g(b)=b'$. Since $(\mathfrak{A}', a, b')_{a\in|\mathfrak{A}|,b\in|\mathfrak{B}|}\models\Delta_{\mathfrak{B}}^{+ea}$, it is easily seen that g is an isomorphism in the algebraic sense (leaving relations out of consideration) and that g is a model theoretic homomorphism (when relations are considered, too). By Lemma 3.5(iv), there is a weak substructure $\mathfrak{B}'$ of $\mathfrak{A}'$, such that $\mathfrak{B}'$ and $\mathfrak{B}$ are isomorphic by g.

Now, $\mathfrak{B}\models T$ implies $\mathfrak{B}'\models T$. T is preserved under weak extensions, hence $\mathfrak{A}'\models T$. By $\mathfrak{A}\prec\mathfrak{A}'$, we have $\mathfrak{A}\models T$, which was to be proved. □

Corollary 4.4. *Let T be an arbitrary first order theory. Then, the two assertions below are equivalent:*

(i) *T is preserved under weak extensions;*

(ii) *T has a set of existential* EAP *axioms.*

Proof. Immediate by Lemma 4.2 and Theorem 4.3. □

The dual of Corollary 4.4 has a somewhat simpler proof; in fact, we need the compactness property only, and we shall not use elementary submodels.

Theorem 4.5. *Let T be an arbitrary first order theory. Then the two assertions below are equivalent:*

(i) *T is preserved under weak substructures;*

(ii) *T has a set of universal* EAN *axioms.*

Proof. (i)⇒(ii): We may assume that T is consistent for otherwise the set $\{(\forall x)\neg(x\equiv x)\}$ shows that (ii) is true.

Let $\Gamma=\{\varphi|\varphi$ is a universal EAN sentence and $T\models\varphi\}$. Then $T\models\Gamma$ and Γ is consistent.

Let $\mathfrak{A}\models\Gamma$ and consider the set $\Delta_{\mathfrak{A}}^{+ea}$ (defined in the very same way as $\Delta_{\mathfrak{B}}^{+ea}$ was defined in the proof of Theorem 4.3, but, of course, $\mathfrak{B}$ replaced everywhere by $\mathfrak{A}$).

We claim that $\Delta_{\mathfrak{A}}^{+ea}\cup T$ is consistent. To see this let $\{\delta_0, \ldots, \delta_m\}\subset\Delta_{\mathfrak{A}}^{+ea}$. Then, the sentence

$$\chi = (\exists x_1 \ldots \exists x_n)(\delta_0(x_1, \ldots, x_n)\wedge \ldots \wedge\delta_m(x_1, \ldots, x_n))$$

is true in $\mathfrak{A}$ for an appropriate $n\in\omega$. But χ must hold in some model of T, since otherwise (when χ is false in every model of T), we would have $\neg\chi\in\Gamma$, because $\neg\chi$ is a universal EAN sentence, and so, we would arrive to the contradiction $\mathfrak{A}\models\neg\chi$. Thus, $\{\delta_0, \ldots, \delta_m\}$ is consistent with T and, by compactness, $\Delta_{\mathfrak{A}}^{+ea}\cup T$ is consistent.

Let $(\mathfrak{B}, a')_{a\in|\mathfrak{A}|}$ be a model of $\Delta_{\mathfrak{A}}^{+ea}\cup T$ (where a' stands for the interpretation of the newly added constant symbol c_a for each $a\in|\mathfrak{A}|$). Let $g\colon |\mathfrak{A}|\to|\mathfrak{B}|$ be defined by the item $g(a)=a'$. Since $(\mathfrak{B}, a')_{a\in|\mathfrak{A}|}\models\Delta_{\mathfrak{A}}^{+ea}$, it is easy to see that g is an isomorphism in the algebraic sense (relations dropped) and is a homomorphism if we consider relations, too. It follows from Lemma 3.5(iv) that there is a weak substructure $\mathfrak{B}'$ of $\mathfrak{B}$ such that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}'$.

T is preserved under weak substructures, hence $\mathfrak{B}'\models T$ follows from $\mathfrak{B}\models T$ and $\mathfrak{A}'\leqq\mathfrak{B}$. Thus, $\mathfrak{A}\models T$, i.e. Γ is an axiom system for T.

(ii)⇒(i): It suffices to prove that every universal EAN sentence φ is preserved under weak substructures. This can be done by a simple argument; details are omitted. □

The statement of Theorem 4.5 is a slight strengthening of a result due to H. Andréka, I. Németi and I. Sain (cf. [1], § 6. Theorem 1; [6], Theorem 1, Theorem 3). Their proof, however, is purely category theoretic in character and works only if T is assumed to be universal. By Theorem 3.7, the assumption that T is universal, does not mean the loss of generality; nevertheless, this is not clear from the category theoretical framework.

For equality-free languages we prove

Theorem 4.6. *Let T be an equality-free consistent first order theory. Then, the following two assertions are equivalent:*

(i) *T is preserved under weak extensions;*

(ii) *T has a set of existential positive equality-free axioms.*

Proof. (i)⇒(ii): Let $\Gamma=\{\varphi|\varphi$ is an existential positive equality-free sentence and $T\models\varphi\}$. T is assumed to be consistent, hence Γ is consistent, because $T\models\Gamma$. We shall prove that $\Gamma\models T$.

Let $\mathfrak{A}\models\Gamma$. Just as in the proof of Theorem 4.3, we see that there is a structure $\mathfrak{B}$, such that $\mathfrak{B}\models T$, and every existential positive equality-free sentence holding in $\mathfrak{B}$ holds in $\mathfrak{A}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be fixed in the rest of this proof.

For every $b\in|\mathfrak{B}|$, let d_b be a new constant symbol and form the diagram language of $\mathfrak{B}$ (the language constructed from the non-logical symbols of t and the new set of constant symbols $\{d_b | b \in |\mathfrak{B}|\}$). Let $\Delta_{\mathfrak{B}}^{+ef}$ be the set of all (positive) atomic sentences of the form $r(\tau_0, \ldots, \tau_n)$, where $r \in \mathscr{R}$, $t_{\mathscr{R}}(r) = n+1$ and $\tau_0, \ldots, \tau_n$ are terms in the diagram language of $\mathfrak{B}$, which are true in $(\mathfrak{B}, b)_{b \in |\mathfrak{B}|}$. Let Σ be the set of all equality-free sentences (of the original language) which hold in $\mathfrak{A}$.

Following closely the way the consistency of $\Gamma_{\mathfrak{A}} \cup \Delta_{\mathfrak{B}}^{+ea}$ is established in the proof of Theorem 4.3, one proves that $\Sigma \cup \Delta_{\mathfrak{B}}^{+ef}$ is consistent.

Let $(\mathfrak{C}, b')_{b \in |\mathfrak{B}|}$ be a model of $\Sigma \cup \Delta_{\mathfrak{B}}^{+ef}$ (where, as usual, b' denotes the interpretation of d_b for each $b \in |\mathfrak{B}|$). First we show the following statement is true:

(13) For every equality-free first order sentence φ,

$$\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{C} \models \varphi.$$

Indeed, if $\mathfrak{A} \models \varphi$, then $\varphi \in \Sigma$ and thus $(\mathfrak{C}, b')_{b \in |\mathfrak{B}|} \models \varphi$, from which $\mathfrak{C} \models \varphi$ follows, because the elements of the set $\{d_b | b \in |\mathfrak{B}|\}$ cannot appear in φ. On the other hand, if $\mathfrak{A} \not\models \varphi$, i.e. $\mathfrak{A} \models \neg\varphi$, then $\neg\varphi \in \Sigma$, and so $\mathfrak{C} \models \neg\varphi$ is obtained. Thus (13) is proved.

Let X be an arbitrary set such that card $X \geqq$ card $|\mathfrak{C}|$. Let h: $X \cup \{d_b | b \in |\mathfrak{B}|\} \to |\mathfrak{C}|$ and g: $\{d_b | b \in |\mathfrak{B}|\} \to |\mathfrak{B}|$ be two onto mappings, such that for all $b \in |\mathfrak{B}|$, $h(d_b) = b'$ and $g(d_b) = b$. Such mappings h and g exist. Let us form the free structures $\mathfrak{C}' = \mathfrak{Fr}_h(\mathfrak{C}, b')_{b \in |\mathfrak{B}|}$ and $\mathfrak{B}' = \mathfrak{Fr}_g(\mathfrak{B}, b)_{b \in |\mathfrak{B}|}$. By Theorem 2.4, $\mathfrak{C}' \models \Sigma \cup \Delta_{\mathfrak{B}}^{+ef}$ and $\mathfrak{B}' \models T \cup \Delta_{\mathfrak{B}}^{+ef}$. We shall show that $\mathfrak{B}' \leqq \mathfrak{C}'$. Obviously, $|\mathfrak{B}'| \subset |\mathfrak{C}'|$, and for all $b \in |\mathfrak{B}|$

(14) $$C_{c_b}^{(\mathfrak{B}')} = C_{c_b}^{(\mathfrak{C}')}$$

is immediate by Definition 2.3. Similarly, for every $f \in \mathscr{F}$, $t_{\mathscr{F}}(f) = n+1$, and $b_0, \ldots, b_n \in |\mathfrak{B}'|$, we have

(15) $$F_f^{(\mathfrak{B}')}(b_0, \ldots, b_n) = F_f^{(\mathfrak{C}')}(b_0, \ldots, b_n).$$

It follows from (14) and (15), that for any closed term τ in the diagram language of $\mathfrak{B}$, the equation

(16) $$\tau^{(\mathfrak{B}')} = \tau^{(\mathfrak{C}')}$$

holds.

Let $r \in \mathscr{R}$, $t_{\mathscr{R}}(r) = n+1$, $b_0, \ldots, b_n \in |\mathfrak{B}'|$. By the definition of $\mathfrak{B}'$, we can find closed terms $\tau_0, \ldots, \tau_n$ of the diagram language of $\mathfrak{B}$, such that $b_0 = \tau_0^{(\mathfrak{B}')}, \ldots, b_n = \tau_n^{(\mathfrak{B}')}$. Hence, the following chain of implications is obtained:

$$\langle b_0, \ldots, b_n \rangle \in R_r^{(\mathfrak{B}')} \Rightarrow \langle \tau_0^{(\mathfrak{B}')}, \ldots, \tau_n^{(\mathfrak{B}')} \rangle \in R_r^{(\mathfrak{B}')} \Rightarrow \mathfrak{B}' \models r(\tau_0, \ldots, \tau_n) \Rightarrow$$
$$\Rightarrow r(\tau_0, \ldots, \tau_n) \in \Delta_{\mathfrak{B}}^{+ef} \Rightarrow (\mathfrak{C}, b')_{b \in |\mathfrak{B}|} \models r(\tau_0, \ldots, \tau_n).$$

Using Theorem 2.4 again, we can continue:

$$(\mathfrak{C}, b')_{b \in |\mathfrak{B}|} \models r(\tau_0, \ldots, \tau_n) \Rightarrow \mathfrak{C}' \models r(\tau_0, \ldots, \tau_n) \Rightarrow \langle \tau_0^{(\mathfrak{C}')}, \ldots, \tau_n^{(\mathfrak{C}')} \rangle \in R_r^{(\mathfrak{C}')}$$

from which $\langle b_0, ..., b_n \rangle \in R_r^{(\mathfrak{C}')}$ follows.

By Lemma 3.5(iv), we see that $\mathfrak{B}' \leqq \mathfrak{C}'$.

Since $\mathfrak{B}' \models T$ and T is preserved under weak extensions, we have $\mathfrak{C}' \models T$, and by Theorem 2.4, $\mathfrak{C} \models T$. By (13), $\mathfrak{A} \models T$, which was to be proved.

(ii)⇒(i): Immediate by Lemma 4.2. □

Using a similar (but somewhat simpler) argument, one can prove the dual of this theorem

Theorem 4.7. *Let T be an equality-free consistent first order theory. Then, the following assertions are equivalent:*

(i) *T is preserved under weak substructures;*

(ii) *T has a set of universal negative equality-free axioms.*

5. This section is devoted to answering question (2) in the particular case when equality is excluded from the language.

Definition 5.1. Let $K \subset \mathfrak{M}^t$.

(i) K is said to be *closed under finite meets* iff for arbitrary $\mathfrak{A}_0, ..., \mathfrak{A}_n \in K$, if $\mathfrak{A}_0 \sqcap ... \sqcap \mathfrak{A}_n$ exists, then $\mathfrak{A}_0 \sqcap ... \sqcap \mathfrak{A}_n \in K$.

(ii) K is *closed under extensions (weak extensions)* iff for arbitrary $\mathfrak{A} \in K$ and $\mathfrak{B} \in \mathfrak{M}^t$, $\mathfrak{A} \subset \mathfrak{B}$ ($\mathfrak{A} \leqq \mathfrak{B}$) entails $\mathfrak{B} \in K$.

Obviously, if T is an arbitrary first order theory and "OPERATION" stands for one of the following items: "finite meets", "extensions", and "weak extensions", then the assertion "T is preserved under OPERATION" is equivalent to the assertion "K is closed under OPERATION where $K = \{\mathfrak{A} | \mathfrak{A} \models T\}$".

Definition 5.2. By a *filter of structures* we shall mean a nonvoid class $K \subset \mathfrak{M}^t$ such that K is closed under both finite meets and weak extensions.

The following assertion characterizes filters of structures from a model theoretical point of view.

Theorem 5.3. *Let T be an arbitrary equality-free first order theory and let K be the class of all models of T. Then the following two assertions are equivalent:*

(i) *T has a set of quantifier-free atomic equality-free axioms;*

(ii) *K is a filter of structures.*

Proof. First we note that both (i) and (ii) imply that T is consistent.

(i)⇒(ii): It is obvious that every equality-free quantifier-free atomic sentence can be considered as an existential positive equality-free sentence and as a universal equality-free Horn sentence, simultaneously. Thus, T is preserved under both weak extensions and finite meets by Lemma 4.2 and Theorem 1.1, respectively; whence

K is closed under both weak extensions and finite meets; i.e. K is a filter of structures (for $K \neq \emptyset$).

(ii)$\Rightarrow$(i): Let us suppose, that K is a filter of structures, i.e. that K is closed under finite meets and weak extensions. It follows that T is preserved under finite meets and weak extensions.

Let $\Gamma = \{\varphi | \varphi$ is an equality-free, quantifier-free atomic sentence, $T \models \varphi\}$. Obviously, $T \models \Gamma$. We shall prove that $\Gamma \models T$.

Let $\mathfrak{C} \models \Gamma$ be arbitrary and set $\Sigma = \{\neg\sigma | \sigma$ is an equality-free, quantifier-free atomic sentence such that $\mathfrak{C} \models \neg\sigma\}$.

Let $\neg\sigma \in \Sigma$ be arbitrary. Then $\{\neg\sigma\} \cup T$ is consistent, for otherwise we would have $T \models \neg(\neg\sigma)$, i.e. $T \models \sigma$, and so $\sigma \in \Gamma$; from which the contradiction $\mathfrak{C} \models \sigma$ would follow.

Let $\{\neg\sigma_0, \ldots, \neg\sigma_m\} \subset \Sigma$, and for every i, $0 \leqq i \leqq m$, let $\mathfrak{B}_i$ be a model of $\{\neg\sigma_i\} \cup T$. Let X be any set such that $\operatorname{card} X \geqq \operatorname{card} |\mathfrak{B}_0| \cup \ldots \cup \operatorname{card} |\mathfrak{B}_m|$, and let $g_i \colon X \cup \mathscr{C} \to |\mathfrak{B}_i|$ be an onto mapping for each i, $0 \leqq i \leqq m$. Let us consider the free structures $\mathfrak{Fr}_{g_i} \mathfrak{B}_i$, $0 \leqq i \leqq m$. It is immediate by Definition 2.3, that $\mathfrak{B} = \mathfrak{Fr}_{g_0} \mathfrak{B}_0 \cap \ldots \cap \mathfrak{Fr}_{g_m} \mathfrak{B}_m$ exists; moreover, for any i, $0 \leqq i \leqq m$, $\mathfrak{Fr}_{g_i} \mathfrak{B}_i \models \{\neg\sigma_i\} \cup T$, by Theorem 2.4. Since T is preserved under finite meets, and σ_i is atomic, we have for every i, $0 \leqq i \leqq m$ that $\mathfrak{B} \models \{\neg\sigma_i\} \cup T$, i.e. $\mathfrak{B} \models \{\neg\sigma_0, \ldots, \neg\sigma_m\} \cup T$. By compactness, $\Sigma \cup T$ is consistent.

Let $\mathfrak{D}$ be a model of $\Sigma \cup T$. If ψ is an arbitrary equality-free, quantifier-free atomic sentence such that $\mathfrak{C} \not\models \psi$, then $\neg\psi \in \Sigma$, hence $\mathfrak{D} \not\models \psi$. It follows that for any equality-free, quantifier-free atomic sentence ψ, $\mathfrak{D} \models \psi$ implies $\mathfrak{C} \models \psi$.

Let Y be an arbitrary set such that $\operatorname{card} Y \geqq \operatorname{card} |\mathfrak{C}| \cup \operatorname{card} |\mathfrak{D}|$ and let $h_1 \colon Y \cup \mathscr{C} \to |\mathfrak{C}|$, $h_2 \colon Y \cup \mathscr{C} \to |\mathfrak{D}|$ be two onto mappings for which $h_1(c) = C_c^{(\mathfrak{C})}$, and $h_2(c) = C_c^{(\mathfrak{D})}$, for any $c \in \mathscr{C}$. Considering the free structures $\mathfrak{Fr}_{h_1} \mathfrak{C}$ and $\mathfrak{Fr}_{h_2} \mathfrak{D}$ we still have for any equality-free, quantifier-free atomic sentence ψ, that $\mathfrak{Fr}_{h_2} \mathfrak{D} \models \psi$ entails $\mathfrak{Fr}_{h_1} \mathfrak{C} \models \psi$. By Definition 2.3 and Lemma 3.5(iv), $\mathfrak{Fr}_{h_2} \mathfrak{D} \leqq \mathfrak{Fr}_{h_1} \mathfrak{C}$. But $\mathfrak{Fr}_{h_2} \mathfrak{D} \models T$ (by Theorem 2.4) and T is preserved under weak extensions, hence $\mathfrak{Fr}_{h_1} \mathfrak{C} \models T$. By Theorem 2.4, $\mathfrak{C} \models T$. $\square$

From a purely formalist point of view one may adopt the following notion:

Definition 5.4. By a *quasi-filter of structures* we mean a class $K \subset \mathfrak{M}^t$ such that K is closed under finite meets and ordinary extensions.

The analogue of Theorem 5.3 for this concept reads as follows.

Theorem 5.5. *Let T be an arbitrary equality-free first order theory and let K be the class of all models of T. Then, the following two assertions are equivalent:*

(i) *T has a set of quantifier-free equality-free Horn axioms;*

(ii) *K is a quasi-filter of structures.*

Proof. Similar to the proof of Theorem 5.3. □

We note that none of Theorems 5.3 and 5.5 generalize for theories with equality. Let us consider for example the theory $T=\{c_1=d_1 \vee c_2=d_2\}$, where c_1, c_2, d_1, d_2 are constant symbols. It is trivial that T is preserved under finite meets and weak extensions, by definition. Hence, K, the class of all models of T, is a filter of structures. T, however, has neither an atomic nor a Horn set of axioms in general, thus Theorem 5.3 is not true for this theory. Since every filter of structures is a quasi-filter of structures, Theorem 5.5 is false for T, too.

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SZKI
DONÁTI u. 35—45
H—1015 BUDAPEST, HUNGARY