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Complete congruence relations of concept lattices

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To the memory of András Huhn

1. Introduction. Although complete lattices have been a main subject of lattice theory for a long time, complete congruence relations of complete lattices have only rarely been studied. In this paper we describe a general approach to complete congruence relations generalizing ideas introduced in [3]. In our approach we understand complete lattices as concept lattices. This enables us to establish a one-to-one correspondence between complete congruence relations and compatible saturated subcontexts of suitable contexts. The use of this correspondence is demonstrated by proving that every distributive complete lattice in which each element is the supremum of $\vee$-irreducible elements is isomorphic to the lattice of all complete congruence relations of some complete lattice. The question remains open which complete lattices are isomorphic to such lattices of complete congruence relations. Examples are given that they need not be distributive.

2. Compatible and saturated subcontexts. A *subcontext* of a context (G, M, I) is understood as a triple $(H, N, I\cap(H\times N))$ with $H\subseteq G$ and $N\subseteq M$; we often write (H, N) instead of $(H, N, I\cap(H\times N))$. Throughout this section, (G, M, I) will be a context and (H, N) a subcontext of (G, M, I). For $g\in G$ and $m\in M$, g' and m' stands for $\{g\}'$ and $\{m\}'$, respectively. By $\pi(H, N)(A, B):=(A\cap H, B\cap N)$ for any concept (A, B) of (G, M, I), we define a map $\pi(H, N)$ from $\mathfrak{B}(G, M, I)$ into $\mathfrak{P}(H)\times\mathfrak{P}(N)$ where, in general, $\mathfrak{P}(S)$ is the complete lattice of all subsets of a set S. The subcontext (H, N) of (G, M, I) is said to be *compatible* if the following conditions are satisfied:

(1a) For all $h\in H$ and $m\in M\setminus h'$ there exists an $n\in N\setminus h'$ with $n'\supseteq m'$;

(1b) for all $n\in N$ and $g\in G\setminus n'$ there exists an $h\in H\setminus n'$ with $h'\supseteq g'$.

The notion of a compatible subcontext is the same as in [3] which follows from Proposition 1.

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Proposition 1. *(H, N) is compatible if and only if $\pi(H, N)$ is a complete lattice homomorphism from $\mathfrak{B}(G, M, I)$ onto $\mathfrak{B}(H, N, I\cap(H\times N))$.*

Proof. Let (H, N) be compatible. By the basic theorem in [2], it must only be shown that $(A\cap H, B\cap N)$ is a concept of $(H, N, I\cap(H\times N))$ for $(A, B)\in\mathfrak{B}(G, M, I)$. Let $h\in H\setminus A$. Then there is an $m\in B$ with $(h, m)\notin I$, i.e. $m\in M\setminus h'$. By (1a), there exists an $n\in N\setminus h'$ with $n'\supseteq m'$. Hence $n\in B\cap N$ and so $h\notin(B\cap N)'$. It follows that $A\cap H=$ $=(B\cap N)'\cap H$ and dually that $B\cap N=(A\cap H)'\cap N$. Thus, $(A\cap H, B\cap N)$ is a concept of $(H, N, I\cap H\times N))$. Conversely, let $(A\cap H, B\cap N)\in\mathfrak{B}(H, N, I\cap(H\times N))$ for all $(A, B)\in\mathfrak{B}(G, M, I)$. Now, let $h\in H$ and $m\in M\setminus h'$. Since $(m'\cap H, m''\cap N)$ is a concept of $(H, N, I\cap(H\times N))$ and $h\notin m'$, there exists an $n\in m''\cap N$ with $(h, n)\notin I$, i.e. $n\in N\setminus h'$ and $n'\supseteq m'$. Hence (H, N) satisfies (1a). Dually we obtain (1b). Thus, (H, N) is compatible.

Let $\Theta(H, N)$ be the set of all pairs of concepts (A, B) and (C, D) of (G, M, I) such that $\pi(H, N)(A, B)=\pi(H, N)(C, D)$, i.e. $\Theta(H, N)$ is the kernel of $\pi(H, N)$. If (H, N) is compatible, Proposition 1 yields that $\Theta(H, N)$ is a complete congruence relation on $\mathfrak{B}(G, M, I)$ and that $\mathfrak{B}(H, N, I\cap(H\times N))\cong\mathfrak{B}(G, M, I)/\Theta(H, N)$. Let us recall that a *complete congruence relation* of a complete lattice L is an equivalence relation Θ on L satisfying $(\bigwedge_{j\in J} x_j)\Theta(\bigwedge_{j\in J} y_j)$ and $(\bigvee_{j\in J} x_j)\Theta(\bigvee_{j\in J} y_j)$ if $x_j\Theta y_j$ for all $j\in J$.

The question arises how to reconstruct the compatible subcontext (H, N) from the complete congruence relation $\Theta(H, N)$. By the following definition, a complete congruence relation Θ of $\mathfrak{B}(G, M, I)$ is naturally transformed into a subcontext of (G, M, I):

$$G(\Theta):=\{g\in G\mid \gamma g:=(g'', g') \text{ is the smallest element of a } \Theta\text{-class}\},$$

$$M(\Theta):=\{m\in M\mid \mu m:=(m', m'') \text{ is the greatest element of a } \Theta\text{-class}\}.$$

To obtain (H, N) from $\Theta(H, N)$ via this definition, (H, N) has to be *saturated*, i.e. (H, N) must satisfy the following conditions:

(2a) For $g\in G$ and $X\subseteq H$, $g'=X'$ implies $g\in H$;

(2b) for $m\in M$ and $Y\subseteq N$, $m'=Y'$ implies $m\in N$.

Proposition 2. *Let (H, N) be compatible. Then (H, N) is saturated if and only if $H=G(\Theta(H, N))$ and $N=M(\Theta(H, N))$.*

Proof. First we assume that (H, N) is saturated. Let $h\in H$ and $(A, B)\in\mathfrak{B}(G, M, I)$. Then $h''\cap H=A\cap H$ implies $h\in A$ and so $h''\subseteq A$; hence $h\in G(\Theta(H, N))$. Now, let $g\in G(\Theta(H, N))$. Since γg is the smallest element of a $\Theta(H, N)$-class, it follows that $\gamma g=((H\cap g'')'', (H\cap g'')')$ and therefore $g'=(H\cap g'')'$. Hence $g\in H$ by (2a).

This proves that $H=G(\mathbf{\Theta}(H, N))$ and dually $N=M(\mathbf{\Theta}(H, N))$. Let us assume these equalities for the opposite direction of the proof. Now we use that for a complete congruence relation Θ of a complete lattice L the set of the smallest elements of the Θ-classes is closed under suprema and the set of the greatest elements of the Θ-classes is closed under infima. Let $g'=X'$ for $g\in G$ and $X\subseteq H=G(\mathbf{\Theta}(H, N))$. Then, by the basic theorem in [2], $\gamma g=\bigvee_{x\in X}\gamma x$ and so $g\in G(\mathbf{\Theta}(H, N))=H$. In this way we obtain (2a) and dually (2b). Thus, (H, N) is saturated.

The next proposition clarifies the nature of the complete congruence relation $\mathbf{\Theta}(H, N)$. In the formulation we use the notation $[z]\,\Theta$ for the equivalence class of Θ represented by z.

Proposition 3. *Let Θ be a complete congruence relation of $\mathfrak{B}(G, M, I)$. Then $(G(\Theta), M(\Theta))$ is a compatible and saturated subcontext of (G, M, I) satisfying $\Theta=\mathbf{\Theta}(G(\Theta), M(\Theta))$ if and only if $\{[\gamma h]\,\Theta \mid h\in G(\Theta)\}$ is a supremum-dense and $\{[\mu n]\,\Theta \mid n\in M(\Theta)\}$ is infimum-dense in $\mathfrak{B}(G, M, I)/\Theta$.*

Proof. Assume that $(G(\Theta), M(\Theta))$ is a compatible and saturated subcontext of (G, M, I) satisfying $\Theta=\mathbf{\Theta}(G(\Theta), M(\Theta))$. By Proposition 1, $\{(h''\cap G(\Theta), h'\cap M(\Theta)) \mid h\in G(\Theta)\}$ is supremum-dense in $\mathfrak{B}(G(\Theta), M(\Theta), I\cap(G(\Theta)\times M(\Theta)))$. Since Θ is the kernel of $\pi(G(\Theta), M(\Theta))$, it follows that $\{[\gamma h]\Theta \mid h\in G(\Theta)\}$ is supremum-dense in $\mathfrak{B}(G, M, I)/\Theta$ and dually that $\{[\mu n]\Theta \mid n\in M(\Theta)\}$ is infimum-dense in $\mathfrak{B}(G, M, I)/\Theta$. Let us assume these properties for the opposite direction of the proof. First we show that $(G(\Theta), M(\Theta))$ is compatible. Let $h\in G(\Theta)$ and $m\in M\setminus h'$. Then $[\gamma h]\Theta\not\leqq[\mu m]\Theta$. Since $\{[\mu n]\Theta \mid n\in M(\Theta)\}$ is infimum-dense in $\mathfrak{B}(G, M, I)$, there exists an $n\in M(\Theta)\setminus h'$ with $\mu n\geqq\mu m$, i.e. $n'\supseteq m'$. This proves (1a) and dually (1b). From the fact that the smallest and greatest elements of the Θ-classes are closed under suprema and infima, respectively, it follows that $(G(\Theta), M(\Theta))$ is saturated. Now, let $(A, B)\in\mathfrak{B}(G, M, I)$ and let (A_-, B_-) and (A^-, B^-) be the smallest and greatest concept in the Θ-class containing (A, B). Then $\gamma g\leqq(A, B)$ for $g\in G(\Theta)$ implies $\gamma g\leqq(A_-, B_-)$. Therefore $A\cap G(\Theta)=A_-\cap G(\Theta)$ and dually $B\cap M(\Theta)=B^-\cap M(\Theta)$. Hence $(A, B)\,\Theta\,(C, D)$ implies $A\cap G(\Theta)=C\cap G(\Theta)$ and $B\cap M(\Theta)=D\cap M(\Theta)$, i.e. $(A, B)\,\mathbf{\Theta}(G(\Theta), M(\Theta))\,(C, D)$. Thus, we have $\Theta\subseteq\mathbf{\Theta}(G(\Theta), M(\Theta))$. The equality follows from $(A_-, B_-)=\gamma(A\cap G(\Theta))$ and $(A^-, B^-)=\mu(B\cap M(\Theta))$.

For subcontexts (H_1, N_1) and (H_2, N_2) of (G, M, I) we define $(H_1, N_1)\leqq(H_2, N_2):\Leftrightarrow H_1\subseteq H_2$ and $N_1\subseteq N_2$. The set of all compatible and saturated subcontexts of (G, M, I) together with this order relation is denoted by $\mathfrak{S}(G, M, I)$. For the complete lattice of all complete congruence relations of a complete lattice L we use the notation $\mathfrak{C}(L)$. From Propositions 1, 2, and 3 we obtain the following theorem:

Theorem 4. *Let* (G, M, I) *be a context such that, for all complete congruence relations* Θ *of* $\mathfrak{B}(G, M, I)$, $\{[\gamma h]\Theta \mid h \in G(\Theta)\}$ *is supremum-dense and* $\{[\mu n]\Theta \mid n \in M(\Theta)\}$ *is infimum-dense in* $\mathfrak{B}(G, M, I)/\Theta$. *Then* $\Theta \mapsto (G(\Theta), M(\Theta))$ *describes an anti-isomorphism from* $\mathfrak{C}(\mathfrak{B}(G, M, I))$ *onto* $\mathfrak{S}(G, M, I)$.

For the study of $\mathfrak{C}(L)$ it is interesting to find suitable contexts (G, M, I) with $L \cong \mathfrak{B}(G, M, I)$ satisfying the assumption of Theorem 4. Obviously, $(L, L, \leqq)$ will do, but it would be better to find smaller contexts. The following lemma serves us with one method recognizing such contexts. Another method is given by Lemma 7.

Lemma 5. *Let* $\{\gamma g \mid g \in G\} \cup \{(M', M)\}$ *be an order ideal and let* $\{\mu m \mid m \in M\} \cup \cup \{(G, G')\}$ *be an order filter of* $\mathfrak{B}(G, M, I)$. *Then* $\{[\gamma h]\Theta \mid h \in G(\Theta)\}$ *is supremum-dense and* $\{[\mu n]\Theta \mid n \in M(\Theta)\}$ *is infimum-dense in* $\mathfrak{B}(G, M, I)/\Theta$ *for each complete congruence relation* Θ *of* $\mathfrak{B}(G, M, I)$.

Proof. For a complete congruence relation Θ of $\mathfrak{B}(G, M, I)$ let $(A, B)_\Theta$ be the smallest concept in the Θ-class containing the concept (A, B). It can be easily seen that $(A, B) \mapsto (A, B)_\Theta$ describes a $\bigvee$-preserving map from $\mathfrak{B}(G, M, I)$ into itself. From $(A, B) = \bigvee_{g \in A} \gamma g$ we obtain $(A, B)_\Theta = \bigvee_{g \in A} (\gamma g)_\Theta$ and so $[(A, B)]\Theta = \bigvee_{g \in A} [(\gamma g)_\Theta]\Theta$. Since $(\gamma g)_\Theta = \gamma h$ for all $g \in G$ with $(\gamma g)_\Theta \neq (M', M)$ by assumption, the first assertion follows (and dually the second).

3. Closed subcontexts. After establishing the correspondence between complete congruence relations and compatible saturated subcontexts, the question arises how to construct compatible and saturated subcontexts. In general, this question seems difficult to answer. But there is a method which can be successfully applied in special cases. This method is based on the relations $\nearrow$ and $\swarrow$ of a context (G, M, I) which have been introduced in [3] as follows $(g \in G, m \in M)$:

$$g \nearrow m :\Leftrightarrow (g, m) \notin I \text{ and } m' \text{ is maximal in } \{n' \mid n \in M \setminus g'\},$$

$$g \swarrow m :\Leftrightarrow (g, m) \notin I \text{ and } g' \text{ is maximal in } \{h' \mid h \in G \setminus m'\}.$$

It has been useful to fill in the arrows in the cross-table describing the given context. An example is shown in Figure 1. A subcontext (H, N) of (G, M, I) is called *(arrow-) closed* if $h \nearrow m$ implies $m \in N$ for $h \in H$ and $m \in M$ and if $g \swarrow n$ implies $g \in H$ for $g \in G$ and $n \in N$. For example $(\{1, 4\}, \{a, d\})$ is a closed subcontext of the context described in Figure 1. A context (G, M, I) is called *doubly founded* if for all $(g, m) \in G \times M \setminus I$ there exists $h \in G$ and $n \in M$ with $g \nearrow n$, $n' \supseteq m'$ and $h \swarrow m$, $h' \supseteq g'$ (cf. [4]).

Lemma 6. *A compatible subcontext of a context* (G, M, I) *for which* $g_1' = g_2'$ *implies* $g_1 = g_2$ *for* $g_1, g_2 \in G$ *and* $m_1' = m_2'$ *implies* $m_1 = m_2$ *for* $m_1, m_2 \in M$, *is closed. Conversely, a closed subcontext of a doubly founded context is compatible.*

	a	b	c	d	e
1	$\swarrow\!\!\nearrow$	×	×	×	×
2	$\nearrow$	×	$\swarrow\!\!\nearrow$	×	$\swarrow$
3	×	$\swarrow\!\!\nearrow$	$\swarrow\!\!\nearrow$	×	$\swarrow$
4		$\swarrow$	×	$\swarrow\!\!\nearrow$	$\swarrow$

Figure 1

Lemma 6 is an immediate consequence of the definitions. Let us recall that a context (G, M, I) is said to be *reduced* if $g'=X'$ implies $g\in X$ for $g\in G$, $X\subseteq G$ and if $m'=Y'$ implies $m\in Y$ for $m\in M$, $Y\subseteq M$. Observe that each subcontext of a reduced context is saturated. A complete lattice L is called *doubly founded* if, for every pair $x<y$ in L, there exists a minimal element $s\in L$ with $s\leqq y$ and $s\not\leqq x$ and a maximal element $t\in L$ with $x\leqq t$ and $y\not\leqq t$. Such minimal and maximal elements are just the $\bigvee$-irreducible and $\bigwedge$-irreducible elements of L, respectively, and every element of L is the supremum of $\bigvee$-irreducible elements and the infimum of $\bigwedge$-irreducible elements of L. If $J(L)$ denotes the set of all $\bigvee$-irreducible elements of L and $M(L)$ the set of all $\bigwedge$-irreducible elements of L, then $L\cong\mathfrak{B}(J(L), M(L), \leqq)$ by the basic theorem in [2], and $(J(L), M(L), \leqq)$ is a reduced context. For a doubly founded lattice L, $(J(L), M(L), \leqq)$ is a doubly founded context; but the concept lattice of a doubly founded context need not be doubly founded (take $(\mathbf{N}, \mathbf{N}, \leqq)$).

Lemma 7. *Let* $L:=\mathfrak{B}(G, M, I)$ *be doubly founded and let* Θ *be a complete congruence relation of* L. *Then* $\{[\gamma h]\Theta \mid h\in G(\Theta)\}$ *is supremum-dense and* $\{[\mu n]\Theta \mid n\in M(\Theta)\}$ *is infimum-dense in* L/Θ.

Proof. Suppose there is a concept (A, B) with $[\bigvee\gamma(A\cap G(\Theta))]\Theta<[(A, B)]\Theta$. Let $(\bar{A}, \bar{B})$ and $(\bar{C}, \bar{D})$ be the greatest element in $[(A, B)]\Theta$ and $[\bigvee\gamma(A\cap G(\Theta))]\Theta$, respectively. Because of $(\bar{C}, \bar{D})<(\bar{A}, \bar{B})$, there exists a minimal concept (E, F) in L with $(E, F)\leqq(\bar{A}, \bar{B})$ and $(E, F)\not\leqq(\bar{C}, \bar{D})$. Since (E, F) is $\bigvee$-irreducible in L, there is a $g\in G$ with $\gamma g=(E, F)$. Moreover, (E, F) must be the smallest element of $[(E, F)]\Theta$ and so $g\in A\cap G(\Theta)$. This contradicts $\gamma g\not\leqq\bigvee\gamma(A\cap G(\Theta))$. Thus, the first assertion is proved and dually the second.

Lemmas 6 and 7 together with Theorem 4 yield the following theorem:

Theorem 8. *For a doubly founded complete lattice* L, $\mathfrak{C}(L)$ *is antiisomorphic to the complete lattice of all closed subcontexts of* $(J(L), M(L), \leqq)$.

Notice that the supremum and infimum of closed subcontexts (H_k, N_k) with $k\in K$ are just given by $\left(\bigcup_{k\in K} H_k, \bigcup_{k\in K} N_k\right)$ and $\left(\bigcap_{k\in K} H_k, \bigcap_{k\in K} N_k\right)$.

Corollary. *For a doubly founded complete lattice L, $\mathfrak{C}(L)$ is completely distributive.*

4. Lattices of complete congruence relations. It is a challenging problem to determine the class of all complete lattices which are isomorphic to some $\mathfrak{C}(L)$. Up to now, no complete lattice is known which does not belong to this class. As a positive result we prove that every distributive complete lattice with enough $\vee$-irreducible elements is isomorphic to some $\mathfrak{C}(L)$.

Theorem 9. *Let D be a distributive complete lattice in which each element is a supremum of $\vee$-irreducible elements. Then there exists a complete lattice L with $D \cong \mathfrak{C}(L)$.*

Proof. Let $J(D)$ be the set of all $\vee$-irreducible elements of D (notice that $0 \notin J(D)$). The following construction of a context (G, M, I) was stimulated by an (unpublished) idea of E. T. Schmidt:

$$G := J(D) \times \{1, 2, 3\}, \quad M := J(D) \times \{4, 6\} \cup D \times \{5\},$$

and

$$I := (J(D) \times \{1\}) \times (J(D) \times \{4\}) \cup (J(D) \times \{2\}) \times (D \times \{5\}) \cup (J(D) \times \{3\}) \times (J(D) \times \{6\}) \cup$$

$$\cup \{((s_1, i), (s_2, j)) \mid s_1, s_2 \in J(D),\ s_1 \neq s_2,\ (i, j) \in \{(2, 4), (3, 4), (1, 6), (2, 6)\}\} \cup$$

$$\cup \{((s, i), (x, 5)) \mid s \in J(D),\ x \in D,\ s \leqq x,\ i \in \{1, 3\}\}.$$

For a concept (A, B) of the context $(J(D), D, \leqq)$ we define

$$\varrho(A, B) := (\bar{A} \times \{1, 2, 3\},\ \bar{A} \times \{4, 6\} \cup B \times \{5\})$$

where $\bar{A} := J(D) \setminus A$. We shall show that ϱ is an antiisomorphism from $\mathfrak{B}(J(D), D, \leqq)$ onto $\mathfrak{S}(G, M, I)$ which leads to $D \cong \mathfrak{C}(\mathfrak{B}(G, M, I))$ using Theorem 4 and the fact that $D \cong \mathfrak{B}(J(D), D, \leqq)$. Figure 2 visualizes the foregoing definitions.

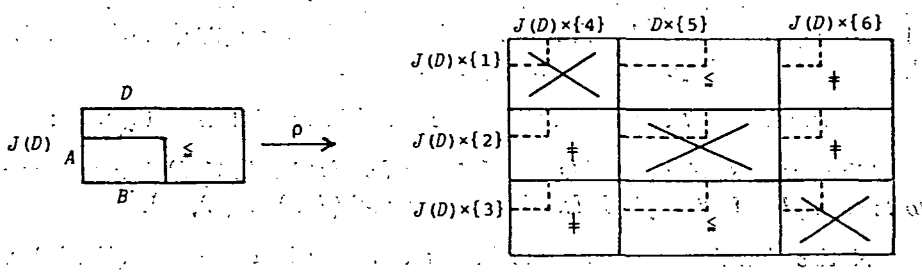


Figure 2

Let (A, B) be a concept of $(J(D), D, \leqq)$. First we show that the subcontext $(\bar{A}, B)$ of $(J(D), D, \leqq)$ satisfies the conditions (1a) and (2b). Let $h \in \bar{A}$ and $m \in D \setminus h'$. Then $\gamma h \not\leqq (A, B)$ and $\gamma h \not\leqq \mu m$. Since γh is $\vee$-irreducible and since $\mathfrak{B}(J(D), D, \leqq)$ is distributive, it follows that $\gamma h \not\leqq (A, B) \vee \mu m$. Hence there exists an $n \in B \setminus h'$ with $n' \supseteq m'$; this proves (1a). Let $m \in D$ and $Y \subseteq B$ with $m' = Y'$. As $A \subseteq Y'$ we get $m \in B$ and so (2b). Now we shall verify the conditions (1b), (2a), (1a), and (2b) for the subcontext $\varrho(A, B)$ of (G, M, I). Define $(H, N) := \varrho(A, B)$. For $g \in G$ and $m \in M$ we write g^I and m^I instead of g' and m', respectively, to avoid confusion; the prime symbol is used in this proof with respect to the context $(J(D), D, \leqq)$. Because of $G \setminus n^I \subseteq H$ for all $n \in N$, (H, N) satisfies (1b). (2a) follows from the fact that $g_1^I \supseteq g_2^I$ implies $g_1 = g_2$ for all $g_1, g_2 \in G$. For $h \in H$ we have $M \setminus h^I \subseteq N \cup D \times \{5\}$. Therefore (1a) holds because $(\bar{A}, B)$ satisfies (1a) in $(J(D), D, \leqq)$ as shown above. Since $m_1^I \supset m_2^I$ for $m_1, m_2 \in M$ and $m_1 \neq (1_D, 5)$ implies $m_1, m_2 \in D \times \{5\}$, (H, N) satisfies (2b) because this condition holds for $(\bar{A}, B)$ in $(J(D), D, \leqq)$. Thus, (H, N) is a compatible and saturated subcontext of (G, M, I).

Now we shall show that a compatible and saturated subcontext of (G, M, I) equals $\varrho(A, B)$ for some $(A, B) \in \mathfrak{B}(J(D), D, \leqq)$. It can be easily seen that $g \swarrow m$ for all $(g, m) \in G \times M \setminus I$ and $g \nearrow m$ for all $(g, m) \in G \times (J(D) \times \{4, 6\}) \setminus I$. By Lemma 6, a compatible subcontext of (G, M, I) must be of the form $(C \times \{1, 2, 3\}, C \times \{4, 6\} \cup B \times \{5\})$ with $C \subseteq J(D)$ and $B \subseteq D$; in addition, $s \leqq x$ has to be hold for all $s \in \bar{C} := J(D) \setminus C$ and $x \in B$. It remains to show that $(\bar{C}, B)$ is a concept of $(J(D), D, \leqq)$ if $(C \times \{1, 2, 3\}, C \times \{4, 6\} \cup B \times \{5\})$ is a compatible and saturated subcontext of (G, M, I). Suppose there is an $s \in C$ with $s \in B'$. Because of $s' \neq D$, we can choose an $x \in D \setminus s'$. By (1a), there exists a $(y, i) \in C \times \{4, 6\} \cup B \times \{5\} \setminus (s, 1)^I$ with $(y, i)^I \supseteq (x, 5)^I$. This implies $y \in B$ which contradicts $s \in B'$. Thus, $\bar{C} = B'$ is shown. Let $x \in D$ with $\bar{C} \subseteq x'$. For each $g \in G \setminus (x, 5)^I$ we have $g \in C \times \{1, 2, 3\}$ and $(x, 5) \in M \setminus g^I$. Hence, by (1a), there exists an $\alpha g \in (C \times \{4, 6\} \cup B \times \{5\}) \setminus g^I$ with $(\alpha g)^I \supseteq (x, 5)^I$. It follows that $\alpha g \in B \times \{5\}$ and $(x, 5)^I = (\alpha(G \setminus (x, 5)^I))^I$. Now (2b) yields $x \in B$ and therefore $B = \bar{C}'$.

Since $(A_1, B_1) \leqq (A_2, B_2) \Leftrightarrow \varrho(A_1, B_1) \geqq \varrho(A_2, B_2)$, it is shown that D is anti-isomorphic to $\mathfrak{S}(G, M, I)$. We apply Lemma 5 to see that (G, M, I) satisfies the assumption of Theorem 4. Obviously, $\{\gamma g \mid g \in G\} \cup \{(M', M)\}$ is an order ideal of $\mathfrak{B}(G, M, I)$. Let $\mu m < (A, B)$ for $m \in M$ and $(A, B) \in \mathfrak{B}(G, M, I) \setminus \{(G, G')\}$. Then $B = \tilde{B} \times \{5\}$ and so $\mu(\wedge \tilde{B}, 5) = (A, B)$. Hence $\{\mu m \mid m \in M\} \cup \{(G, G')\}$ is an order filter of $\mathfrak{B}(G, M, I)$. Finally, Theorem 4 yields $D \cong \mathfrak{C}(\mathfrak{B}(G, M, I))$.

The assumptions of Theorem 9 are fulfilled by distributive dually continuous lattices [1; p. 69] and, in particular, completely distributive complete lattices [1; p. 58]. Since the construction in the proof yields a finite context for a finite lattice D, the assumption of distributivity is unavoidable for this kind of construction. Nevertheless, there are non-distributive lattices $\mathfrak{C}(L)$ for certain infinite complete lattices L where $\mathfrak{C}(L)$ might even be finite. This we show by two examples.

Example 1. Let $\mathbf{Z}$ be the set of all integers and let $\mathbf{E}$ and $\mathbf{O}$ be the set of all even and odd integers, respectively. We define a context (G, M, I) as follows:

$$G := \mathbf{Z}\times\{1,2,3\}, \quad M := \mathbf{Z}\times\{4,5,6\},$$
$$I := \{((x,i),(y,j)) \mid x, y\in\mathbf{Z}, x \leqq y, (i,j)\in\{(1,4),(2,5),(3,6)\}\}.$$

Now, we consider the following subcontexts:

$$(H_1, N_1) := (\mathbf{Z}\times\{1\}\cup\mathbf{E}\times\{2\}\cup\mathbf{O}\times\{3\},\ \mathbf{Z}\times\{4\}\cup\mathbf{O}\times\{5\}\cup\mathbf{E}\times\{6\}),$$
$$(H_2, N_2) := (\mathbf{O}\times\{1\}\cup\mathbf{Z}\times\{2\}\cup\mathbf{E}\times\{3\},\ \mathbf{E}\times\{4\}\cup\mathbf{Z}\times\{5\}\cup\mathbf{O}\times\{6\}),$$
$$(H_3, N_3) := (\mathbf{E}\times\{1\}\cup\mathbf{O}\times\{2\}\cup\mathbf{Z}\times\{3\},\ \mathbf{O}\times\{4\}\cup\mathbf{E}\times\{5\}\cup\mathbf{Z}\times\{6\}).$$

It can be easily checked that (H_i, N_i) is a compatible and saturated subcontext of (G, M, I) for $i=1,2,3$; furthermore, the subcontexts $(\emptyset,\emptyset)$, (H_1, N_1), (H_2, N_2), (H_3, N_3), and (G, M) form a sublattice of $\mathfrak{S}(G, M, I)$ isomorphic to M_3. This shows that $\mathfrak{C}(\mathfrak{B}(G, M, I))$ is not distributive.

Example 2. Let L_n be the complete lattice described by Figure 3.

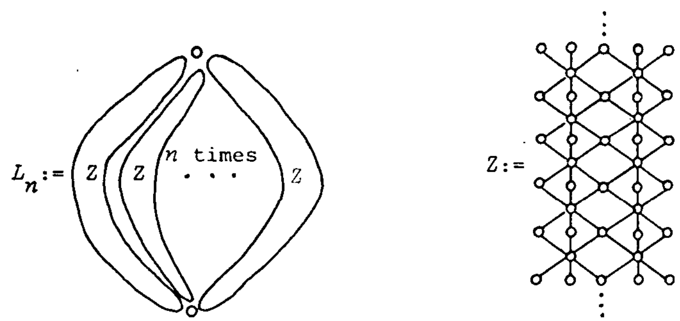
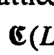


Figure 3

The lattice Z has only two non-trivial complete congruence relations. It follows that $\mathfrak{C}(L_n)\cong\big($  $\big)^n\oplus 1$. For $n\geqq 2$, $\mathfrak{C}(L_n)$ is not distributive. Let us remark that $\mathfrak{C}(L_n)$ is antiisomorphic to the face lattice of an n-cube. The diagram of $\mathfrak{C}(L_2)$ is shown in Figure 4.

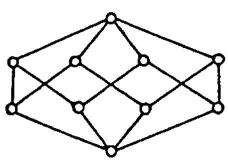


Figure 4

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