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Non-Arguesian configurations in a modular lattice

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To the memory of András Huhn

1. Introduction. In [1] we showed that if L is non-Arguesian, then there exist, in the ideal lattice of L, elements p_α, $\alpha\in 5^{[2]}$, and q_β, $\beta\in 5^{[3]}$, that are related to each other in a manner similar to the ten points and ten lines in a non-Arguesian configuration in a projective plane. In the lattice case, however, each p_α is a point in a plane P_α, and each q_β is a line in the plane Q_β, with all of these planes being intervals in the ideal lattice of L. Actually our construction yielded thirty two intervals $I_\mu=u_\mu/z_\mu$, $\mu\subseteqq 5$, and it was shown that, with at most two exceptions, these intervals are non-degenerate projective planes. The exceptional intervals, $I_\emptyset$ and I_5, were shown to be projective geometries of dimension three or less.

Our present objective is to describe in greater detail how the various intervals I_μ fit together. The notation and terminology of [1] will be in effect. A non-Arguesian perspectivity configuration (or PC), $\mathbf{d}$, will be called *prime* if $\mathbf{d}$ covers $\mathbf{d}_*$ in $\mathbf{PC}(L)$. These PC's and their associated intervals $I_\mu=u_\mu/z_\mu$, $\mu\subseteqq 5$, will be the primary objects of our investigation. To simplify the notation, we write I_i for $I_{\{i\}}$, I_{ij} for $I_{\{ij\}}$, $I_{\neg i}$ for $I_{5\setminus\{i\}}$, etc.

It is easy to see that if, $\emptyset\neq\mu\prec\nu\neq 5$ ($\prec$ means "is covered by"), then the planes I_μ and I_ν are either transposes of each other (possibly equal) or else they are connected by a two dimensional gluing (either loose or tight). Much less is known about the intervals $I_\emptyset$ and I_5. In the examples that have been constructed so far, these too are non-degenerate projective planes, but we do not know if this is always the case. We do however show that, if $I_\emptyset$ is either 2 or 3 dimensional, then it is non-degenerate. By duality, the same holds for I_5.

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There are two further technical conditions that apply to PC's. A PC, **d**, is called *stable* if, whenever two intervals of the form I_i and I_{ij} are transposes of each other, they are equal. This supposed restriction causes no real loss of generality since we will show that, for every non-stable **d**, there exists a stable prime **e** with $\mathbf{e} < \mathbf{d}$. A PC, **d**, is called *Boolean* if the two functions $\mu \to z_\mu$ and $\mu \to u_\mu$ of 2^5 into L are both lattice homomorphisms. Clearly, if **d** is Boolean, then $L' := \bigcup \{I_\mu: \mu \subseteqq 5\}$ is a sublattice of L of finite length. A fundamental result states that, if **d** is both Boolean and stable, $\{I_\mu: \mu \subseteqq 5\}$ consists of 2^r planes where $0 \leqq r \leqq 3$. In this case the length of L' is at most 9, and each simple subdirect factor of L' has length 6 or less.

Much less is known about the case when **d** is stable but not Boolean. We do show however that in this case the twenty planes, I_μ, $2 \leqq |\mu| \leqq 3$, are distinct from each other and from the planes of the form I_i or $I_{\neg i}$. Hopefully this case will be broken down eventually into subcases for which reasonable descriptions can be found.

Some examples of the above cases can be found in [3].

2. The gluings. Throughout this section we work with a fixed prime PC, **d**, in a modular lattice, L.

Lemma 2.1. *For distinct* $i, j \in 5$, $z_i z_j = z_\emptyset$.

Proof. By definition, $z_\emptyset$ is the meet of all the entries in the matrix **d**. Since each diagonal entry is the meet of all entries in its row (or column), it follows that $z_\emptyset$ is the meet of the diagonal entries in **d**. For distinct $i, j, k \in 5$, we have

$$z_i z_j = (d_{*ij} d_{*ik})(d_{*ij} d_{*jk}) \leqq d_{*ik} d_{*jk} = z_k.$$

Consequently $z_i z_j = z_\emptyset$.

Lemma 2.2. *For all* $\mu, \nu \subseteqq 5$,

(1) $z_\mu + z_\nu = z_{\mu \cup \nu}$, *if* $\mu \cap \nu \neq \emptyset$;
(2) $z_\mu z_\nu = z_{\mu \cap \nu}$, *if* $\mu \cup \nu \neq 5$;
(3) $u_\mu + u_\nu = u_{\mu \cup \nu}$, *if* $\mu \cap \nu \neq \emptyset$;
(4) $u_\mu u_\nu = u_{\mu \cap \nu}$, *if* $\mu \cup \nu \neq 5$.

Proof. Statement (1) and its dual (4) are, respectively, parts (2) and (1) of [1; Lemma 5.2]. It therefore suffices to prove (2). Moreover we may assume that $|\mu| \leqq |\nu|$, $\mu \cap \nu \subset \nu$, and $|\mu \cup \nu| = 4$. We consider four cases:

(A) $|\mu| = 2$; $|\nu| = 3$; $|\mu \cap \nu| = 1$. We may assume $\mu = \{i, j\}$ and $\nu = \{i, k, m\}$. Then

$$z_\mu z_\nu = d_{*ij}(d_{*ik} + d_{*im}) = d_{*ii} = z_i = z_{\mu \cap \nu}.$$

(B) $|\mu| = 3$; $|\nu| = 3$; $|\mu \cap \nu| = 2$. We may assume $\mu = \{i, j, k\}$ and $\nu = \{i, j, m\}$.

Then:

$$z_\mu z_\nu = (d_{*ij}+d_{*jk})(d_{*ij}+d_{*jm}) = d_{*ij} = z_{ij} = z_{\mu\cap\nu}.$$

(C) $|\mu|=2$; $|\nu|=2$; $|\mu\cap\nu|=0$. We may assume $\mu=\{i,j\}$ and $\nu=\{k,m\}$. Then

$$z_\mu z_\nu = d_{*ij}d_{*km} \leqq d_{*ij}(d_{*ik}+d_{*im})d_{*km}(d_{*ik}+d_{*jk}) = d_{*ii}d_{*kk} = z_i z_k = z_\emptyset,$$

by 2.1.

(D) $|\mu|=1$; $|\nu|=3$; $|\mu\cap\nu|=0$. We may assume $\mu=\{i\}$ and $\nu=\{j,k,m\}$. Then

$$z_\mu z_\nu = z_i z_{ij} z_{jkm} = z_i z_j = z_\emptyset,$$

by (A) and 2.1.

Lemma 2.3. *For $i\in 5$, the four elements, $d_{ij}u_i$ with $j\neq i$, are four points in general position in the plane I_i.*

Proof. Let i, j, k, m, n be the distinct members of 5. Then, by computing with intervals,

$$d_{ij}u_i/z_i = d_{ij}u_{ikm}/d_{ij}u_{ikm}(d_{ik}+d_{im}) \cong (d_{ij}u_{ikm}+d_{ik}+d_{im})/(d_{ik}+d_{im}) =$$

(by transposition)

$$= (d_{ij}+d_{ik}+d_{im})u_{ikm}/(d_{ik}+d_{im}) = u_{ikm}/(d_{ik}+d_{im}).$$

Now $d_{ik}+d_{im}$ is a line in I_{ikm}, and is therefore covered by u_{ikm}. Thus $z_i \prec d_{ij}u_i$ for each $j\neq i$. To see that the four points are in general position, we compute

$$(d_{ij}u_i+d_{ik}u_i)d_{im}u_i \leqq (d_{ij}+d_{ik})d_{im} = d_{ii} = z_i.$$

Theorem 2.4. *If μ and ν are non-empty proper subsets of 5 with $\mu\prec\nu$, then either*

$$z_\mu \prec u_\mu z_\nu \quad \textit{and} \quad (u_\mu+z_\nu) \prec u_\nu,$$

or

$$z_\mu = u_\mu z_\nu \quad \textit{and} \quad (u_\mu+z_\nu) = u_\nu.$$

Proof. The intervals I_μ and I_ν are of the same length and have comparable upper and lower endpoints. Consequently, $z_\mu \prec u_\mu z_\nu$ if and only if $(u_\mu+z_\nu)\prec u_\nu$, and $z_\mu=u_\mu z_\nu$ holds just in case $(u_\mu+z_\nu)=u_\nu$ is true. Therefore we need only show that for each $\mu\prec\nu$, at least one of the four conditions holds. By duality, we need only consider $|\mu|=1$ or 2.

Assume that $\mu=\{i\}$ and $\nu=\{i,j\}$. By 2.3, $z_i\prec u_i d_{ij}$ whence $u_i z_{ij}$ must equal one of those two elements. Thus $z_\mu\prec u_\mu z_\nu$ or $z_\mu=u_\mu z_\nu$.

Assume now that $\mu=\{i,j\}$ and $\nu=\{i,j,k\}$. By the Main Theorem of [1], the element $q=d_{ij}+d_{ik}$ is a line in the plane I_ν, and qu_μ is a line on the point d_{ij} in I_μ. Now $z_\mu\leqq u_\mu z_\nu \leqq qu_\mu$. This last inequality must be strict since

$$d_{ij}z_{ijk} = d_{ij}(z_{ij}+z_{ik}) = z_{ij}+d_{ij}z_{ik} = z_{ij}+z_i = z_{ij} \prec d_{ij}.$$

Therefore the length of $u_\mu z_\nu / z_\mu$ is at most 2 and one of our relations must again hold.

Lemma 2.5. *For all* $i \in 5$, $u_\emptyset z_i$ *either covers or equals* $z_\emptyset$.

Proof. For distinct i, j, k, m in 5,

$$u_\emptyset z_i = z_i u_i u_{jkm} = z_i u_{jkm}, \quad \text{and} \quad z_\emptyset = z_i z_j = z_i d_{ij}(d_{jk} + d_{jm}) = z_i(d_{jk} + d_{jm}).$$

Since $(d_{jk} + d_{jm}) \prec u_{jkm}$, the conclusion follows.

Lemma 2.6. *Any four of the five elements,* z_i, $i \in 5$, *are independent over* $z_\emptyset$.

Proof. If $i, j, k, m \in 5$ are distinct, then

$$z_i(z_j + z_k + z_m) \leqq z_i z_{jkm} = z_\emptyset.$$

Theorem 2.7. *The following conditions are equivalent:*

(1) *The five elements,* $z_i u_\emptyset$, $i \in 5$, *are points in general position in* $I_\emptyset$;

(2) $I_\emptyset$ *is a non-degenerate 3-space;*

(3) length $(I_\emptyset) = 4$;

(4) $z_\emptyset < z_i u_\emptyset$, *for all* $i \in 5$.

Proof. Now [1; Theorem 5.4] gives us that length $(I_\emptyset) \leqq 4$. Thus (1)$\Rightarrow$(2) and (2)$\Rightarrow$(3) are trivial. If $z_\emptyset = z_i u_\emptyset$, for some $i \in 5$, then $I_\emptyset \cong (z_i + u_\emptyset)/z_i$, a subinterval of a length 3 lattice. Therefore (3)$\Rightarrow$(4). Finally if (4) holds, then the $z_i u_\emptyset$ are five points in $I_\emptyset$ by 2.5. By 2.6, any four of these points are independent. From length $(I_\emptyset) \leqq 4$, we deduce (1).

Corollary 2.8. *If the conditions of the theorem hold, then for each* $i \in 5$, I_i *transposes down onto the interval* $u_\emptyset / z_i u_\emptyset$ *and* $u_\emptyset = \sum (z_s u_\emptyset \colon s \neq i)$.

Theorem 2.9. *If* length $(I_\emptyset) = 3$, *then at least two of the intervals* I_i *transpose down onto* $I_\emptyset$. *Thus in this case as well,* $I_\emptyset$ *is a non-degenerate projective space (i.e. a plane).*

Proof. Since any four of the elements $z_i u_\emptyset$ are independent, at least one out of each four must be $z_\emptyset$. Therefore at least two of the five such elements must be $z_\emptyset$. But this forces, for these i, I_i to transpose down onto $I_\emptyset$ since both intervals are the same length.

Theorem 2.10. *The duals of 2.7, 2.8, and 2.9 also hold. In particular, if* I_5 *is of length 3 or 4, it is a non-degenerate projective space.*

3. Boolean configurations. The definition of a Boolean configuration in Section 1 contains redundancies. We already know, for instance, that whenever $\mu \cap \nu \neq \emptyset$, $z_\mu + z_\nu = z_{\mu \cap \nu}$ holds for any PC. In this section we will reduce the number of conditions needed to be checked in order to show that a PC, **d**, is Boolean. As before, we assume that **d** is a prime PC in a modular lattice, L.

Recall that a subset $U \subseteq L$ is called *distributive* if it generates a distributive sublattice of L. A 3-element subset $U=\{a, b, c\}$ is distributive if either $(a+b)c=$ $=ac+bc$ or $(a+c)(b+c)=ab+c$.

Lemma 3.1. *The following conditions on a prime* PC *are equivalent:*
(1) $z_\mu+z_\nu=z_{\mu\cup\nu}$ *for all* $\mu, \nu\subseteq \mathbf{5}$;
(2) $z_\mu+z_\nu=z_{\mu\cup\nu}$ *for some* $\mu, \nu\neq\emptyset$ *with* $\mu\cap\nu=\emptyset$, *and* $\mu\cup\nu\neq\mathbf{5}$;
(3) $\{z_{ij}, z_{ik}, z_{jk}\}$ *is distributive for all pairwise distinct* $i, j, k\in\mathbf{5}$;
(4) $\{z_{ij}, z_{ik}, z_{jk}\}$ *is distributive for some pairwise distinct* $i, j, k\in\mathbf{5}$;
(5) $\{d_{23}, d_{24}, d_{34}\}$ *is distributive.*

Proof. By 2.2, (1) is equivalent to $z_\mu+z_\nu=z_{\mu\cup\nu}$ with the added condition that μ and ν are disjoint. By noting that $z_2=d_{23}d_{24}$, $z_3=d_{23}d_{34}$, and $z_{23}=d_{23}(d_{24}+d_{34})$, (5) is equivalent to $z_\mu+z_\nu=z_{\mu\cup\nu}$ with $\mu=\{2\}$ and $\nu=\{3\}$. By using the special automorphisms of $\mathbf{PC}(L)$, we get that (5) is equivalent to $z_\mu+z_\nu=z_{\mu\cup\nu}$ with the added constraint that μ and ν are disjoint singletons. This last property and 2.2 however easily imply that $z_\mu=\sum(z_i\colon i\in\mu)$ for all $\mu\subseteq\mathbf{5}$ and this implies (1). Therefore (1) is equivalent to (5).

A priori, (1) implies (2). Conversely, assume that (2) holds with μ or ν a non-singleton. If $\mu=\{i\}$ and $\nu\supseteq\{j, k\}$, then

$$z_{ij} = z_{ij}(z_\mu+z_\nu) = z_i+z_{ij}z_\nu = z_i+z_j.$$

If $\mu=\{i, j\}$ and $\nu=\{k, m\}$, then

$$z_{ijk} = z_{ijk}(z_\mu+z_\nu) = z_{ij}+z_{ijk}z_\nu = z_{ij}+z_k.$$

Thus this case reduces to the previous one. Therefore (1)$\Leftrightarrow$(2)$\Leftrightarrow$(5).

Now for distinct $i, j, k\in\mathbf{5}$, $\{z_{ij}, z_{ik}, z_{jk}\}$ is distributive if and only if

$$z_{ij}(z_{ik}+z_{jk}) = z_{ij}z_{ik}+z_{ij}z_{jk}.$$

The left side of this equation is z_{ij}, and the right side is z_i+z_j. Thus (4) implies (2) and (1) implies (3). This completes the proof.

Lemma 3.2. *For a* PC, **d**, *the following are equivalent:*
(1) $z_\mu z_\nu=z_{\mu\cap\nu}$ *for all* $\mu, \nu\subseteq\mathbf{5}$;
(2) $z_\mu z_\nu=z_{\mu\cap\nu}$ *for some* $\mu, \nu\neq\mathbf{5}$ *with* $\mu\cup\nu=\mathbf{5}$ *and* $\mu\cap\nu\neq\emptyset$;
(3) $\{z_{ijk}, z_{ijm}, z_{ijn}\}$ *is distributive for all distinct* $i, j, k, m\in\mathbf{5}$;
(4) $\{z_{ijk}, z_{ijm}, z_{ijn}\}$ *is distributive for some distinct* $i, j, k, m\in\mathbf{5}$.

Proof. In considering (2), we may assume that $|\mu|\leqq|\nu|$. The possible values for $s=|\mu|$ and $t=|\nu|$ are therefore

$$(s, t) = (4, 4), (3, 4), (3, 3), \text{ and } (2, 4).$$

For each of these four ordered pairs, (s, t), we define:

$(\forall_{st})$ $z_\mu z_\nu = z_{\mu\cap\nu}$ for all μ, ν with $\mu\cup\nu = 5$, $|\mu| = s$, and $|\nu| = t$;

$(\exists_{st})$ $z_\mu z_\nu = z_{\mu\cap\nu}$ for some μ, ν with $\mu\cup\nu = 5$, $|\mu| = s$, and $|\nu| = t$.

We claim that (3), (4), and each of the eight statements above are equivalent to each other.

Assume that i, j, k, m, n are all distinct in 5, and consider the equation

$$(*)\qquad z_{ijkm}z_{ijkn} = z_{ijk}.$$

This can be rewritten as

$$(z_{ijk}+z_{ijm})(z_{ijk}+z_{ijn}) = z_{ijk}$$

and since $z_{ijm}z_{ijn}=z_{ij}\leqq z_{ijk}$, this is equivalent to

$$(**)\qquad \{z_{ijk}, z_{ijm}, z_{ijn}\} \text{ is distributive.}$$

Since $(*)$ is $\{i, j, k\}$-symmetric and $(**)$ is $\{k, m, n\}$-symmetric, it follows that both conditions are invariant under all symmetries of the indices. Therefore (3), (4), $(\forall_{44})$, and $(\exists_{44})$ are equivalent.

If $(\forall_{44})$ holds, then

$$z_{ijk}z_{ijmn} = z_{ijkm}z_{ijkn}z_{ijmn} = z_{ijm}z_{ijn} = z_{ij},$$

and thus $(\forall_{34})$ holds. On the other hand if $(\exists_{34})$ holds, say $z_{ijk}z_{ijmn}=z_{ij}$, then

$$z_{ijkm}z_{ijmn} = z_{im}+z_{ijk}z_{ijmn} = z_{im}+z_{ij} = z_{ijm}$$

and $(\exists_{44})$ holds. Consequently, $(\forall_{44})$ is equivalent to both $(\exists_{34})$ and $(\forall_{34})$.

If $(\forall_{34})$ holds, then

$$z_{ijk}z_{imn} = z_{ijk}z_{ikmn}z_{ijmn} = z_{ij}z_{ik} = z_{ij},$$

and thus $(\forall_{33})$ holds. On the other hand if $(\exists_{33})$ holds, say $z_{ijk}z_{imn}=z_i$, then

$$z_{ijk}z_{ijmn} = z_{ij}+z_{ijk}z_{imn} = z_{ij}+z_i = z_{ij}$$

and $(\exists_{33})$ holds. Consequently, $(\forall_{44})$ is equivalent to both $(\exists_{33})$ and $(\forall_{33})$.

A similar argument shows that each of the statements $(\exists_{24})$ and $(\forall_{24})$ is equivalent to $(\forall_{44})$. Therefore (2), (3), and (4) are equivalent.

To obtain (2) implies (1) we need only consider complementary subsets of 5. Assuming (2), we obtain

$$z_i z_{jkmn} = z_{ij}z_{ik}z_{jkmn} = z_j z_k = z_\emptyset, \quad\text{and}\quad z_{ij}z_{kmn} = z_{ijk}z_{ijm}z_{kmn} = z_k z_m = z_\emptyset.$$

Thus (2) implies (1) and the proof is complete.

Corollary 3.3. *If, for some non-empty proper subsets,* $\mu\subset\nu\subseteqq 5$, $z_\mu=z_\nu$, *then* **d** *is Boolean.*

Proof. The inclusion $\mu\subset\nu$ implies that $u_\mu\leqq u_\nu$, and since I_μ and I_ν are both projective planes we get equality here as well. Now let $\varkappa=\nu\setminus\mu$ and $\lambda=5\setminus\varkappa=$ $=(5\setminus\nu)\cup\mu$. We compute

$$z_{\mu\cup\varkappa}=z_\nu=z_\nu+z_\varkappa=z_\mu+z_\varkappa,\quad \text{and}\quad z_{\nu\cap\lambda}=z_\mu=z_\mu z_\lambda=z_\nu z_\lambda.$$

By Lemmas 3.1, 3.2 and their duals, it follows that **d** is Boolean.

The above argument works in general to produce:

Theorem 3.4. *A* PC, **d**, *is Boolean if and only if, for some distinct* $i,j\in 5$,

$$z_i+z_j=z_{ij},\quad \textit{and}\quad z_{\neg i}+z_{\neg j}=z_{\neg ij},\quad \textit{and}\quad u_i+u_j=u_{ij},\quad \textit{and}\quad u_{\neg i}+u_{\neg j}=u_{\neg ij}.$$

4. Stable configurations. We still assume that **d** is a prime PC in a modular lattice, L.

Theorem 4.1. *Let* **d** *be prime and stable. Then* **d** *is either Boolean or satisfies*

(***) *For all* $\mu,\nu\subseteqq 5$, *if* $\emptyset\subset\mu\prec\nu\subset 5$, *then* $z_\mu\prec u_\mu z_\nu$ *and* $u_\mu+z_\nu\prec u_\nu$.

Proof. Let **d** be stable, and take $\emptyset\subset\mu\prec\nu\subset 5$. By 2.4 we must have I_μ transposing up to I_ν, or $z_\mu\prec z_\nu u_\mu$ and $u_\mu+z_\nu\prec u_\nu$, If the first property holds, then let $\{i\}=\nu\setminus\mu$ and take $j\in\mu$. Now

$$u_j z_{ij}=u_j z_{ij} u_\mu z_\nu=u_j z_\mu z_{ij}=z_i.$$

Since **d** is stable, this implies $I_{ij}=I_i$, and hence **d** is Boolean by 3.3.

Lemma 4.2. *Let* **d** *be a prime* PC. *For any* $x\in d_{02}/z_0$, *there exists a unique* PC, **e**, *such that*

$$e_{01}=d_{01}(x+d_{12}),\quad e_{02}=x,\quad e_{12}=d_{12}(x+d_{01}),$$

and for $\{i,j\}=\{0,1\}$ *and* $k\in\{3,4\}$,

$$e_{ik}=d_{ik}(d_{jk}+e_{01}).$$

Moreover if x *is not less than or equal to* z_{02}, *then* **e** *is non-Arguesian.*

Proof. The uniqueness of **e** is obvious for, by [1; Theorem 3.2], every PC in L is completely determined by the elements listed above. Thus we are left with showing the existence. This however also follows from [1; Lemma 2.4] and the quoted theorem. If **e** were Arguesian, then $\mathbf{e}=\mathbf{e}_*$, and

$$x=e_{02}=e_{*02}\leqq d_{*02}=z_{02}.$$

Lemma 4.3. *If* **d** *is not stable, then there exists a prime* PC, $\mathbf{e}<\mathbf{d}$ *that is both stable and Boolean.*

Proof. If **d** is a prime PC that is not stable then there exists $i, j \in \mathbf{5}$ such that I_i transposes up to I_{ij} but is not equal to I_{ij}. Thus for these i and j we have

$$z_{ij}u_j = z_j, \quad z_{ij}+u_i = u_j, \quad \text{and} \quad z_i < z_j.$$

By using the special automorphisms of $\mathbf{PC}(L)$, we may assume that $i=0$, and $j=2$. Using $x=d_{02}u_0$ in the previous lemma, we obtain a non-Arguesian PC, **e**, with $\mathbf{e}<\mathbf{d}$ and $e_{02}=d_{02}u_0$. To see that **e** is prime, we note that $z_0=z_0(\mathbf{d})\leqq z_{02}(\mathbf{e})<e_{02}$ (and that $z_0 \prec e_{02}$). Therefore $z_0=z_0(\mathbf{e})=z_{02}(\mathbf{e})\prec e_{02}$.

That **e** is Boolean follows from 3.3 and the fact that $z_0(\mathbf{e})=z_{02}(\mathbf{e})$, but **e** may not be stable. What this **e** has done is replace the transpose $I_0(\mathbf{d})$ up to $I_{02}(\mathbf{d})$ with the equality $I_0(\mathbf{e})=I_{02}(\mathbf{e})$. But for all $i\in\mathbf{5}$, direct calculations show that

$$z_i(\mathbf{d}) = z_i(\mathbf{e}) \leqq z_{ij}(\mathbf{e}) \leqq z_{ij}(\mathbf{d}).$$

Therefore this **e** preserves all equalities of the form $I_i(\mathbf{d})=I_{ij}(\mathbf{d})$. This means that after finitely many steps (at most 5^2) all transpositions are replaced by equalities and the resultant PC is both Boolean and stable.

Thus if L is a non-Arguesian modular lattice, we can find, in the lattice of ideals of L, a prime (non-Arguesian) PC, **d**. If **d** is stable, then **d** is either Boolean or satisfies ($***$). If **d** is not stable, we can find a smaller PC, **e**, that is both stable and Boolean. Therefore every non-Arguesian variety of modular lattices contains a non-Arguesian lattice with a stable (non-Arguesian) PC. The Boolean case has a nice finite solution which we present in the next section. By [3], there exists infinitely many distinct stable PC's satisfying ($***$), and these authors at least have found no classification of them. Our only general result is the following.

Theorem 4.4. *Let* **d** *be a stable non-Boolean* PC. *Then the twenty planes,* I_μ, $2\leqq|\mu|\leqq 3$, *are distinct from each other, and from the planes,* I_i *and* $I_{\neg i}$, $i\in\mathbf{5}$.

Proof. Let $\mu\neq\nu\subseteq\mathbf{5}$ satisfy:

$$1 \leqq |\mu|, |\nu| \leqq 4, \quad \min\{|\mu|, |\nu|\} \leqq 3, \quad \text{and} \quad \max\{|\mu|, |\nu|\} \geqq 2.$$

We wish to show that the assumption, $z_\mu=z_\nu$, leads to a contradiction. We obtain this contradiction by producing a covering pair of subsets, $\varkappa\prec\lambda$, with $z_\varkappa=z_\lambda$, and invoking ($***$).

If $\mu\cap\nu\neq\emptyset$, then $z_\mu+z_\nu=z_{\mu\cup\nu}$, and we may choose $\varkappa$ to be the set of smallest cardinality and λ to be any cover contained in $\mu\cup\nu$. This produces our contradiction on ($***$). Therefore we may conclude that

$$[0] \quad \mu\cap\nu = \emptyset.$$

Therefore there exists $i\in\mu\setminus\nu$. But now we have for all $i\in\mu$

$$z_{\nu\cup\{i\}} = z_{\nu\cup\{i\}}+z_\nu = z_{\nu\cup\{i\}}+z_\mu = z_{\mu\cup\nu}.$$

To avoid conflict with $(***)$, we must have:

$$i\in\mu \text{ implies } [1] \quad \mu\cup\nu=\mathbf{5} \text{ and } 4\leqq \nu\cup|\{i\}|, \text{ or}$$
$$[2] \quad \mu\cup\nu=\nu\cup\{i\}.$$

We also have $j\in\nu\setminus\mu$ and the trick above can be applied again to produce

$$j\in\nu \text{ implies } [3] \quad \mu\cup\nu=\mathbf{5} \text{ and } 4\leqq |\mu\cup\{j\}|, \text{ or}$$
$$[4] \quad \mu\cup\nu=\mu\cup\{j\}.$$

Now [0] makes [2] equivalent to $\mu=\{i\}$, and [4] equivalent to $\nu=\{j\}$. Thus [1] and [4] are incompatible as well as [2] and [3]. Our initial assumptions deny the conjunction of [2] and [4], so we must have [1] and [3]. But this forces $3\leqq|\nu|$ and $|\mu|$ which contradicts [0]. This concludes the proof.

5. Stable Boolean configurations. Throughout this section, **d** will be a prime, Boolean, and stable PC in a modular lattice, L. The lattice homomorphisms,

$$z, u\colon \mathbf{2^5} \to L,$$

produce Boolean congruences on $\mathbf{2}^5$ which are, of course, determined by their respective ideals, $\mathrm{Id}(z)$ and $\mathrm{Id}(u)$, of subsets congruent to $\emptyset$. Now $\{i\}\in\mathrm{Id}(z)\Leftrightarrow z_i=z_\emptyset\Leftrightarrow$for all $j\neq i$, $z_{ij}=z_j\Leftrightarrow$for all $j\neq i$, $u_{ij}=u_j\Leftrightarrow u_i=u_\emptyset\Leftrightarrow\{i\}\in\mathrm{Id}(u)$. Therefore $\mathrm{Id}(z)=\mathrm{Id}(u)$, and by factoring out this ideal we produce, for some r with $0\leqq r\leqq 5$, lattice embeddings

$$z', u'\colon \mathbf{2^r} \to L.$$

Our first result shows that this r can be further restricted.

Lemma 5.1. *If* **d** *is Boolean and stable, then the set* $\{I_\mu\colon \mu\subseteqq\mathbf{5}\}$ *consists of* 2^r *planes for some* r, $0\leqq r\leqq 3$.

Proof. Let i, j, k, m, n be distinct members of $\mathbf{5}$, and assume that for all $s\neq n$, $z_{sn}>z_n$. From 2.3 and stability, this implies that for all $s\neq n$, $u_n z_{sn}=u_n d_{sn}$. 2.3 also says that $\{u_n d_{sn}\colon s\neq n\}$ are points in general position in I_n. But **d** is Boolean, and therefore

$$u_i d_{in} \leqq u_i z_{in}(u_j z_{jn}+u_k z_{kn}+u_m z_{mn}) \leqq z_{in} z_{jkmn} = z_n.$$

This is a contradiction.

Thus for every $n\in\mathbf{5}$, there exists an $s\neq n$ such that $z_{sn}=z_n$. Again since **d** is Boolean this implies that for every $n\in\mathbf{5}$, there exists an $s\neq n$ such that $z_s=z_\emptyset$. Elementary counting now produces two distinct $s\in\mathbf{5}$ with $z_s=z_\emptyset$.

We may therefore replace $\mathbf{5}$ by $\mathbf{r}$ for $0\leqq r\leqq 3$, and assume that we have lattice monomorphisms,

$$z, u\colon \mathbf{2^r} \to L,$$

that satisfy:

(1) $I_\mu := u_\mu/z_\mu$ is a non-degenerate projective plane for all $\mu \subseteq \mathbf{r}$;

(2) For all $\emptyset \subseteq \mu \prec \nu \subseteq \mathbf{r}$, $z_\mu u_\nu \prec z_\nu$ and $u_\mu + z_\nu \prec u_\nu$.

We define $L' := \bigcup \{I_\mu : \mu \subseteq \mathbf{r}\}$. Clearly L' is a sublattice of L of finite length.

Lemma 5.2. *L' is simple if and only if for all $i \in \mathbf{r}$, $z_i \leqq u_{\neg i}$.*

Proof. If our condition fails, then, for the offending $i \in \mathbf{r}$, L' is the disjoint union of the filter, $\uparrow z_i$ and the ideal, $\downarrow u_{\neg i}$. Thus L' is not simple.

Conversely, assume the condition holds. We proceed by induction on r. If $r=0$, then L' is a non-degenerate projective plane and hence simple. If $0<r\leqq 3$, take a prime quotient q/p in L', and let θ be the congruence it generates. Since $L' = \uparrow z_i \cup \downarrow u_{\neg i}$ and $z_i \leqq u_{\neg i}$, we must have this quotient in $\uparrow z_i$ or in $\downarrow u_{\neg i}$. By induction, θ collapses either the filter or the ideal. Since $z_i < u_{\neg i}$, induction applies also to the other part and θ collapses all of L'. Therefore L' is simple.

Theorem 5.3. *Suppose **V** is a variety of modular lattices and assume that there exists a Boolean, prime* PC *in some member of **V**. Then there exists in **V** a simple non-Arguesian lattice of length $3+r$, with $0\leqq r\leqq 3$, and a Boolean, stable, prime* PC, $\mathbf{d}$, *in L with the following properties:*

(1) *L is generated by $\{d_{ij}: i \neq j \text{ in } \mathbf{5}\}$;*

(2) *The set $\{I_\mu: \mu \subseteq \mathbf{5}\}$ consists of precisely 2^r planes.*

Proof. By 4.3. there exists in some member L of **V** a PC, $\mathbf{d}$, that is prime, Boolean, and stable. By 5.1, the set $\{I_\mu: \mu \subseteq \mathbf{5}\}$ consists of 2^r distinct planes for some r, with $0\leqq r\leqq 3$. We may assume without loss of generality that L is generated by the PC and is therefore the union of the planes I_μ.

Since L is obviously of finite length, we may assume that its length is as small as possible. We claim that in this case L is simple. To see this, we consider a homomorphism $\varphi: L \to S$, where S is simple and φ does not identify d_{01} and d_{*01}. Clearly $\varphi(\mathbf{d})$ is a (non-Arguesian) PC in S and in fact also prime and Boolean (since $\varphi(z_\mu(\mathbf{d})) = z_\mu(\varphi(\mathbf{d}))$ and similarly for the u's). The length of S therefore cannot be less than that of L. This makes φ an isomorphism and L simple.

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