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Idempotent algebras with restrictions on subalgebras

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Dedicated to the memory of András P. Huhn

We study some consequences of an interesting property of idempotent algebras, namely, that their direct squares have enough reduced subalgebras in the following sense: For arbitrary idempotent algebra $\mathfrak{A}$, every reduced subalgebra $\mathfrak{B}$ of any finite power $\mathfrak{A}^n$ $(n>1)$ of $\mathfrak{A}$ produces reduced subalgebras in $\mathfrak{A}^2$, unless $\mathfrak{B}$ is a subdirect product of pairwise isomorphic, simple, locally affine subalgebras of $\mathfrak{A}$ (see Theorem 1.1). Section 1 contains also some applications. It follows that an idempotent algebra is locally quasi-primal if and only if it has no nonsingleton, locally affine subalgebras; and its square has no reduced subalgebras (Corollary 1.2). More generally, an idempotent algebra is locally para-primal if and only if its square has no reduced subalgebras (Corollary 1.3). For comparison, recall Rosenberg's Theorem [9] implying that in order to verify a finite algebra $\mathfrak{A}=(A;\ F)$ to be primal, one has to exclude the existence of certain types of subalgebras in $\mathfrak{A}^n$ with n running up to $n=|A|$.

In Section 2 we determine, up to local term equivalence, all idempotent algebras (of cardinality greater than 2) having no nonsingleton proper subalgebras (Theorem 2.1). They turn out to fall into three types: (a) locally quasi-primal algebras, (b) algebras locally term equivalent to the full idempotent reduct of a simple module, and (c) algebras whose clones of local term operations form a family of disjoint descending $(\omega+1)$-chains; these $(\omega+1)$-chains are related to "higher dimensional crosses" among the subalgebras of finite powers of the corresponding algebras.

This description is applied in Section 3 to finite algebras with minimal clones. It is proved that a finite algebra $(A;p)$ with p a Mal'tsev operation has a minimal clone if and only if p arises from an elementary Abelian group on A (Theorem 3.1); furthermore, a finite idempotent groupoid with minimal clone is term equivalent

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to an algebra of this form if and only if it has a minimal nonsingleton subgroupoid of cardinality greater than 2 (Theorem 3.2).

The investigations in Sections 1 and 2 were partly inspired by the problem of determining the maximal subclones of the clone $\mathcal{I}_A$ of all idempotent operations on a finite set A, which was raised by I. G. Rosenberg during the Séminaire de Mathématiques Supérieures on "Universal algebra and relations" (Montreal, 1984), and was solved independently by several participants. Here the solution is derived from Theorem 1.1 (see Corollary 1.4 in the case when A is finite).

0. Preliminaries

For a nonempty set A, the clone of all operations on A will be denoted by $\mathcal{O}_A$, and for $n \geqq 1$, $\mathcal{O}_A^{(n)}$ will designate the set of n-ary operations on A. We write $|A|$ for the cardinality of A. Recall that an operation $f \in \mathcal{O}_A^{(n)}$ is said to *preserve* a subset B of A^k $(k \geqq 1)$ iff B is a subuniverse of the algebra $(A; f)^k$. For arbitrary mapping $g\colon A_1 \times \ldots \times A_n \to A_{n+1}$ $(n \geqq 1,\ A_1, \ldots, A_{n+1} \subseteq A)$ we define a subset $g_\square$ of A^{n+1} by

$$g_\square = \{(g(x_1, \ldots, x_n), x_1, \ldots, x_n)\colon\ x_1 \in A_1, \ldots, x_n \in A_n\}.$$

For arbitrary operations $f, f' \in \mathcal{O}_A$, f is said to *commute* with f' iff f preserves $(f')_\square$. It is easy to see that the commutativity of operations is a symmetric relation. If $\pi\colon B \to C$ is a bijection [or $\pi\colon A \to A$ is a permutation], then $\pi_\square$ will also be called a bijection [or permutation, respectively].

We now introduce some notation for constructions that will be used to produce subuniverses from subuniverses. Let B be a subset of A^k $(k \geqq 1)$. We will write $\mathbf{k}$ for the set $\{1, \ldots, k\}$ indexing the components of B. For an l-tuple $(i_1, \ldots, i_l) \in \mathbf{k}$ we define the projection of B onto its components $i_1, \ldots, i_l$ by

$$\mathrm{pr}_{i_1, \ldots, i_l} B = \{(x_{i_1}, \ldots, x_{i_l})\colon\ (x_1, \ldots, x_k) \in B\}.$$

In particular, if $l = k$ and $i_1, \ldots, i_k$ is a permutation of $1, \ldots, k$, then $\mathrm{pr}_{i_1, \ldots, i_k} B$ arises from B by rearranging the components. The property that, up to the order of their components, the subsets B and B' of A^k coincide, will be denoted by $B \approx B'$. For a nonvoid subset I of $\mathbf{k}$ with $I = \{i_1, \ldots, i_l\}$, $i_1 < \ldots < i_l$, we write $\mathrm{pr}_I B$ for $\mathrm{pr}_{i_1, \ldots, i_l} B$. The symbol $B \leqq B_1 \times \ldots \times B_k$ will be used to designate that $\mathrm{pr}_i B = B_i$ for all $i \in \mathbf{k}$. For $B \leqq B_1 \times \ldots \times B_k$ and for arbitrary bijections $\pi_i\colon B_i \to C_i$ $(C_i \subseteq A,\ i \in \mathbf{k})$ we set

$$B[\pi_1, \ldots, \pi_k] = \{(x_1\pi_1, \ldots, x_k\pi_k)\colon\ (x_1, \ldots, x_k) \in B\}.$$

If $1 \leqq l \leqq k$ and $(a_{l+1}, \ldots, a_k) \in A^{k-l}$, then we define the subset of A^l arising from B by "substituting the constants $a_{l+1}, \ldots, a_k$ for the $(l+1)$-st up to the k-th components" as follows:

$$B(x_1, \dots, x_l, a_{l+1}, \dots, a_k) = \{(x_1, \dots, x_l)\in A^l\colon (x_1, \dots, x_l, a_{l+1}, \dots, a_k)\in B\}.$$

Let $\mathscr{C}$ be a clone on A. We say that an operation $g\in\mathcal{O}_A^{(n)}$ *can be interpolated* by operations from $\mathscr{C}$ iff for every finite subset S of A^n there exists an operation $f\in\mathscr{C}$ agreeing with g on S. The clone $\mathscr{C}$ is called *locally closed* iff it contains every operation that can be interpolated by its members. For an algebra $\mathfrak{A}=(A; F)$ the operations that can be interpolated by term [polynomial] operations of $\mathfrak{A}$ are called *local term [polynomial] operations* of $\mathfrak{A}$. It is easy to see that the local term [polynomial] operations of $\mathfrak{A}$ form a locally closed clone, which will be denoted by $\mathscr{T}_{\mathrm{loc}}(\mathfrak{A})$ $[\mathscr{P}_{\mathrm{loc}}(\mathfrak{A})]$; moreover, $\mathscr{T}_{\mathrm{loc}}(\mathfrak{A})$ $[\mathscr{P}_{\mathrm{loc}}(\mathfrak{A})]$ is the least locally closed clone containing the clone $\mathscr{T}(\mathfrak{A})$ [resp., $\mathscr{P}(\mathfrak{A})$] of term [polynomial] operations of $\mathfrak{A}$. Clearly, if $\mathfrak{A}$ is finite, then $\mathscr{T}_{\mathrm{loc}}(\mathfrak{A})=\mathscr{T}(\mathfrak{A})$ and $\mathscr{P}_{\mathrm{loc}}(\mathfrak{A})=\mathscr{P}(\mathfrak{A})$. Two algebras with the same universe are called *term equivalent [locally term equivalent]* iff their clones of term operations [local term operations] coincide. It is well known that the algebras are determined, up to local term equivalence, by the subuniverses of their finite powers in the following sense: For an algebra $\mathfrak{A}=(A; F)$ and for $f\in\mathcal{O}_A$ we have $f\in\mathscr{T}_{\mathrm{loc}}(\mathfrak{A})$ if and only if f preserves every subuniverse of each finite power of $\mathfrak{A}$.

Let $\mathbf{A}=(A; +, -, 0)$ be an Abelian group. An algebra $\mathfrak{A}=(A; F)$ is said to be *affine [locally affine]* with respect to $\mathbf{A}$ iff the Mal'tsev operation $x-y+z$ is a term operation [resp., local term operation] of $\mathfrak{A}$, and every operation (hence every local term operation) of $\mathfrak{A}$ commutes with $x-y+z$. The Abelian group $\mathbf{A}$ is called *elementary* (or more precisely, an *elementary Abelian q-group*) iff for some prime q, $qa=0$ holds for all $a\in A$.

1. Reduced subalgebras of finite powers of idempotent algebras

Let A be a nonempty set. A subset B of A^k $(k\geq 1)$ is said to be *directly indecomposable* iff $B\not\approx(\mathrm{pr}_I B)\times(\mathrm{pr}_{\bar I} B)$ holds for all partitions $\{I, \bar I\}$ of $\mathbf{k}$, and B is *reduced* iff it is directly indecomposable and no projection $\mathrm{pr}_{i,j}B$ $(1\leq i<j\leq k)$ of B is a bijection. A subalgebra of some finite power of an algebra is called reduced iff its universe has this property. The *size* of B is the cardinal $\max\{|\mathrm{pr}_i B|\colon 1\leq i\leq k\}$.

The main result of this section is

1.1. Theorem. *Let $\mathfrak{A}=(A; F)$ be an idempotent algebra. For any $n\geq 2$ and for arbitrary reduced subuniverse $B\leq B_1\times\dots\times B_n$ $(B_1, \dots, B_n\subseteq A)$ of $\mathfrak{A}^n$ one of the following conditions holds:*

(1.1.1) *$\mathfrak{A}^2$ has a reduced subuniverse of the same size as B; or*

(1.1.2) *$\mathfrak{B}_i=(B_i; F)$ $(1\leq i\leq n)$ are isomorphic locally affine subalgebras of $\mathfrak{A}$, moreover, there exist a division ring K and a vector space ${}_K\mathbf{B}_1=(B_1; +, K)$ such* *that* $\mathfrak{B}_1$ *is locally term equivalent to the full idempotent reduct of the module* ${}_{(\mathrm{End}_K \mathbf{B}_1)}\mathbf{B}_1$. *For arbitrary isomorphisms* $\pi_i\colon \mathfrak{B}_i \to \mathfrak{B}_1$ $(1 \leqq i \leqq n)$, *the subuniverse* $B[\pi_1, \ldots, \pi_n]$ *of* $\mathfrak{A}^n$ *has the form*

(1) $$B[\pi_1, \ldots, \pi_n] \approx$$

$$\approx \{(y_1, \ldots, y_{u-1}, g_u(y_1, \ldots, y_{u-1}), \ldots, g_n(y_1, \ldots, y_{u-1}))\colon y_1, \ldots, y_{u-1} \in B_1\}$$

for some $u \in \mathbf{n}$ $(u \geqq 2)$ *and for some operations* $g_u, \ldots, g_n \in \mathscr{P}({}_K\mathbf{B}_1)$.

This is an extension of Theorem 4.3 (see also the remark following its proof) in [13] to not necessarily finite algebras. Before sketching the proof, which is quite similar to that of the finite version, we present several applications.

Theorem 1.1 yields nice criteria for idempotent algebras to be locally quasi-primal or para-primal, respectively. Recall that an algebra $\mathfrak{A} = (A; F)$ is called *locally quasi-primal,* or *quasi-primal* if $\mathfrak{A}$ is finite, iff every operation preserving all isomorphisms between subalgebras of $\mathfrak{A}$ is a local term operation of $\mathfrak{A}$ (A. F. Pixley [5], [6]). Equivalently, $\mathfrak{A}$ is locally quasi-primal iff $\mathfrak{A}^k$ has no reduced subuniverses for $k \geqq 2$ (see P. H. Krauss [2]). Combining the latter characterization with Theorem 1.1 we immediately get

1.2. Corollary. *An idempotent algebra* $\mathfrak{A} = (A; F)$ *is locally quasi-primal if and only if* $\mathfrak{A}$ *has no nonsingleton, locally affine subalgebras and* $\mathfrak{A}^2$ *has no reduced subalgebras.*

An algebra $\mathfrak{A} = (A; F)$ is called *locally para-primal* iff for every $k \geqq 1$, for every subuniverse B of $\mathfrak{A}^k$, and for every set $I \subseteq \mathbf{k}$ which is minimal with respect to the property that the projection $B \to \mathrm{pr}_I B$ is one-to-one, the equality $\mathrm{pr}_I B = \prod_{i \in I} \mathrm{pr}_i B$ holds (see [14]). This concept arises from the definition of *para-primal* algebras, introduced by D. M. Clark and P. H. Krauss [1], by simply omitting the requirement that $\mathfrak{A}$ be finite. Thus the finite locally para-primal algebras are exactly the para-primal algebras. It is easy to see that every locally quasi-primal algebra is locally para-primal.

1.3. Corollary. *An idempotent algebra* $\mathfrak{A} = (A; F)$ *is locally para-primal if and only if* $\mathfrak{A}^2$ *has no reduced subalgebras.*

Proof. The necessity is an immediate consequence of the definition of local para-primality. Conversely, if $\mathfrak{A}^2$ has no reduced subalgebras, then by Theorem 1.1 every reduced subuniverse B of any finite power of $\mathfrak{A}$ is as described in (1.1.2). Now it can be checked without difficulty that B satisfies the condition required in the definition of local para-primality. This implies that the same holds also for arbitrary subuniverses of finite powers of $\mathfrak{A}$. Hence $\mathfrak{A}$ is locally para-primal.

We note that for finite algebras $\mathfrak{A}$ Corollaries 1.2 and 1.3 can be strengthened further; see Corollaries 4.5 and 4.13 in [13].

Let $\mathcal{I}_A$ denote the clone of idempotent operations on A. Clearly, $\mathcal{I}_A$ is locally closed. Applying Theorem 1.1 we can determine the locally closed clones sitting "high up" in the lattice of locally closed subclones of $\mathcal{I}_A$ (in the terminology of [11] these clones, or more precisely the corresponding relations, form a generic system for $\mathcal{I}_A$, which is as irredundant as possible). We call the subsets

$$X^{a_1,a_2}=(A\times\{a_2\})\cup(\{a_1\}\times A)\quad(a_1,a_2\in A)$$

of A^2 *crosses*, and we write X^a for $X^{a,a}$ $(a\in A)$. For a subset B of some power of A the clone of all operations on A preserving B is denoted by $\mathrm{Pol}_A\{B\}$.

1.4. Corollary. *Let A be a set with $|A|\geqq 2$. Every locally closed, proper subclone of $\mathcal{I}_A$ is contained in one of the clones $\mathcal{I}_A\cap\mathrm{Pol}_A\{B\}$ where*

(1.4.1) $B\subset A$, $|B|\geqq 2$; *or*

$(1.4.2)_1$ $B=\pi_\square$ *for some permutation π of A with at most one fixed point such that all nontrivial cycles of π are of the same length q for some prime q; or*

$(1.4.2)_2$ $B=\pi_\square$ *for some permutation π of A with at most one fixed point such that all nontrivial cycles of π are of infinite length; or*

$(1.4.3)_1$ $B=X^a$ *for some $a\in A$; or*

$(1.4.3)_2$ $B=X^{a_1,a_2}$ *with $\{a_1,a_2\}=A$.*

These clones are locally closed, proper subclones of $\mathcal{I}_A$. The maximal locally closed subclones of $\mathcal{I}_A$ are exactly those of types (1.4.1), $(1.4.2)_1$, $(1.4.3)_1$ *and* $(1.4.3)_2$.

Proof. To prove the first claim let $\mathcal{C}$ be a proper subclone of $\mathcal{I}_A$, and assume $\mathcal{C}$ is locally closed. Then $\mathcal{C}=\mathcal{T}_{\mathrm{loc}}(\mathfrak{A})$ for the idempotent algebra $\mathfrak{A}=(A;\mathcal{C})$. If $\mathfrak{A}$ has a proper subuniverse B with $|B|\geqq 2$, then $\mathcal{C}\subseteq\mathcal{I}_A\cap\mathrm{Pol}_A\{B\}$ with B of type (1.4.1), and we are done. Therefore we suppose from now on that the singletons and A are the only subuniverses of $\mathfrak{A}$. Since $\mathcal{C}\subset\mathcal{I}_A$, if $\mathfrak{A}$ is locally quasi-primal, then $\mathfrak{A}$ has a nonidentity automorphism σ. As the set of fixed points of each automorphism of $\mathfrak{A}$ is a subuniverse of $\mathfrak{A}$, it follows that every nonidentity power of σ has at most one fixed point. Thus, either $\pi=\sigma$ is of type $(1.4.2)_2$, or some power π of σ is of type $(1.4.2)_1$, implying in both cases that $\mathcal{C}\subseteq\mathcal{I}_A\cap\mathrm{Pol}_A\{\pi_\square\}$. If $\mathfrak{A}$ is locally para-primal but not locally quasi-primal, then by Corollary 1.3 and Theorem 1.1 $\mathfrak{A}$ is locally term equivalent to the full idempotent reduct of the module $_{(\mathrm{End}_K\mathbf{A})}\mathbf{A}$ for some vector space $_K\mathbf{A}$ over a division ring K. Therefore every translation $x\pi=x+a$ with $a\neq 0$ is an automorphism of $\mathfrak{A}$ which is of type $(1.4.2)_1$ or $(1.4.2)_2$ according to whether the characteristic of K is prime or zero. Hence we conclude again that $\mathcal{C}\subseteq\mathcal{I}_A\cap\mathrm{Pol}_A\{\pi_\square\}$. Finally, if $\mathfrak{A}$ is not locally para-primal, then by Corollary 1.3 $\mathfrak{A}^2$ has a reduced subuniverse B. Obviously, $B\lneqq A\times A$. Since the sets $B(a,x)$ and $B(x,a)$ are nonempty subuniverses of $\mathfrak{A}$ for all $a\in A$, it follows that B is a cross;

say $B=X^{a_1,a_2}$. Clearly, $X^{a_2,a_1}\approx X^{a_1,a_2}$, hence X^{a_2,a_1} is also a subuniverse of $\mathfrak{A}^2$. In case $a_1\neq a_2$ the set $\mathrm{pr}_1(X^{a_1,a_2}\cap X^{a_2,a_1})=\{a_1, a_2\}$ is a subuniverse of A, hence $a_1=a_2$ or $|A|=2$. Thus B is of type $(1.4.3)_1$ or $(1.4.3)_2$, and obviously $\mathscr{C}\subseteq\mathscr{I}_A\cap\cap\mathrm{Pol}_A\{B\}$. This concludes the proof of the first claim.

As regards the second assertion, it is straightforward to check that the clones $\mathscr{I}_A\cap\mathrm{Pol}_A\{B\}$ listed in the corollary are locally closed and are properly contained in $\mathscr{I}_A$. A case-by-case analysis shows also that, except for the obvious coincidences implied by the equalities

$$\mathrm{Pol}_A\{\pi_\square\}=\mathrm{Pol}_A\{(\pi^k)_\square\} \quad \text{if } \pi \text{ is as in } (1.4.2)_1, \quad 1<k<q,$$

and

$$\mathrm{Pol}_A\{X^{a_1,a_2}\}=\mathrm{Pol}_A\{X^{a_2,a_1}\} \quad (a_1, a_2\in A),$$

any two clones listed in the corollary and such that not both are of type $(1.4.2)_2$ are incomparable. Therefore the clones $\mathscr{I}_A\cap\mathrm{Pol}_A\{B\}$ with B of type (1.4.1), $(1.4.2)_1$, $(1.4.3)_1$ or $(1.4.3)_2$ are indeed maximal among the locally closed, proper subclones of $\mathscr{I}_A$. Finally, if $B=\pi_\square$ is of type $(1.4.2)_2$, then $\mathscr{I}_A\cap\mathrm{Pol}_A\{\pi_\square\}$ is not maximal, since

$$\mathscr{I}_A\cap\mathrm{Pol}_A\{\pi_\square\}\subset\mathscr{I}_A\cap\mathrm{Pol}_A\{(\pi^l)_\square\} \quad (\subset\mathscr{I}_A) \quad \text{for every integer } l>1.$$

The proof of the corollary is complete.

We now sketch the proof of Theorem 1.1. The first lemma is the same as Lemma 4.4 in [13], excepting that the algebra is not assumed to be finite. As the proof carries over without change to this more general situation, we do not go into the details here.

1.5. Lemma. *Let* $\mathfrak{A}=(A; F)$ *be an idempotent algebra with* $|A|>1$, *and assume* $\mathfrak{A}^2$ *has no reduced subuniverses of size* m *for some cardinal* m $(1<m\leq|A|)$. *Furthermore, let* $B\leq B_1\times\ldots\times B_n$ $(n\geq 2)$ *be a directly indecomposable subuniverse of* $\mathfrak{A}^n$ *of size* m. *Then*

(1.5.1) $\mathfrak{B}_i=(B_i; F)$ $(1\leq i\leq n)$ *are isomorphic subalgebras of* $\mathfrak{A}$, *and*

(1.5.2) *for arbitrary isomorphisms* $\pi_i\colon \mathfrak{B}_i\to\mathfrak{B}_1$ $(1\leq i\leq n)$ *the subuniverse* $B[\pi_1, \ldots, \pi_n]$ *of* $\mathfrak{A}^n$ *has the form* (1) *for some* $u\in\mathbf{n}$ $(u\geq 2)$ *and for some operations* $g_u, \ldots, g_n\in\mathcal{O}_{B_1}^{(u-1)}$.

The following statement is an important intermediate step in the proof of Lemma 1.5 (cf. Claim 1 in the proof of Lemma 4.4 in [13]).

1.6. Lemma. *Under the assumptions of Lemma* 1.5 B *has a projection*

$$\bar{B}=\mathrm{pr}_{i_1,\ldots,i_k}B\leq B_{i_1}\times\ldots\times B_{i_k} \quad (\{i_1, \ldots, i_k\}\subseteq\mathbf{n})$$

with $k\geq 2$ *such that*

(1.6.1) $|B_{i_l}|=m$ *for all* l $(1\leqq l\leqq k)$.

(1.6.2) $\mathrm{pr}_{\mathbf{k}-\{j\}}\bar{B}=\prod_{l\in\mathbf{k}-\{j\}} B_{i_l}$ *for all* j $(1\leqq j\leqq k)$, *and*

(1.6.3) $\bar{B}(x_1, b_2, \ldots, b_{j-1}, x_2, b_{j+1}, \ldots, b_k)$ *is a bijection* $B_{i_1}\to B_{i_j}$ *for all* j $(2\leqq j\leqq k)$ *and for all elements* $b_l\in B_{i_l}$ $(2\leqq l\leqq k,\ l\neq j)$.

Notice that condition (1.6.3) implies the existence of a function $g\colon B_{i_2}\times\ldots\times B_{i_k}\to B_{i_1}$ such that $\bar{B}=g_\square$, moreover, $g(b_2, \ldots, b_{j-1}, x, b_{j+1}, \ldots, b_k)\colon B_{i_j}\to B_{i_1}$ is a bijection for all j $(2\leqq j\leqq k)$ and for all elements $b_l\in B_{i_l}$ $(2\leqq l\leqq k,\ l\neq j)$. This leads to the following definition. A function $h\colon C_2\times\ldots\times C_m\to C_1$ $(m\geqq 2, C_1, \ldots, C_m\subseteq A)$ is said to have the *constant substitution property* iff for every j $(2\leqq j\leqq m)$ such that h depends on its j-th variable, and for arbitrary elements $c_l\in C_l$ $(2\leqq l\leqq m,\ l\neq j)$, the unary function

$$h(c_2, \ldots, c_{j-1}, x, c_{j+1}, \ldots, c_m)\colon C_j \to C_1$$

is a bijection.

The next result is a special case of Theorem 2.1 in [14] (cf. also Proposition 3.4 in [13]).

1.7. Proposition. *Let* B *be a set with* $|B|>1$, *and* $\mathscr{C}$ *a clone on* B *containing all the constants. Assume*

(1.7.1) *every surjective operation in* $\mathscr{C}$ *has the constant substitution property,*

(1.7.2) $\mathscr{C}$ *contains a surjective operation depending on at least two of its variables, and*

(1.7.3) *for every quasigroup operation in* $\mathscr{C}$, $\mathscr{C}$ *also contains the corresponding left and right divisions.*

Then there exist a division ring K *and a vector space* ${}_K\mathbf{B}=(B;+,K)$ *such that* $\mathscr{C}=\mathscr{P}({}_K\mathbf{B})$.

Now we are in a position to prove the theorem.

Proof of Theorem 1.1. Let B be of size m, and assume (1.1.1) fails for $\mathfrak{A}$. Then $n>2$ and the conclusions of Lemma 1.5 hold for B. Therefore we have to prove only the claims for $\mathfrak{B}_1$ and that in the representation (1) of $B[\pi_1, \ldots, \pi_n]$ we have $g_u, \ldots, g_n\in\mathscr{P}({}_K\mathbf{B}_1)$.

In what follows, all operations occurring are defined on B_1. Let $\mathscr{C}$ denote the set of all operations commuting with every basic operation of $\mathfrak{B}_1$ (and hence with every local term operation of $\mathfrak{B}_1$). It is easy to see that $\mathscr{C}$ is a clone on B_1 satisfying (1.7.3). Since $\mathfrak{B}_1$ is idempotent, $\mathscr{C}$ contains all the constants. Furthermore, every operation g_j $(u\leqq j\leqq n)$ occurring in the representation (1) of $B[\pi_1, \ldots, \pi_n]$ described in Lemma 1.5 belongs to $\mathscr{C}$, as $(g_j)_\square^{\cdot}$ is a projection of $B[\pi_1, \ldots, \pi_n]$. These operations g_j $(u\leqq j\leqq n)$ are obviously surjective; moreover, since (1.1.1) fails and B is reduced, therefore each of them depends on at least two of its variables. Finally, we show that every surjective operation $g\in\mathscr{C}$ has the constant substitution property. We may assume without loss of generality that g depends on all of its variables. If, say, g is k-ary, then $g_\Box$ is a directly indecomposable subuniverse of $\mathfrak{A}^{k+1}$ of size m. As no proper projection of $g_\Box$ can satisfy condition (1.6.3), we conclude that (1.6.3) holds for $g_\Box$, implying that g has the constant substitution property.

Thus Proposition 1.7 applies for the clone $\mathscr{C}$. Consequently there exists a vector space ${}_K\mathbf{B}_1=(B_1;\,+,K)$ over some division ring K such that $\mathscr{C}=\mathscr{P}({}_K\mathbf{B}_1)$. Hence $g_u,\dots,g_n\in\mathscr{P}({}_K\mathbf{B}_1)$. Furthermore, it is easy to see that the clone of the full idempotent reduct of the module ${}_{(\mathrm{End}\,{}_K\mathbf{B}_1)}\mathbf{B}_1$ coincides with the clone $\mathscr{C}^*$ of all operations commuting with each member of $\mathscr{C}$. Therefore it remains to prove that $\mathscr{T}_{\mathrm{loc}}(\mathfrak{B}_1)=\mathscr{C}^*$. The inclusion $\subseteqq$ is trivial by the definition of $\mathscr{C}$.

Before verifying the reverse inclusion observe that the singletons are the only proper subuniverses of $\mathfrak{B}_1$. Indeed, if $S\subset B_1$ $(S\neq\emptyset)$ is a proper subuniverse of $\mathfrak{B}_1$, then $x_1-x_2\in\mathscr{C}$ implies that

$$U=\{(x_1,x_2)\in B_1^2\colon\ x_1-x_2\in S\}$$

is a subuniverse of $\mathfrak{B}_1^2$ (and hence of $\mathfrak{A}^2$). Since $\mathrm{pr}_1U=\mathrm{pr}_2U=B_1$, $U\neq B_1^2$, and by assumption U is not reduced, therefore it follows that U is a bijection. Hence $|S|=1$.

Now let $f\in\mathscr{C}^*$, and let C be an arbitrary directly indecomposable subuniverse of $\mathfrak{B}_1^t$ for some integer $t\geqq1$. Since $\mathfrak{B}_1$ has no nonsingleton proper subalgebras, we have either $C\leqq B_1^t$ so that C is of size m, or $|C|=t=1$. If $t=1$, then f obviously preserves C. Suppose $t\geqq2$. Lemma 1.5 implies then that C has the form

$$C\approx\{(y_1,\dots,y_{v-1},f_v(y_1,\dots,y_{v-1}),\dots,f_t(y_1,\dots,y_{v-1}))\colon\ y_1,\dots,y_{v-1}\in B_1\}$$

for some v $(2\leqq v\leqq t)$ and some operations $f_v,\dots,f_t\in\mathscr{O}_{B_1}$. The sets $(f_j)_\Box$ $(v\leqq j\leqq t)$ are projections of C, yielding that $f_v,\dots,f_t\in\mathscr{C}$. Thus f commutes with $f_v,\dots,f_t$, implying that f preserves C as well. This means that f preserves every directly indecomposable subuniverse of each finite power of $\mathfrak{B}_1$. Hence it preserves also all subuniverses of finite powers of $\mathfrak{B}_1$, that is, $f\in\mathscr{T}_{\mathrm{loc}}(\mathfrak{B}_1)$. Therefore $\mathscr{C}^*=\mathscr{T}_{\mathrm{loc}}(\mathfrak{B}_1)$, as was to be proved.

2. Plain idempotent algebras

Recall that an algebra is called *plain* iff it is simple and has no nonsingleton proper subalgebras. Clearly, for an idempotent algebra the property of having no nonsingleton proper subalgebras implies simplicity. As we shall see in this section, having no nonsingleton proper subalgebras is a rather strong constraint on idempotent algebras: up to local term equivalence, there are only "a few" plain idempotent algebras.

In the description of plain idempotent algebras an important role will be played by the "higher dimensional crosses"

$$X_n^a = \bigcup_{i=1}^{n} (A\times\ldots\times A\times\overset{i}{\{a\}}\times A\times\ldots\times A), \quad n \geqq 2,$$

where a is a fixed element of the set A. Obviously, $X_2^a = X^a$. For $k \geqq 2$ and $a \in A$ let $\mathscr{F}_k^a$ denote the clone of all idempotent operations on A preserving X_k^a. Furthermore, put $\mathscr{F}_\omega^a = \bigcap_{2\leqq k<\omega} \mathscr{F}_k^a$. Since for arbitrary element $b \in A$, $b \neq a$, we have $X_n^a(x_1, \ldots, x_{n-1}, b) = X_{n-1}^a$, therefore

$$\mathscr{F}_2^a \supseteqq \mathscr{F}_3^a \supseteqq \ldots \supseteqq \mathscr{F}_k^a \supseteqq \mathscr{F}_{k+1}^a \supseteqq \ldots \supseteqq \mathscr{F}_\omega^a.$$

For a permutation group G acting on A we will denote by $\mathscr{I}_A(G)$ the clone of all idempotent operations on A commuting with every member of G.

2.1. Theorem. *Every plain idempotent algebra* $\mathfrak{A} = (A; F)$ *with* $|A| \geqq 3$ *is locally term equivalent to one of the following algebras:*

(2.1.1) $(A; \mathscr{I}_A(G))$ *for a permutation group* G *acting on* A *such that every nonidentity member of* G *has at most one fixed point;*

(2.1.2) *the full idempotent reduct of the module* $_{(\mathrm{End}\,_K\mathbf{A})}\mathbf{A}$ *for some vector space* $_K\mathbf{A} = (A; +, K)$ *over a division ring* K;

(2.1.3) $(A; \mathscr{I}_A(G) \cap \mathscr{F}_k^0)$ *for some* k $(2 \leqq k \leqq \omega)$, *some element* $0 \in A$, *and a permutation group* G *acting on* A *such that* 0 *is the unique fixed point of every nonidentity member of* G.

Remarks. 1. It is not hard to show that every algebra locally term equivalent to an algebra in (2.1.1) or (2.1.3) is plain. The same is well known to hold for (2.1.2), too. Note that the algebras in (2.1.1) are locally quasi-primal.

2. The conclusion of the theorem fails for 2-element algebras. Obviously, every 2-element algebra is plain, and Post's description [8] of all clones on a 2-element set (or Corollary 1.4 above) shows that, up to term equivalence, there are more 2-element idempotent algebras than those of types (2.1.1)—(2.1.3) listed in the theorem.

The first step of the proof of Theorem 2.1 is based on Theorem 1.1.

2.2. Proposition. *For a plain idempotent algebra* $\mathfrak{A} = (A; F)$ *with* $|A| \geqq 3$ *one of the following conditions holds:*

(2.2.1) $\mathfrak{A}$ *is locally quasi-primal, or*

(2.2.2) *there exist a division ring* K *and a vector space* $_K\mathbf{A} = (A; +, K)$ *such that* $\mathfrak{A}$ *is locally term equivalent to the full idempotent reduct of the module* $_{(\mathrm{End}\,_K\mathbf{A})}\mathbf{A}$, *or*

(2.2.3) *there exists an element* $0 \in A$ *such that* X^0 *is the only reduced subuniverse of* $\mathfrak{A}^2$, *moreover,* 0 *is the unique fixed point of each nonidentity automorphism of* $\mathfrak{A}$.

Proof. Since $\mathfrak{A}$ has no nonsingleton proper subalgebras, Theorem 1.1 implies that $\mathfrak{A}$ is of type (2.2.1) or (2.2.2), or $\mathfrak{A}^2$ has a reduced subuniverse. Assume the last possibility holds for $\mathfrak{A}$, and let D be a reduced subuniverse of $\mathfrak{A}^2$. Obviously, $D \leqq A \times A$. Using that the subuniverses $D(x, a)$, $D(a, x)$ $(a \in A)$ of $\mathfrak{A}$ are singletons or equal to A, we can conclude that D is a cross, say $D = X^{a_1, a_2}$ $(a_1, a_2 \in A)$. We must have $a_1 = a_2$, since otherwise $\mathrm{pr}_1(X^{a_1, a_2} \cap X^{a_2, a_1}) = \{a_1, a_2\}$ would be a nonsingleton proper subuniverse of $\mathfrak{A}$. It follows now that there is at most one cross among the subuniverses of $\mathfrak{A}^2$. Indeed, if X^b and X^c $(b, c \in A, b \neq c)$ were subuniverses of $\mathfrak{A}^2$, then $\mathrm{pr}_1(X^b \cap X^c) = \{b, c\}$ would again be a nonsingleton proper subuniverse of $\mathfrak{A}$. Thus there is an element $0 \in A$ such that X^0 is the unique reduced subuniverse of $\mathfrak{A}^2$. This implies in particular that $X^0[\pi, \pi] = X^0$ for arbitrary automorphism π of $\mathfrak{A}$; hence π fixes 0. Furthermore, since the fixed points of an automorphism of $\mathfrak{A}$ form a subuniverse in $\mathfrak{A}$, 0 is the only fixed point of each nonidentity automorphism of $\mathfrak{A}$.

Now we discuss in more detail the algebras $\mathfrak{A}$ of type (2.2.3). To show that every such $\mathfrak{A}$ is locally term equivalent to an algebra in (2.1.3), we determine the subuniverses of finite powers of $\mathfrak{A}$. For a natural number $n \geqq 2$ and for a family P of subsets of $\mathbf{n}$ we set

$$Y^0_{n,P} = \bigcup_{I \in P} A^{(n,I)}$$

where

$$A^{(n,I)} = A_1 \times \ldots \times A_n \quad \text{with} \quad A_i = \begin{cases} A & \text{if } i \in I \\ \{0\} & \text{if } i \in \mathbf{n} - I. \end{cases}$$

Since the element 0 is fixed throughout this discussion, we omit the superscript ^{0} in $Y^0_{n,P}$ and X^0_n. Clearly, $Y_{n,P} \subseteqq X_n$ unless $\mathbf{n} \in P$, and equality holds if P is the set of $(n-1)$-element subsets of $\mathbf{n}$. Let us call a subset C of A^n *irredundant* iff $\mathrm{pr}_i C = A$ for all $i \in \mathbf{n}$ and no projection $\mathrm{pr}_{i,j} C$ $(i, j \in \mathbf{n},\ i \neq j)$ of C is a permutation of A. Clearly, all reduced subsets of A^n are irredundant.

2.3. Proposition. *Let $\mathfrak{A} = (A; F)$ $(|A| \geqq 3)$ be a plain idempotent algebra satisfying condition* (2.2.3). *Then for every integer $n \geqq 2$, every irredundant subuniverse of $\mathfrak{A}^n$ is of the form $Y_{n,P}$ for some family P of subsets of $\mathbf{n}$.*

The case $n=2$ is a consequence of (2.2.3): A^2 and X_2 are the only irredundant subuniverses of $\mathfrak{A}^2$. The next three steps of the proof will be carried out in Lemmas 2.4 through 2.6.

2.4. Lemma. *The claim of Proposition* 2.3 *is true for* $n=3$.

Proof. Let C be an irredundant subuniverse of $\mathfrak{A}^3$. Clearly, its projections $\mathrm{pr}_{i,j} C$ $(i, j \in 3,\ i \neq j)$ are also irredundant, and hence equal A^2 or X_2.

If $\mathrm{pr}_{1,2}C=\mathrm{pr}_{1,3}C=\mathrm{pr}_{2,3}C=X_2$, then every triple from C has at least two components 0. Thus

(2) $$C\subseteq (A\times\{0\}\times\{0\})\cup(\{0\}\times A\times\{0\})\cup(\{0\}\times\{0\}\times A).$$

Since $\mathrm{pr}_1 C=A$, the subuniverse $C(x_1, 0, 0)$ of $\mathfrak{A}$ must contain $A-\{0\}$. Thus the assumption $|A|\geqq 3$ and the plainness of $\mathfrak{A}$ imply that $C(x_1, 0, 0)=A$, hence $A\times\{0\}\times\{0\}\subseteq C$. By symmetry it follows that equality holds in (2).

Suppose $\mathrm{pr}_{1,2}C=\mathrm{pr}_{1,3}C=X_2$, $\mathrm{pr}_{2,3}C=A^2$. Then, clearly,

(3) $$C\subseteq (A\times\{0\}\times\{0\})\cup(\{0\}\times A\times A).$$

Since for arbitrary $(a, b)\in A^2$ with $a\neq 0$ we have $(a, b)\in \mathrm{pr}_{2,3}C$, therefore $(x, a, b)\in C$ for some $x\in A$. However, $\mathrm{pr}_{1,2}C=X_2$ yields that $x=0$. Hence for all $b\in A$ the subuniverse $C(0, x_1, b)$ contains the set $A-\{0\}$, implying that $C(0, x_1, b)=A$. Thus $\{0\}\times A\times A\subseteq C$, which together with $\mathrm{pr}_1 C=A$ shows that we have equality in (3).

Assume now that $\mathrm{pr}_{1,2}C=X_2$ and $\mathrm{pr}_{1,3}C=\mathrm{pr}_{2,3}C=A^2$. Then

$$C\subseteq X_2\times A = (A\times\{0\}\times A)\cup(\{0\}\times A\times A).$$

As in the previous case, we get that $\{0\}\times A\times A\subseteq C$, and similarly (interchanging the role of the first and second components) $A\times\{0\}\times A\subseteq C$. Thus $C=X_2\times A$.

By symmetry it remains to consider the case $\mathrm{pr}_{1,2}C=\mathrm{pr}_{1,3}C=\mathrm{pr}_{2,3}C=A^2$. If $C=A^3$, we are done, so assume that $C\neq A^3$. First we show the required equality $C=X_3$ under the additional assumption that there is an element $c\in A$ with $C(c, x_1, x_2)=A^2$. In this case we have $C(x_1, x_2, b)\supseteq\{c\}\times A$ for all $b\in A$. Taking into account that $C(x_1, x_2, b)$ is a subuniverse of $\mathfrak{A}^2$ and $\mathrm{pr}_i C(x_1, x_2, b)=A$ for $i\in\mathbf{2}$ (the latter follows from $\mathrm{pr}_{1,3}C=\mathrm{pr}_{2,3}C=A^2$), we get that $C(x_1, x_2, b)$ equals X_2 or A^2 for every $b\in A$. Since $C\neq A^3$, the former has to hold for at least one $b\in A$, implying that $c=0$. On the other hand, $\mathrm{pr}_{1,2}C=A^2$ ensures that there is a $b'\in A$ with $C(x_1, x_2, b')=A^2$. Therefore the same argument as before (with c replaced by b') yields that $C(x_1, x_2, b)=A^2$ if and only if $b=0$. Hence

$$C = \bigcup_{b\in A}(C(x_1, x_2, b)\times\{b\}) = (A^2\times\{0\})\cup\bigcup_{\substack{b\in A\\ b\neq 0}}(X_2\times\{b\}) = X_3.$$

Finally, suppose that for all elements $c\in A$ the subuniverses $C(c, x_1, x_2)$, and symmetrically also $C(x_1, c, x_2)$, $C(x_1, x_2, c)$, of $\mathfrak{A}^2$ are distinct from A^2. Since their projections onto each component are equal to A, we get that each of these subuniverses is either X_2 or an automorphism of $\mathfrak{A}$. Since all automorphisms of $\mathfrak{A}$ fix 0, and $\mathrm{pr}_{1,2}C=A^2$, we conclude that there exists an element $b\in A$ such that $C(x_1, x_2, b)=X_2$. Then $C(0, x_1, x_2)\supseteq A\times\{b\}$, implying that $C(0, x_1, x_2)=X_2$ and

$b=0$. This shows that $C(x_1, x_2, c)=X_2$ if and only if $c=0$, and by symmetry the same holds for the subuniverses $C(x_1, c, x_2)$, $C(c, x_1, x_2)$ as well.

Consider now the set

$$D = \{(x, y, z)\in A^3\colon \text{there is a } u\in A \text{ such that } (x, y, u),\ (u, y, z)\in C\}.$$

It is easy to check that D is a subuniverse of $\mathfrak{A}^3$. Furthermore, $\mathrm{pr}_{1,2}C=\mathrm{pr}_{2,3}C=A^2$ implies that $\mathrm{pr}_{1,2}D=\mathrm{pr}_{2,3}D=A^2$. Choosing $u=0$ in the definition of D we get that $A\times\{0\}\times A\subseteq D$. Hence D satisfies the additional assumption under which we can conclude that D equals X_3 or A^3. Then for arbitrary elements $a, b\in A-\{0\}$ we have $(0, a, b)\in D$, that is $(0, a, c), (c, a, b)\in C$ for some $c\in A$. However, $C(0, x_1, x_2)=X_2$ implies that $c=0$ and $(a, b)\in X_2$, a contradiction. Thus this case cannot occur, completing the proof of Lemma 2.4.

2.5. Lemma. *For arbitrary integer $n\geqq 3$ and for every subuniverse C of $\mathfrak{A}^n$ satisfying $\mathrm{pr}_{\mathbf{n}-\{i\}}C=A^{n-1}$ for all $i\in\mathbf{n}$, we have $C=A^n$ or $C=X_n$.*

Proof. We proceed by induction on n. Clearly C is irredundant, therefore by Lemma 2.4 the claim is true if $n=3$. Let now $n\geqq 4$, and suppose $C\neq A^n$. Since the subuniverses $C(x_1, \ldots, x_{i-1}, a, x_i, \ldots, x_{n-1})$ $(i\in\mathbf{n},\ a\in A)$ of $\mathfrak{A}^{n-1}$ satisfy the assumption of the lemma, we get from the induction hypothesis that

$$C(x_1, \ldots, x_{i-1}, a, x_i, \ldots, x_{n-1})=A^{n-1} \text{ or } X_{n-1} \text{ for all } a\in A \text{ and } i\in\mathbf{n}.$$

We have $C(b, x_1, \ldots, x_{n-1})=A^{n-1}$ for at least one $b\in A$, because $\mathrm{pr}_{\mathbf{n}-1}C=A^{n-1}$. Since $C\neq A^n$, there also exists an element $c\in A$ such that $C(x_1, \ldots, x_{n-1}, c)=X_{n-1}$. Thus

$$X_{n-1} = C(x_1, \ldots, x_{n-1}, c) \supseteq \{b\}\times A^{n-2},$$

yielding $b=0$. Consequently

$$C(b, x_1, \ldots, x_{n-1}) = \begin{cases} A^{n-1} & \text{if } b = 0 \\ X_{n-1} & \text{otherwise} \end{cases} \quad (b\in A),$$

whence $C=X_n$, as required.

2.6. Lemma. *For arbitrary irredundant subuniverse C of $\mathfrak{A}^n$ $(n\geqq 3)$ the subuniverses $C(x_1, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$ $(i\in\mathbf{n})$ of $\mathfrak{A}^{n-1}$ are also irredundant.*

Proof. By symmetry it suffices to prove the statement for $\tilde{C}=C(x_1, \ldots, x_{n-1}, 0)$. Firstly, let $1\leqq i\leqq n-1$. As $D=\mathrm{pr}_{i,n}C$ is an irredundant subuniverse of $\mathfrak{A}^2$, we must have $D=A^2$ or $D=X_2$. Thus $\mathrm{pr}_i\tilde{C}=D(x, 0)=A$. Secondly, let $1\leqq i<j\leqq n-1$. Clearly, $D'=\mathrm{pr}_{i,j,n}C$ is an irredundant subuniverse of $\mathfrak{A}^3$. Thus, making use of Lemma 2.4, one can easily see that the set $\mathrm{pr}_{i,j}\tilde{C}=D'(x_1, x_2, 0)$ is not a permutation of A. This shows that $\tilde{C}$ is irredundant.

Proof of Proposition 2.3. The case $n=3$ is settled in Lemma 2.4, so we may assume that $n\geqq 4$ and the claim is true for the irredundant subuniverses of $\mathfrak{A}^{n-1}$. Consider an irredundant subuniverse C of $\mathfrak{A}^n$. If C contains an n-tuple with all components distinct from 0, then by the induction hypothesis $\mathrm{pr}_{\mathbf{n}-\{i\}}C=A^{n-1}$ for all $i\in\mathbf{n}$. Hence, by Lemma 2.5, $C=A^n$. Otherwise, if every n-tuple in C has at least one component 0, then Lemma 2.6 and the induction hypothesis ensure for each $i\in\mathbf{n}$ the existence of a family P_i of subsets of $\mathbf{n}-\{i\}$ such that

$$C(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = Y_{n-1, P_i}.$$

Putting $P=\bigcup_{i\in\mathbf{n}} P_i$ we get that $C=Y_{n,P}$.

Making use of Proposition 2.3 we can now conclude the proof of Theorem 2.1.

Proof of Theorem 2.1 Let G denote the automorphism group of $\mathfrak{A}$. According to Proposition 2.2 we have to distinguish three cases. Suppose first (2.2.1), that is, $\mathfrak{A}$ is locally quasi-primal. Since $\mathfrak{A}$ is plain, every internal isomorphism of $\mathfrak{A}$ is either an automorphism of $\mathfrak{A}$, or an isomorphism between two singleton subalgebras of $\mathfrak{A}$. Therefore $\mathscr{T}_{\mathrm{loc}}(\mathfrak{A})=\mathscr{I}_A(G)$. Furthermore, since the set of fixed points of each automorphism of $\mathfrak{A}$ is a subuniverse of $\mathfrak{A}$, therefore each nonidentity member of G has at most one fixed point. Consequently $\mathfrak{A}$ is locally term equivalent to an algebra of type (2.1.1).

In case (2.2.2) we have nothing to prove.

Finally, if (2.2.3) holds for $\mathfrak{A}$, then we can apply Proposition 2.3. Observe first that for arbitrary irredundant subuniverse C of $\mathfrak{A}^n$ $(n\geqq 3)$ such that $C\subset X_n$ we have

$$C = \{(x_1, \ldots, x_n)\in A^n\colon (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\in \mathrm{pr}_{\mathbf{n}-\{i\}}C \text{ for all } i\in\mathbf{n}\}.$$

Indeed, the inclusion $\subseteqq$ being trivial, suppose $(x_1, \ldots, x_n)\in A^n$ belongs to the set on the right hand side. Since $C\subset X_n$, by Lemma 2.5 some projection of C onto $n-1$ components is distinct from A^{n-1}, say $\mathrm{pr}_{\mathbf{n}-1}C\neq A^{n-1}$. However, $\mathrm{pr}_{\mathbf{n}-1}C$ is an irredundant subuniverse of $\mathfrak{A}^{n-1}$, therefore by Proposition 2.3 at least one component of $(x_2, \ldots, x_n)\in\mathrm{pr}_{\mathbf{n}-1}C$ equals 0, say $x_2=0$. Then $(x_1, x_3, \ldots, x_n)\in\mathrm{pr}_{\mathbf{n}-\{2\}}C$ implies that $(x_1, y, x_3, \ldots, x_n)\in C$ for some $y\in A$. Since C is of the form described in Proposition 2.3, we have also $(x_1, x_2, x_3, \ldots, x_n)=(x_1, 0, x_3, \ldots, x_n)\in C$.

Since $\mathfrak{A}$ is a plain idempotent algebra, an idempotent operation on A is a local term operation of $\mathfrak{A}$ if and only if it preserves the automorphisms of $\mathfrak{A}$ and the irredundant subuniverses of finite powers of $\mathfrak{A}$. By repeated application of the observation made in the previous paragraph it follows that for every irredundant subuniverse C of $\mathfrak{A}^n$ $(n\geqq 1)$, an operation preserves C if and only if it preserves all those projections of C which are of the form X_j for some j $(2\leqq j\leqq n)$. Thus $\mathscr{T}_{\mathrm{loc}}(\mathfrak{A})=\mathscr{I}_A(G)\cap\mathscr{F}_\omega^0$ if all X_j $(j\geqq 2)$ occur among the subuniverses of finite powers of $\mathfrak{A}$, and $\mathcal{T}_{\text{loc}}(\mathfrak{A})=\mathcal{I}_A(G)\cap\mathcal{F}_k^0$ $(2\leqq k<\omega)$ if k is the largest j such that X_j is a subuniverse of $\mathfrak{A}^j$. The proof of Theorem 2.1 is complete.

Recall that an algebra $\mathfrak{A}=(A;F)$ is *locally functionally complete* iff $\mathcal{P}_{\text{loc}}(\mathfrak{A})=\mathcal{O}_A$. In terms of subuniverses this condition is equivalent to requiring that for every integer $n\geqq 1$, A^n is the only irredundant reflexive subuniverse of $\mathfrak{A}^n$. ($B\subseteq A^n$ is said to be *reflexive* iff $(a,\ldots,a)\in B$ for all $a\in A$.) Thus a slight improvement of a result of L. Szabó [12] can easily be derived from Propositions 2.2 and 2.3.

2.7. Corollary. *If $\mathfrak{A}=(A;F)$ is a plain idempotent algebra with $|A|\geqq 3$, then either $\mathfrak{A}$ is locally functionally complete, or there exist a division ring K and a vector space ${}_K\mathbf{A}=(A;+,K)$ such that $\mathfrak{A}$ is locally term equivalent to the full idempotent reduct of the module ${}_{(\text{End}\,_K\mathbf{A})}\mathbf{A}$.*

Proof. By Proposition 2.2 we have to show that if $\mathfrak{A}$ is of type (2.2.1) or (2.2.3), then for every integer $n\geqq 1$, A^n is the only irredundant reflexive subuniverse of $\mathfrak{A}^n$. For type (2.2.1) this is well known (see P. H. Krauss [2]), while for type (2.2.3) it follows from Proposition 2.3.

3. Two results on minimal clones

Throughout this section projections will be called *trivial operations*, and the term *trivial clone* will mean the clone of projections. It is obvious that a minimal clone is generated by each of its nontrivial members. Thus the most natural way of classifying minimal clones is by their nontrivial members of least possible arity. Accordingly, by a result of I. G. Rosenberg [10], the algebras $(A;f)$ with minimal clones fall into five types: (i) f is a nontrivial unary operation, (ii) f is a nontrivial idempotent binary operation, (iii) f is a majority operation, (iv) f is a nontrivial semiprojection, or (v) $f(x,y,z)=x+y+z$ for an elementary Abelian 2-group.

It is well known (see J. Płonka [7]) that for every elementary Abelian q-group $\mathbf{A}=(A;+)$ (q prime) the algebra $(A;x-y+z)$ has a minimal clone. If $q>2$, then these algebras are of type (ii), since they have nontrivial binary term operations. In this section we present two conditions ensuring that a finite algebra with minimal clone be term equivalent to an algebra of this form.

As was observed by I. G. Rosenberg [10], an algebra $(A;f)$ where f is a minority operation has a minimal clone if and only if $f(x,y,z)=x+y+z$ for some elementary Abelian 2-group $\mathbf{A}=(A;+)$. P. P. Pálfy posed a more general question: For which Mal'tsev operations p is the clone of the algebra $(A;p)$ minimal? The following result answers this question in the finite case.

3.1. Theorem. *A finite algebra* $(A; p)$ *where* p *is a Mal'tsev operation has a minimal clone if and only if there exists an elementary Abelian group* $\mathbf{A}=(A; +)$ *such that* $p(x, y, z)=x-y+z$.

It remains open whether the same holds true for infinite algebras $(A; p)$ as well.

The next result gives a characterization for those idempotent groupoids with minimal clones which are term equivalent to $(A; x-y+z)$.

3.2. Theorem. *A finite idempotent groupoid* $(A; \cdot)$ *with minimal clone is term equivalent to an algebra* $(A; x-y+z)$ *for some elementary Abelian q-group* (q *is an odd prime*) *if and only if it has a minimal nonsingleton subgroupoid of cardinality greater than* 2.

The if part of this statement can be rephrased as follows: In a finite idempotent groupoid $(A; \cdot)$ with minimal clone every minimal nonsingleton subgroupoid is 2-element, unless $(A; \cdot)$ is term equivalent to $(A; x-y+z)$ for some elementary Abelian q-group (q is an odd prime). Since the clone of every subgroupoid of $(A; \cdot)$ is minimal or trivial, it follows that every 2-element subgroupoid of $(A; \cdot)$ is either a left zero semigroup, or a right zero semigroup, or a semilattice. This suggests that in trying to determine the finite idempotent groupoids with minimal clones it may be useful to classify these groupoids according to the types of their 2-element subgroupoids. P. P. Pálfy [4] has made an interesting observation in this direction by proving that if an idempotent groupoid $(A; \cdot)$ with minimal clone has a left zero semigroup as well as a right zero semigroup among its 2-element subgroupoids, then $(A; \cdot)$ is a rectangular band.

We now turn to the proof of Theorems 3.1 and 3.2. As a preparation, we state two lemmas.

3.3. Lemma. *A finite plain idempotent algebra* $\mathfrak{B}=(B; F)$ *with* $|B|\geqq 3$ *has a minimal clone if and only if it is term equivalent to* $(B; x-y+z)$ *for some cyclic group* $\mathbf{B}=(B; +)$ *of prime order.*

Proof. Let $\mathfrak{B}=(B; F)$ be a finite plain idempotent algebra with $|B|\geqq 3$. By Theorem 2.1 we have one of the following three possibilities for $\mathfrak{B}$:

(a) $\mathfrak{B}$ is quasi-primal, or

(b) there exist a prime q and an elementary Abelian q-group $\mathbf{B}=(B; +)$ such that $\mathfrak{B}$ is affine with respect to $\mathbf{B}$, or

(c) $\mathscr{I}_A(G)\cap\mathscr{F}_\omega^0\subseteq\mathscr{T}(\mathfrak{B})$ for some element $0\in B$ and some permutation group G acting on B such that 0 is the unique fixed point of each nonidentity permutation in G.

Suppose $\mathfrak{B}$ has a minimal clone. Since the clone of a quasi-primal algebra cannot be minimal, case (a) does not hold for $\mathfrak{B}$. We prove that case (c) is also excluded. Assume $\mathfrak{B}$ satisfies the conditions in (c), and define a binary operation $*$ on B as follows:

$$x*y=\begin{cases}0 & \text{if } x=0 \text{ or } y=0\\ x & \text{otherwise}\end{cases}\qquad (x,y\in B).$$

It is easy to check that $*\in\mathscr{I}_A(G)\cap\mathscr{F}^0_\omega$. Hence $*$ is a nontrivial term operation of $\mathfrak{B}$. However, the algebra $(B;*)$ is not term equivalent to $\mathfrak{B}$, because it is not plain (every 2-element subset is a subalgebra). Therefore the clone of $\mathfrak{B}$ is not minimal. Thus $\mathfrak{B}$ satisfies condition (b), implying that $x-y+z$ is a term operation of $\mathfrak{B}$. Since $\mathscr{T}(\mathfrak{B})$ is a minimal clone and $\mathfrak{B}$ is plain, we conclude that $\mathfrak{B}$ is term equivalent to $(B;x-y+z)$ and $|B|=q$. This completes the proof of the only if part. The converse is well known (cf. J. Płonka [7]).

3.4. Lemma. *Let $\mathfrak{A}=(A;f)$ be a finite idempotent algebra whose clone is minimal, and let $\mathfrak{B}=(B;f)$ be a subalgebra of $\mathfrak{A}$. If $\mathfrak{B}$ is term equivalent to $(B;x-y+z)$ for some cyclic group $\mathbf{B}=(B;+)$ of prime order, then there exists an elementary Abelian group $\mathbf{A}=(A;+)$ such that $\mathfrak{A}$ is term equivalent to $(A;x-y+z)$.*

Proof. Let $|B|=q$ (q prime). For arbitrary term $\mathbf{t}$ let $\mathbf{t}_\mathfrak{A}$, resp. $\mathbf{t}_\mathfrak{B}$, denote the term operations induced by $\mathbf{t}$ in $\mathfrak{A}$, resp. $\mathfrak{B}$. We claim that for arbitrary term $\mathbf{s}$, if $\mathbf{s}_\mathfrak{B}$ is a projection, then $\mathbf{s}_\mathfrak{A}$ is also a projection. Indeed, suppose $\mathbf{s}_\mathfrak{A}$ is not a projection. Then by the minimality of $\mathscr{T}(\mathfrak{A})$ we get that $(A;\mathbf{s}_\mathfrak{A})$ is term equivalent to $\mathfrak{A}$. Hence $(B;\mathbf{s}_\mathfrak{B})$ must be term equivalent to $\mathfrak{B}$, implying that $\mathbf{s}_\mathfrak{B}$ is not a projection. Clearly, if $\mathbf{s}_\mathfrak{B}$ is an i-th projection, then $\mathbf{s}_\mathfrak{A}$ is also an i-th projection. Thus, for arbitrary term $\mathbf{p}$ inducing $x-y+z$ in $\mathfrak{B}$, the identities

$$\mathbf{p}(x,y,y)=x=\mathbf{p}(y,y,x), \tag{4}$$

$$\mathbf{p}(\mathbf{p}(z,y,x),z,y)=x, \tag{5}$$

$$\mathbf{p}(\mathbf{p}(\mathbf{p}(x,y,z),z,u),u,y)=x, \tag{6}$$

$$\underbrace{\mathbf{p}(\mathbf{p}(\ldots(\mathbf{p}}_{q\text{ times}}(x,y,z),y,z)\ldots),y,z)=x, \tag{7}$$

which obviously hold in $\mathfrak{B}$, are satisfied in $\mathfrak{A}$ as well. By (4), $\mathbf{p}_\mathfrak{A}$ is a Mal'tsev operation. Identifying the variables u, y in (6) we get the identity

$$\mathbf{p}(\mathbf{p}(x,y,z),z,y)=x,$$

which shows that (5) and (6) are equivalent to

$$\mathbf{p}(z,y,x)=\mathbf{p}(x,y,z)\quad\text{and}\quad \mathbf{p}(\mathbf{p}(x,y,z),z,u)=\mathbf{p}(x,y,u),$$

respectively. These identites imply that for arbitrary element $0\in A$ the operations

$$x+y=\mathbf{p}_\mathfrak{A}(x,0,y)\quad\text{and}\quad -x=\mathbf{p}_\mathfrak{A}(0,x,0)\quad(x,y\in A)$$

define an Abelian group $\mathbf{A}=(A;\ +,\ -,0)$, and $\mathbf{p}_{\mathfrak{A}}(x,y,z)=x-y+z$ (see Proposition 2.2 in [13]). Now the identity (7) ensures that $\mathbf{A}$ is an elementary Abelian q-group. Since $\mathcal{T}(\mathfrak{A})$ is a minimal clone, $\mathfrak{A}$ is term equivalent to the algebra $(A;\mathbf{p}_{\mathfrak{A}})=$ $=(A;x-y+z)$, as required.

Proof of Theorem 3.1. The sufficiency is well known. To prove the necessity consider a finite algebra $\mathfrak{A}=(A;p)$ with minimal clone, where p is a Mal'tsev operation. Let $\mathfrak{B}=(B;p)$ be a minimal nonsingleton subalgebra of $\mathfrak{A}$. Clearly, p is a Mal'tsev operation on B, too, and $\mathcal{T}(\mathfrak{B})$ is a minimal clone. Furthermore, $\mathfrak{B}$ is a plain idempotent algebra. If $|B|\geqq 3$, then by Lemma 3.3 $\mathfrak{B}$ is term equivalent to $(B;x-y+z)$ for some cyclic group $\mathbf{B}=(B;\ +)$ of prime order. Using that p is a Mal'tsev operation, one can easily verify that this is true also when $|B|=2$. (Alternatively, we can draw the same conclusion for $\mathfrak{B}$ by applying R. McKenzie's Theorem [3] stating that every finite plain Mal'tsev algebra is either quasi-primal or affine with respect to an elementary Abelian group.) Thus, by Lemma 3.4, $(A;p)$ is term equivalent to the algebra $(A;x-y+z)$ for some elementary Abelian group $(A;\ +)$. As $x-y+z$ is the unique Mal'tsev operation in the clone of $(A;x-y+z)$, the operation p coincides with $x-y+z$.

Proof of Theorem 3.2. The necessity is obvious. Conversely, suppose that $\mathfrak{A}=(A;\ \cdot)$ is a finite idempotent groupoid with minimal clone such that $\mathfrak{A}$ has a minimal nonsingleton subgroupoid $\mathfrak{B}=(B;\ \cdot)$ with $|B|\geqq 3$. Clearly, $\mathcal{T}(\mathfrak{B})$ is nontrivial, therefore it is a minimal clone. Furthermore, $\mathfrak{B}$ is a plain idempotent algebra. In the same way as in the previous proof, Lemmas 3.3 and 3.4 yield that $\mathfrak{A}$ is term equivalent to the algebra $(A;x-y+z)$ for some elementary Abelian q-group (q prime). Obviously, $q=|B|$ $(\geqq 3)$, which completes the proof.

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