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A generalization of McAlister's P-theorem for E-unitary regular semigroups

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To the memory of András Huhn

A regular semigroup is called E-unitary if its set of idempotents is a unitary subset. One can easily show that E-unitary regular semigroups are necessarily orthodox.

In 1974 McALISTER [5], [6] proved that every inverse semigroup is an idempotent separating homomorphic image of an E-unitary inverse semigroup and described E-unitary inverse semigroups by means of groups, partially ordered sets and semilattices. This structure theorem is referred to as the "P-theorem". By making use of McAlister's P-theorem O'CARROLL [8] proved that every E-unitary inverse semigroup can be embedded into a semidirect product of a semilattice by a group.

These results have opened up new perspectives not only in the theory of inverse semigroups but in the theory of regular semigroups. McAlister's first result was generalized for orthodox semigroups independently by TAKIZAWA [15] and the author [10]. TAKIZAWA [14] generalized the P-theorem, too, but only for E-unitary $\mathcal{R}$-unipotent semigroups. This structure theorem was applied in [12] to prove the analogue of O'Carroll's embedding theorem for E-unitary $\mathcal{R}$-unipotent semigroups.

The aim of this paper is to present a generalization of the P-theorem for E-unitary regular semigroups. It has to be pointed out in advance that our main result which is proved in Sections 2 and 3 cannot be considered as a structure theorem in the sense that E-unitary regular semigroups are constructed in it from "simpler" objects. Indeed, it is doubtful that strictly combinatorial semigroups which play an important role in the construction are "simpler" than E-unitary regular semigroups. However, the strictly combinatorial partial semigroup introduced in Section 2 is applied in a forthcoming paper [13] to prove that every E-unitary regular semigroup with regular band of idempotents can be embedded into a semidirect product of a band by a group.

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MARGOLIS and PIN [4] generalized McAlister's P-theorem in another direction, namely for E-unitary not necessarily regular semigroups with commuting idempotents. It turns out that in the special case of E-unitary regular semigroups with commuting idempotents, that is, in the case of E-unitary inverse semigroups the main theorem of [4] asserts almost the same result as a part of our main theorem. In Section 4 we deduce a characterization of E-unitary regular semigroups which is similar to that formulated in the main theorem of [4].

1. Preliminaries

Let S be a semigroup. The set of idempotents in S is denoted by E_S and the set of inverses of an element s in S by $V_S(s)$. For the least group congruence on S we use the notation σ_S and the factor semigroup S/σ_S will be denoted by G_S. If it causes no confusion we omit S from E_S, $V_S(s)$ and σ_S.

A regular semigroup S is called *E-unitary* if E is a unitary subset in S. It is easy to see that E-unitary regular semigroups are necessarily orthodox.

Result 1.1 (HOWIE and LALLEMENT [3] and SAITÔ [9]). *For a regular semigroup S, the following conditions are equivalent:*

(i) *S is E-unitary,*

(ii) *E is a left unitary subset in S,*

(iii) *E is a right unitary subset in S,*

(iv) *E constitutes a σ-class.*

Let $\varphi\colon S\to T$ be a homomorphism where S and T are regular semigroups. We denote by $\ker\varphi$ the congruence on S induced by φ and by $\operatorname{Ker}\varphi$ the union of idempotent $\ker\varphi$-classes. If $\varkappa$ is a congruence on S then instead of $\operatorname{Ker}\varkappa^{\natural}$ we simply write $\operatorname{Ker}\varkappa$.

Now let S be an orthodox semigroup with 0. Assume that S is categorical at 0. It is obvious that the least inverse semigroup congruence γ on S is 0-restricted and S/γ is also categorical at 0. Hence it follows by Theorem 7.66 [1] that there exists a least 0-restricted congruence β on S/γ such that $(S/\gamma)/\beta$ is a primitive inverse semigroup. It is easily seen that $\ker\gamma^{\natural}\beta^{\natural}$ is the least 0-restricted primitive inverse semigroup congruence on S which will be denoted by ϱ_S or, simply, by ϱ.

Proposition 1.2. *Let $S=S^0$ be an orthodox semigroup which is categorical at 0. Then the following conditions are equivalent for $s, t\in S$:*

(i) *$s, t\neq 0$ and $s\varrho t$;*

(ii) *$set'\in E\setminus 0$ for some $e\in E$ and $t'\in V(t)$;*

(iii) *$s'et\in E\setminus 0$ for some $e\in E$ and $s'\in V(s)$;*

(iv) *$se=ft\neq 0$ for some $e, f\in E\setminus 0$;*

(v) *$EsE\cap EtE\neq\{0\}$.*

Proof. One can prove the equivalence of conditions (ii)—(v) in the same fashion as the equivalence of conditions (2), (3), (6) and (8) in Lemma 1.3 [14]. One needs only to investigate whether products are 0 or not. Let us see, for example, the proof of the implication (v)$\Rightarrow$(ii). Suppose that $esf=gth\neq 0$ for some $e, f, g, h\in E$, and let $s'\in V(s)$, $t'\in V(t)$. Then $(s'es)f\in E$ and $s((s'es)f)t'=(ss')g(tht')\in E$. If it were 0 then

$$0 = e(ss')g(tht')th = e(ss')gt(t'th)^2 = e(ss')gth =$$

$$= e(ss')esf = (ess')^2 sf = (ess')sf = esf$$

would follow, a contradiction. Thus $s((s'es)f)t'\in E\setminus 0$.

Similarly to the proof of Lemma 1.3 [14], one can check that the relation $\varkappa$ consisting of the pair (0, 0) and the pairs (s, t) satisfying (ii)—(v) is a 0-restricted congruence on S. We intend to show that $\varkappa=\varrho$. First observe that $S/\varkappa$ is a primitive inverse semigroup. Indeed, if $e, f\in E$ with $ef\neq 0$ then $ef=eef=eff\in EeE\cap EfE$ and hence $e\varkappa f$. Now let τ be any 0-restricted primitive inverse semigroup congruence on S and let $e, f\in E$ with $se=ft\neq 0$. Then $s\tau\cdot e\tau=f\tau\cdot t\tau\neq 0$ in the primitive inverse semigroup S/τ. Hence we infer that $(s\tau)^{-1}\cdot s\tau=e\tau=f\tau=t\tau\cdot(t\tau)^{-1}$ which implies $s\tau=s\tau\cdot e\tau=f\tau\cdot t\tau=t\tau$. Thus $\varkappa\subseteq\tau$, completing the proof of the fact that $\varkappa=\varrho$.

A regular semigroup S with 0 is called *$E\setminus 0$-unitary* if $E\setminus 0$ is a unitary subset in S. Let S be an $E\setminus 0$-unitary regular semigroup with 0. If $e\in E\setminus 0$ and $e'\in V(e)$ then $ee'\in E\setminus 0$. Since $E\setminus 0$ is a left unitary subset in S we deduce that $e'\in E\setminus 0$. Thus S is orthodox.

Proposition 1.3. *Every $E\setminus 0$-unitary regular semigroup with* 0 *is orthodox.*

Thus there exists a least 0-restricted primitive inverse semigroup congruence on every $E\setminus 0$-unitary regular semigroup being categorical at 0. The analogue of Result 1.1 holds:

Proposition 1.4: *Let $S=S^0$ be an orthodox semigroup which is categorical at* 0. *Then the following conditions are equivalent:*

(i) *S is $E\setminus 0$-unitary;*

(ii) *$E\setminus 0$ is a left unitary subset in S;*

(iii) *$E\setminus 0$ is a right unitary subset in S;*

(iv) Ker $\varrho = E$.

Proof. The equivalence of conditions (ii) and (iv) is easily verified by making use of the equivalence of (i) and (iii), (iv) in Proposition 1.2. The equivalence (iii)$\Leftrightarrow$(iv) follows by symmetry, and (i) is equivalent to (iii) and (ii) by definition.

For an $E\setminus 0$-unitary regular semigroup S which is categorical at 0, the congruence ϱ can be described as follows:

Proposition 1.5. *Let* $S=S^0$ *be an* $E\setminus 0$*-unitary regular semigroup which is categorical at* 0. *Then*

$$\varrho = \{(s, t)\colon\ st'\in E\setminus 0 \text{ for some } t'\in V(t)\}\cup\{(0, 0)\}.$$

Proof. Denote the relation on the right hand side of the equality by $\varkappa$. It is clear by Proposition 1.2 that $\varkappa\subseteq\varrho$. Suppose now that $s, t\neq 0$ and $s\varrho t$ in S. Then there exist $e, f\in E$ with $es=tf\neq 0$. This implies $est'=tft'\in E\setminus 0$. Since S is $E\setminus 0$-unitary we obtain that $st'\in E\setminus 0$, that is, $s\varkappa t$. Thus the reverse inclusion $\varrho\subseteq\varkappa$ also holds.

In Sections 2, 3 and 4 we will need the following facts:

Lemma 1.6. *Let* $S=S^0$ *be an orthodox semigroup which is categorical at* 0. *If there exists a* 0*-restricted homomorphism* φ *of* S *onto a primitive inverse semigroup such that* $\operatorname{Ker}\varphi\subseteq E$ *then* S *is* $E\setminus 0$*-unitary and* $\ker\varphi=\varrho$.

Proof. Since $\ker\varphi$ is a 0-restricted primitive inverse semigroup congruence we have $\varrho\subseteq\ker\varphi$. Therefore $\operatorname{Ker}\varrho\subseteq\operatorname{Ker}\varphi\subseteq E$. However, $E\subseteq\operatorname{Ker}\varrho$ trivially holds whence we infer $\operatorname{Ker}\varrho=\operatorname{Ker}\varphi=E$. Then, by Proposition 1.4, it follows that S is $E\setminus 0$-unitary. Let $s, t\in S\setminus 0$ be such that $s\varphi=t\varphi$ and let $t'\in V(t)$. Then $(st')\varphi=$ $=s\varphi\cdot(t\varphi)^{-1}=t\varphi\cdot(t\varphi)^{-1}\in E_{S\varphi}$ which implies $st'\in E_S=E$. Thus, by Proposition 1.5, we have $s\varrho t$, completing the proof of the inclusion $\ker\varphi\subseteq\varrho$.

In order to simplify the notations later on, we will denote by $B(I)$ the $\mathscr{H}$-trivial Brandt semigroup $(I\times I)\cup 0$ with multiplication

$$[i, j][k, l] = \begin{cases}[i, l] & \text{if } j = k\\ 0 & \text{otherwise}\end{cases}$$

and

$$0[i, j] = [i, j]0 = 0\cdot 0 = 0.$$

It is well known that every $\mathscr{H}$-trivial Brandt semigroup is isomorphic to $B(I)$ for some set I.

Lemma 1.7. *Let* $S=S^0$ *be an* $E\setminus 0$*-unitary regular semigroup which is categorical at* 0 *and for which* S/ϱ *is an* $\mathscr{H}$*-trivial Brandt semigroup. Then the only* 0*-restricted primitive inverse semigroup congruence on* S *is* ϱ.

Proof. A 0-restricted primitive inverse semigroup congruence properly containing ϱ cannot exist as $\mathscr{H}$-trivial Brandt semigroups are congruence-free.

If S is a semigroup with 0 then the partial groupoid obtained from S by eliminating 0 and letting products be undefined if they are equal to 0 in S will be denoted by $\hat{S}$.

Given a partial groupoid $(X;\ \cdot)$, let us adjoin a new symbol $0(\notin X)$ to X and extend the multiplication to $X\cup 0$ in such a way that $x\cdot 0=0\cdot x=0\cdot 0=0$ for every $x\in X$ and $x\cdot y=0$ provided $x, y\in X$ and $x\cdot y$ is not defined in X. The groupoid obtained in this fashion is denoted by $\check{X}$. If $\check{X}$ is a semigroup then we term X a *partial semigroup.*

The basic concepts of semigroup theory such as left, right ideals, Green's relations, inverse of an element, regularity, automorphisms can be defined in a partial semigroup X in the same way as in $\check{X}$. For example, a non-empty subset $R\subseteq X$ is said to be a *right ideal* in X if $\{r\cdot x\colon r\in R,\ x\in X$ and $r\cdot x$ is defined$\}\subseteq R$. Clearly, R is a right ideal in X if and only if $R\cup 0$ is a non-trivial right ideal in $\check{X}$. One can easily see that, for example, the set of all idempotent elements in X is $E_{\check{X}}\setminus 0$, Green's relation $\mathcal{R}$ on X is just the restriction of the $\mathcal{R}$-relation of $\check{X}$ to X, the set of inverses of an element x in X is equal to $V_{\check{X}}(x)$ and $\alpha\colon X\to X$ is an automorphism of X if and only if $\check{\alpha}\colon \check{X}\to\check{X}$ defined by $0\check{\alpha}=0$ and $x\check{\alpha}=x\alpha$ $(x\in X)$ is an automorphism. Therefore it is not ambiguous to write $\mathcal{R}$ or $V(x)$ without indicating whether they are considered on X or on $\check{X}$. If we want to emphasize that the set of inverses is considered in X then we write $V_X(x)$. Moreover, we will use the notation E_Y for the set of all idempotent elements in a subset Y of X and $V_X(Y)$ or, simply, $V(Y)$ for $\bigcup\{V_X(a)\colon a\in Y\}$.

Let G be a group and S a full or partial semigroup. We say that G *acts on* S if a homomorphism $\varphi\colon G\to(\mathrm{Aut}\,S)^d$ is given where $(\mathrm{Aut}\,S)^d$ is the dual of the automorphism group of S. For every $s\in S$ and $g\in G$, we denote $s(g\varphi)$ by gs.

Let G be a group and S a semigroup with 0 on which G acts. Define a multiplication on the set $\big((S\setminus 0)\times G\big)\cup 0$ by

$$(s,g)(t,h)=\begin{cases}(s\cdot gt,\ gh) & \text{if } s\cdot gt\neq 0\\ 0 & \text{otherwise}\end{cases}$$

and

$$0\cdot(s,g)=(s,g)\cdot 0=0\cdot 0=0$$

for every $s, t\in S\setminus 0$ and $g, h\in G$. It is not difficult to check that this multiplication is associative. The semigroup obtained in this way is called the *0-semidirect product of* S *by* G and is denoted by $S*_0G$.

If G is a group acting on a semigroup S without 0 then $(S^0*_0G)\setminus 0$ is a semigroup termed the *semidirect product of* S *by* G and is denoted by $S*G$.

Let X be a partial semigroup and G a group acting on X. Let $\varphi\colon G\to(\mathrm{Aut}\,X)^d$ be the homomorphism defining this action. Then $\check{\varphi}\colon G\to(\mathrm{Aut}\,\check{X})^d$, $g\check{\varphi}=\widecheck{g\varphi}$ is a homomorphism. Since $x(g\check{\varphi})=x(g\varphi)$ for every $x\in X$ and $g\in G$, it is not confusing to denote $x(g\check{\varphi})$ also by gx. By the *semidirect product* $X*G$ we mean the partial semigroup $\widehat{\check{X}*_0G}$.

2. On E-unitary regular semigroups

By McAlister's P-theorem [6], every E-unitary inverse semigroup S is isomorphic to a P-semigroup $P(G, \mathcal{X}, \mathcal{Y})$ where G is a group, $\mathcal{X}$ is a partially ordered set on which G acts by order automorphisms, $\mathcal{Y}$ is an order ideal in $\mathcal{X}$ such that $\mathcal{Y}$ is a lower semilattice and $P(G, \mathcal{X}, \mathcal{Y})$ is, actually, a well-determined subsemigroup in the semidirect product of the "partial semilattice" $\mathcal{X}$ by G. Takizawa ([14]; cf. also [11]) generalized this result by proving that every E-unitary $\mathcal{R}$-unipotent semigroup S is isomorphic to a so-called PL-semigroup constructed in a similar way as a P-semigroup by means of a group, an $\mathcal{R}$-trivial "partial idempotent semigroup" $\mathcal{X}$ on which G acts and by means of a subband $\mathcal{Y}$ of $\mathcal{X}$ forming an order ideal in $\mathcal{X}$ with respect to the natural partial order $\leqq_{\mathcal{X}}$. In both cases the triple $(G, \mathcal{X}, \mathcal{Y})$ can be chosen in such a way that G is isomorphic to G_S and $\mathcal{Y}$ to E_S.

The proofs of McAlister's and Takizawa's results are based on the observation that $\mathcal{R}\cap\sigma=\iota$ (ι is the identity relation) holds in an E-unitary inverse [$\mathcal{R}$-unipotent] semigroup (cf. [6] and [14]). Hence the elements of an E-unitary inverse [$\mathcal{R}$-unipotent] semigroup S can be coordinatized with pairs from $E_S\times G_S$.

When we intend to generalize these results for E-unitary regular semigroups the difficulty lies in the fact that, in an arbitrary E-unitary regular semigroup S, we have no such natural coordinatization of elements as in the case of E-unitary inverse [$\mathcal{R}$-unipotent] semigroups. The analogue of that coordinatization would be the injection $S\to E_S/\mathcal{R}\times G_S\times E_S/\mathcal{L}$ defined by $s\mapsto(R_{ss'}, s\sigma, L_{s's})$ where $s'\in V(s)$. However, it seems very complicated to determine in an abstract way which triples are coordinates of an element, how the coordinates are multiplied and what conditions they have to satisfy in order that the groupoid defined in this way be an E-unitary regular semigroup. Therefore we looked for another way of characterizing E-unitary regular semigroups. We cannot expect to obtain a construction analogous to P-semigroups which produced all E-unitary regular semigroups up to isomorphisms and in which $\mathcal{Y}$ were isomorphic to E_S. In finding a generalization of the P-theorem for E-unitary regular semigroups, we tried to preserve the other main feature of McAlister's and Takizawa's results, namely, we wanted to obtain an E-unitary regular semigroup as a well-determined subsemigroup of a semidirect product of a certain partial groupoid by a group. We imitate the proof of the P-theorem due to Munn [7] and that of Theorem 3.1 in [14]. The new idea in our case is that the partial groupoid $\mathcal{X}$ is defined on $S\times G_S$ instead of $E_S\times G_S$.

Let S be an E-unitary regular semigroup. Define a partial groupoid $\mathcal{X}=(S\times G_S;\circ)$ as follows:

$$(1)\qquad \begin{gathered}(s,g)\circ(t,h)\ \text{ is defined if and only if }\ s\sigma=g^{-1}h,\\ \text{and in this case }\ (s,g)\circ(t,h)=(st,g).\end{gathered}$$

Put $\mathcal{Y}=\{(s,1)\colon s\in S\}$.

In the sequel we prove several properties of the triple $(G_S, \mathscr{X}, \mathscr{Y})$.

(I) *$\check{\mathscr{X}}$ is an orthodox semigroup which is categorical at* 0. *Moreover,* $E_{\check{\mathscr{X}}}=E_S\times G_S$ *and, for every* $(s, g)\in\mathscr{X}$, *we have* $V_{\check{\mathscr{X}}}((s, g))=\{(s', g\cdot s\sigma): s'\in V_S(s)\}$.

Proof. Let $(s, g), (t, h), (u, k)\in\mathscr{X}$. It is clear by (1) that we have $(s, g)\circ(t, h)=0$ in $\check{\mathscr{X}}$ if and only if $s\sigma\neq g^{-1}h$. Suppose first that $(s, g)\circ(t, h)\neq 0$ and $(t, h)\circ(u, k)\neq 0$. Then $s\sigma=g^{-1}h$ and $t\sigma=h^{-1}k$ which imply that $(st)\sigma=g^{-1}k$. Hence it follows by (1) that $((s, g)\circ(t, h))\circ(u, k)=(st, g)\circ(u, k)=(stu, g)=(s, g)\circ(tu, h)=(s, g)\circ((t, h)\circ(u, k))\neq 0$. If $(s, g)\circ(t, h)=0$ and $(t, h)\circ(u, k)\neq 0$, then $s\sigma\neq g^{-1}h$ which implies by (1) that $(s, g)\circ((t, h)\circ(u, k))=(s, g)\circ(tu, h)=0$. If $(s, g)\circ(t, h)\neq 0$ and $(t, h)\circ(u, k)=0$, then $s\sigma=g^{-1}h$ and $t\sigma\neq h^{-1}k$ whence we infer that $(st)\sigma\neq g^{-1}k$. Therefore $((s, g)\circ(t, h))\circ(u, k)=(st, g)\circ(u, k)=0$. Thus we have shown that $\check{\mathscr{X}}$ is a semigroup which is categorical at 0.

Let $(s, g)\in\mathscr{X}$. Now we determine $V_{\check{\mathscr{X}}}((s, g))$. Making use of the fact that $s'\sigma=(s\sigma)^{-1}$ for each $s'\in V_S(s)$, one can easily check that $(s', g\cdot s\sigma)\in V_{\check{\mathscr{X}}}((s, g))$ for every $s'\in V_S(s)$. If $(t, h)\in V_{\check{\mathscr{X}}}((s, g))$ then (1) implies $t\in V_S(s)$ and, since $(s, g)\circ(t, h)\neq 0$, we have $s\sigma=g^{-1}h$. So it is verified that $V_{\check{\mathscr{X}}}((s, g))$ consists of those elements indicated in the assertion. In particular, we obtain that $\check{\mathscr{X}}$ is regular.

It remains to determine $E_{\check{\mathscr{X}}}$. It is obvious that $(e, g)\in E_{\check{\mathscr{X}}}$ for any $e\in E_S$ and $g\in G_S$. Assume that $(e, g)\in E_{\check{\mathscr{X}}}$. Then $(e, g)\circ(e, g)=(e, g)$, that is, $e\sigma=g^{-1}g=1$ and $e^2=e$. Clearly, $E_{\check{\mathscr{X}}}$ is a band because, for every $(e, g), (f, h)\in E_{\check{\mathscr{X}}}$, we have

$$(e, g)\circ(f, h)=\begin{cases}(ef, g) & \text{if } g=h\\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

This implies that $\check{\mathscr{X}}$ is orthodox. The proof is complete.

(II) *The mapping* $\varphi\colon \check{\mathscr{X}}\to B(G_S)$ *defined by* $(s, g)\varphi=[g, g\cdot s\sigma]$ *and* $0\varphi=0$ *is a surjective* 0*-restricted homomorphism with* $\operatorname{Ker}\varphi\subseteq E_{\check{\mathscr{X}}}$. *Consequently,* $\check{\mathscr{X}}$ *is* $E\setminus 0$*-unitary,* $\ker\varphi=\varrho$, *the least* 0*-restricted primitive inverse semigroup congruence on* $\check{\mathscr{X}}$ *and* $\check{\mathscr{X}}/\varrho$ *is an* $\mathscr{H}$*-trivial Brandt semigroup.*

Proof. Let $(s, g), (t, h)\in\mathscr{X}$. If $s\sigma=g^{-1}h$, that is, $g\cdot s\sigma=h$ then

$$((s, g)\circ(t, h))\varphi=(st, g)\varphi=[g, g\cdot(st)\sigma]=$$
$$=[g, h\cdot t\sigma]=[g, g\cdot s\sigma]\cdot[h, h\cdot t\sigma]=(s, g)\varphi\cdot(t, h)\varphi.$$

If $s\sigma\neq g^{-1}h$, that is, $g\cdot s\sigma\neq h$ then

$$((s, g)\circ(t, h))\varphi=0\varphi=0=[g, g\cdot s\sigma][h, h\cdot t\sigma]=(s, g)\varphi\cdot(t, h)\varphi.$$

Thus φ is a 0-restricted homomorphism. It is surjective because $G_S=S/\sigma$. Since S is E-unitary, $s\sigma=1$ implies $s\in E_S$. Therefore, by (I), $\operatorname{Ker}\varphi\subseteq E_{\check{\mathscr{X}}}$. By Lemma

1.6, this ensures that $\check{\mathscr{X}}$ is $E\setminus 0$-unitary and ker φ is the least 0-restricted primitive inverse semigroup congruence ϱ.

(III) *$\mathscr{Y}$ is a maximal right ideal in $\mathscr{X}$ with the property that $E_{\mathscr{Y}}$ is a subband in $\mathscr{Y}$.*

Proof. It is straightforward by (1) and (2) that $\mathscr{Y}$ is a right ideal in $\mathscr{X}$ and $E_{\mathscr{Y}}$ is a subband in $\mathscr{Y}$. Suppose now that $\mathscr{Y}_1$ is a right ideal in $\mathscr{X}$ such that $E_{\mathscr{Y}_1}$ is a subband in $\mathscr{Y}_1$ and $\mathscr{Y}\subseteqq\mathscr{Y}_1$. Then (2) implies $E_{\mathscr{Y}}=E_{\mathscr{Y}_1}$. Let $(s, g)\in\mathscr{Y}_1$. Since $\mathscr{Y}_1$ is a right ideal in $\mathscr{X}$ we infer by (I) that $(ss', g)=(s, g)\circ(s', g\cdot s\sigma)\in\mathscr{Y}_1\cap E_{\mathscr{X}}= =E_{\mathscr{Y}_1}=E_{\mathscr{Y}}$ for every $s'\in V_S(s)$. Thus $g=1$ and $(s, g)\in\mathscr{Y}$ proving that $\mathscr{Y}_1\subseteqq\mathscr{Y}$. The proof is complete.

Let us define an action of G_S on $\mathscr{X}$ as follows: for every (s, g) and $h\in G_S$ let $h(s, g)=(s, hg)$.

(IV) *G_S acts on $\mathscr{X}$ such that $G_S\mathscr{Y}=\mathscr{X}$ and, for every $g\in G_S$, there exists $a\in\mathscr{Y}$ with $ga\in V_{\mathscr{X}}(\mathscr{Y})$.*

Proof. By (1), one can immediately check that, for every $h\in G_S$, the mapping $\tilde{h}\colon \mathscr{X}\to\mathscr{X}$ defined by $(s, g)\tilde{h}=(s, hg)$ is an automorphism and $\widetilde{kh}=\tilde{h}\tilde{k}$ for every $h, k\in G_S$. The equality $G_S\mathscr{Y}=\mathscr{X}$ is a trivial consequence of the definition of the action. In order to verify the last assertion, observe that, by (I), we have $V_{\mathscr{X}}(\mathscr{Y})= =\{(s', s\sigma)\colon s\in S,\ s'\in V_S(s)\}=\{(t, h)\in\mathscr{X}\colon t\sigma=h^{-1}\}$. Since $G_S=S/\sigma$, for any $g\in G_S$, there exists $s\in S$ with $s\sigma=g^{-1}$. For such an s we have $g(s, 1)=(s, g)\in V_{\mathscr{X}}(\mathscr{Y})$.

As an easy consequence of the equality obtained here for $V_{\mathscr{X}}(\mathscr{Y})$ we deduce

(V) *For every $(s, 1)\in\mathscr{Y}$ and $g\in G_S$, we have $g^{-1}(s, 1)\in V_{\mathscr{X}}(\mathscr{Y})$ if and only if $s\sigma=g$.*

(VI) *The mapping $\varepsilon\colon S\to\mathscr{X}*G_S$ defined by $s\varepsilon=((s, 1), s\sigma)$ is an embedding of S into $\mathscr{X}*G_S$. In particular, S is isomorphic to the subsemigroup $\{(a, g)\in\mathscr{Y}\times G_S\colon g^{-1}a\in V_{\mathscr{X}}(\mathscr{Y})\}$.*

Proof. The mapping ε is clearly injective and, by (V), its range is $\{(a, g)\in \in\mathscr{Y}\times G_S\colon g^{-1}a\in V_{\mathscr{X}}(\mathscr{Y})\}$. All we have to check is that ε is a homomorphism. Let $s, t\in S$. Then, by (1), we have

$$s\varepsilon\cdot t\varepsilon=((s, 1), s\sigma)((t, 1), t\sigma)=((s, 1)\circ s\sigma(t, 1), s\sigma\cdot t\sigma)=$$
$$=((s, 1)\circ(t, s\sigma), (st)\sigma)=((st, 1), (st)\sigma)=(st)\varepsilon$$

which completes the proof.

Statement (VI) shows that we succeeded in finding a partial semigroup $\mathscr{X}$ on which G_S acts such that S is isomorphic to a well-determined subsemigroup of $\mathscr{X}*G_S$.

3. PO-semigroups and 0-semidirect products of strictly combinatorial semigroups by groups

In this section we introduce the concept of a PO-triple and a PO-semigroup so as it is inspired by the results of the preceding section and give a description of E-unitary regular semigroups by means of PO-semigroups and by means of 0-semidirect products of strictly combinatorial semigroups by groups.

A regular semigroup S with 0 is called *strictly combinatorial* if (i) S is categorical at 0, (ii) S is $E\setminus 0$-unitary and (iii) S/ϱ is an $\mathcal{H}$-trivial Brandt semigroup.

A partial semigroup X is termed *strictly combinatorial* if $\check{X}$ is a strictly combinatorial semigroup.

An $\mathcal{H}$-trivial semigroup is sometimes called combinatorial. In order to justify the terminology just introduced we show that a strictly combinatorial semigroup is necessarily $\mathcal{H}$-trivial. Let S be a strictly combinatorial semigroup and s an element in a non-zero subgroup of S. Then there exists an inverse s' of s in this subgroup and thus $ss'=s's\neq 0$. Hence we have $(s\varrho)(s\varrho)^{-1}=(s\varrho)^{-1}(s\varrho)\neq 0$ in the factor semigroup S/ϱ which is an $\mathcal{H}$-trivial Brandt semigroup. This implies that $s\varrho$ is idempotent and thus $s\in \operatorname{Ker}\varrho\setminus 0$. Since S is $E\setminus 0$-unitary, we infer by Proposition 1.4 that s is idempotent. Thus we verified that each subgroup in S is trivial which implies that S is $\mathcal{H}$-trivial.

Now we define the notions which will play the role of the McAlister triple and the P-semigroup.

Let G be a group, $(\mathcal{X};\circ)$ a strictly combinatorial partial semigroup and $\mathcal{Y}$ a subset in $\mathcal{X}$. Suppose that

(PO1) $\mathcal{Y}$ is a right ideal in $(\mathcal{X};\circ)$ and $E_{\mathcal{Y}}$ is a subband in $\mathcal{Y}$;

(PO2) G acts on $(\mathcal{X};\circ)$;

(PO3) $G\mathcal{Y}=\mathcal{X}$;

(PO4) for every $g\in G$, there exists $a\in\mathcal{Y}$ with $ga\in V(\mathcal{Y})$.

The triple $(G,\mathcal{X},\mathcal{Y})$ satisfying the above conditions is called a *PO-triple*. If

(M) $\mathcal{Y}$ is a maximal right ideal in $(\mathcal{X};\circ)$ with the property that $E_{\mathcal{Y}}$ is a subband in $\mathcal{Y}$, then $(G,\mathcal{X},\mathcal{Y})$ is termed a *POM-triple*.

Given a PO-triple $(G,\mathcal{X},\mathcal{Y})$, we define a multiplication on the set

$$PO(G,\mathcal{X},\mathcal{Y})=\{(a,g)\in\mathcal{Y}\times G:\ g^{-1}a\in V(\mathcal{Y})\}$$

by

(3) $$(a,g)(b,h)=(a\circ gb,\ gh).$$

Property (PO4) ensures that the image of $PO(G,\mathcal{X},\mathcal{Y})$ under the second projection is just G. The following lemma shows that the image of $PO(G,\mathcal{X},\mathcal{Y})$ under the first projection is $\mathcal{Y}$.

Lemma 3.1 *Let* $(G, \mathscr{X}, \mathscr{Y})$ *be a PO-triple. Then, for every* $a\in\mathscr{Y}$, *there exists* $g\in G$ *with* $ga\in V(\mathscr{Y})$.

Proof. If $a\in\mathscr{Y}$ and $a'\in V(a)$, then, by (PO3), we have $a'=hb$ for some $h\in G$ and $b\in\mathscr{Y}$. Thus, by (PO2), $h^{-1}a\in V(h^{-1}a')=V(b)\subseteq V(\mathscr{Y})$.

Proposition 3.2. *Let* $(G, \mathscr{X}, \mathscr{Y})$ *be a PO-triple.*

(i) $PO(G, \mathscr{X}, \mathscr{Y})$ *is an E-unitary regular semigroup and* $E_{PO(G,\mathscr{X},\mathscr{Y})}=\{(e, 1): e\in E_{\mathscr{Y}}\}$ *is isomorphic to* $E_{\mathscr{Y}}$.

Moreover, for any $(a, g), (b, h)\in PO(G, \mathscr{X}, \mathscr{Y})$, *we have*

(ii) $V_{PO(G,\mathscr{X},\mathscr{Y})}((a, g))=\{(c, g^{-1}):\ c\in\mathscr{Y}\cap V(g^{-1}a)\}$;

(iii) $(a, g)\mathscr{R}(b, h)$ *if and only if* $a\mathscr{R}b$;

(iv) $(a, g)\mathscr{L}(b, h)$ *if and only if* $g^{-1}a\mathscr{L}h^{-1}b$;

(v) $(a, g)\gamma(b, h)$ *if and only if* $g=h$ *and* $V(g^{-1}a)\cap V(h^{-1}b)\cap\mathscr{Y}\neq\square$;

(vi) $(a, g)\sigma(b, h)$ *if and only if* $g=h$.

(vii) $PO(G, \mathscr{X}, \mathscr{Y})/\sigma$ *is isomorphic to* G.

The E-unitary regular semigroup $PO(G, \mathscr{X}, \mathscr{Y})$ is called the *PO-semigroup determined by the PO-triple* $(G, \mathscr{X}, \mathscr{Y})$ or, simply, a *PO-semigroup*.

Proof. For brevity, denote $PO(G, \mathscr{X}, \mathscr{Y})$ by S.

(i) First of all, we have to show that S is closed under the multiplication defined by (3). Let $(a, g), (b, h)\in S$. Then $g^{-1}a\in V(a^+)$ and $h^{-1}b\in V(b^+)$ for some $a^+, b^+\in\mathscr{Y}$. Since $\mathscr{Y}$ is a right ideal in $\mathscr{X}$ this implies by (PO2) that $a^+\circ g^{-1}a$ and $b\circ hb^+$ belong to $E_{\mathscr{Y}}$. As $E_{\mathscr{Y}}$ is a subband in $\mathscr{Y}$ by (PO1), the product $(a^+\circ g^{-1}a)\circ(b\circ hb^+)$ is defined and thus $g^{-1}a\circ(a^+\circ g^{-1}a)\circ(b\circ hb^+)\circ b=g^{-1}a\circ b$ is also defined. From this it follows by (PO2) that $a\circ gb$ is defined and, since $\mathscr{Y}$ is a right ideal in $\mathscr{X}$, it belongs to $\mathscr{Y}$. Moreover, we obtain that $(gh)^{-1}(a\circ gb)=h^{-1}(g^{-1}a)\circ h^{-1}b$ is also defined in $\mathscr{X}$, that is, it is not equal to 0 in $\check{\mathscr{X}}$. Since the strictly combinatorial semigroup $\check{\mathscr{X}}$ is orthodox by Proposition 1.3, we infer that $h^{-1}(g^{-1}a)\circ h^{-1}b\in V(b^+\circ h^{-1}a^+)$. Hence $b^+\circ h^{-1}a^+\neq 0$, that is, the product $b^+\circ h^{-1}a^+$ is defined in $\mathscr{X}$. Since $\mathscr{Y}$ is a right ideal in $\mathscr{X}$ and $b^+\in\mathscr{Y}$, we have $b^+\circ h^{-1}a^+\in\mathscr{Y}$. Thus $(gh)^{-1}(a\circ gb)\in V(\mathscr{Y})$, completing the proof of the fact that S is closed under multiplication (3).

A straightforward calculation shows that the multiplication defined by (3) is associative. Now we turn to proving the regularity of S. Observe that it suffices to verify (ii). For, if $(a, g)\in S$ then $g^{-1}a\in V(\mathscr{Y})$. Therefore there exists $b\in\mathscr{Y}\cap V(g^{-1}a)$ and hence $gb\in V(a)\subseteq V(\mathscr{Y})$. Thus $(b, g^{-1})\in S$ is an inverse of (a, g). The element (b, h) is an inverse of (a, g) if and only if $(a, g)=(a, g)(b, h)(a, g)=(a\circ gb\circ gha, ghg)$ and $(b, h)=(b, h)(a, g)(b, h)=(b\circ ha\circ hgb, hgh)$, that is, if and only if $h=g^{-1}$ and $a\circ gb\circ a=a, b\circ g^{-1}a\circ b=b$. The latter equalities are equivalent by (PO2) to the condition that $b\in V(g^{-1}a)$.

It is easy to see that $E_S = \{(e, 1)\colon e \in E_{\mathscr{Y}}\}$ which is a band with respect to the multiplication defined in (3) and E_S is isomorphic to $E_{\mathscr{Y}}$.

Now we prove that the homomorphism $\varphi\colon S \to G$, $(a, g)\varphi = g$ is onto and $\operatorname{Ker} \varphi \subseteq E_S$. This implies that $\ker \varphi = \sigma$ and thus, by Result 1.1, S is E-unitary and (vi), (vii) hold. Property (PO4) of the PO-triple $(G, \mathscr{X}, \mathscr{Y})$ ensures that φ is onto. Assume that $(a, g) \in \operatorname{Ker} \varphi$. Then $g = 1$ and $(a, 1)^2 \in S$ whence $a \circ a \in \mathscr{Y}$. Since $\check{\mathscr{X}}/\varrho$ is and $\mathscr{H}$-trivial Brandt semigroup and in such a Brandt semigroup a square of an element x is non-zero if and only if x is idempotent we infer that $a\varrho$ is a non-zero idempotent in $\check{\mathscr{X}}/\varrho$. However, $\operatorname{Ker} \varrho = E_{\check{\mathscr{X}}}$ by Proposition 1.4. This implies $a \in E_{\mathscr{X}}$. Thus $(a, g) \in E_S$ which proves that $\operatorname{Ker} \varphi \subseteq E_S$. This completes the proof of (i), (ii), (vi) and (vii).

(iii) We have $(a, g)\mathscr{R}(b, h)$ if and only if there exist inverses $(a, g)'$ and $(b, h)'$ of (a, g) and (b, h), respectively, such that $(a, g)(a, g)'\mathscr{R}(b, h)(b, h)'$ in E_S. By applying (ii) we deduce that this holds if and only if there exist $a^+ \in V(g^{-1}a)$ and $b^+ \in V(h^{-1}b)$ such that $a \circ ga^+ \mathscr{R} b \circ hb^+$ in $E_{\mathscr{Y}}$. Since $a \mathscr{R} a \circ ga^+$ and $b \mathscr{R} b \circ hb^+$ in $\mathscr{X}$, this is equivalent to requiring that $a \mathscr{R} b$.

(iv) is proved dually to (iii).

(v) is an immediate consequence of (ii). The proof is complete.

In the terminology introduced here the results of Section 2 can be formulated in such a way that the triple $(G_S, \mathscr{X}, \mathscr{Y})$ defined there is a POM-triple and S is isomorphic to $PO(G_S, \mathscr{X}, \mathscr{Y})$. Thus we deduce the following

Proposition 3.3. *Every E-unitary regular semigroup is isomorphic to a PO-semigroup defined by a POM-triple.*

It is clear that, for a given PO-triple $(G, \mathscr{X}, \mathscr{Y})$, the PO-semigroup $PO(G, \mathscr{X}, \mathscr{Y})$ is a regular subsemigroup without 0 in the 0-semidirect product $\check{\mathscr{X}} *_0 G$.

In the sequel we investigate the connection between PO-semigroups and regular subsemigroups without 0 in 0-semidirect products of a strictly combinatorial semigroup by a group.

Lemma 3.4. *Let G be a group acting on a semigroup T with 0. Then*

(i) *the 0-semidirect product $T *_0 G$ is a semigroup in which $E_{T *_0 G} = \{(e, 1)\colon e \in E_T \setminus 0\} \cup 0$ and $V_{T *_0 G}((t, g)) = \{(g^{-1}t', g^{-1})\colon t' \in V_T(t)\}$ for every $(t, g) \in T *_0 G$;*

(ii) *$T *_0 G$ is regular if and only if T is regular;*

(iii) *$T *_0 G$ is orthodox if and only if T is orthodox, and in this case, $E_{T *_0 G}$ is isomorphic to E_T;*

(iv) *$T *_0 G$ is categorical at 0 if and only if T is categorical at 0;*

(v) *if T is regular and categorical at 0 then $T *_0 G$ is $E \setminus 0$-unitary if and only if T is $E \setminus 0$-unitary.*

Proof. Statements (i)—(iv) can be easily proved therefore they are left to the reader. In order to prove (v) it suffices to check by Proposition 1.4 that $E_{T*_0G}\setminus 0$ is a left unitary subset in $T*_0G$ if and only if $E_T\setminus 0$ is a left unitary subset in T. Suppose first that $E_{T*_0G}\setminus 0$ is a left unitary subset in $T*_0G$, and let $e\in E_T$, $t\in T$ be such that $et\in E_T\setminus 0$. Then we have $(e, 1)(t, 1)=(et, 1)\in E_{T*_0G}\setminus 0$ and $(e, 1)\in E_{T*_0G}\setminus 0$ by (i), which imply that $(t, 1)\in E_{T*_0G}\setminus 0$. Thus, again applying (i), we infer that $t\in E_T\setminus 0$. Conversely, suppose that $E_T\setminus 0$ is a left unitary subset in T and $(e, 1)\in E_{T*_0G}$, $(t, g)\in T*_0G$ with $(e, 1)(t, g)\in E_{T*_0G}\setminus 0$. Then, by (i), we obtain that $g=1$ and $e, et\in E_T\setminus 0$. Hence it follows that $t\in E_T\setminus 0$, that is, $(t, g)\in E_{T*_0G}\setminus 0$. The proof is complete.

Lemma 3.4 implies that if T is strictly combinatorial then $T*_0G$ is an $E\setminus 0$-unitary regular semigroup which is categorical at 0. Consequently, every regular subsemigroup without 0 in $T*_0G$ is E-unitary. Hence we obtain

Proposition 3.5. *Every regular subsemigroup without 0 in a 0-semidirect product of a strictly combinatorial semigroup by a group is E-unitary.*

Now we turn to investigating the connection between PO-semigroups defined by POM-triples and maximal subsemigroups without 0 in 0-semidirect products of strictly combinatorial semigroups by groups. First of all, we determine the maximal subsemigroups without 0 in a 0-semidirect product of a strictly combinatorial semigroup by a group.

Lemma 3.6. *Let G be a group acting on a strictly combinatorial semigroup T. Then, in $T*_0G$, the maximal subsemigroups without 0 are*

$$M_i = \{(t, g)\in T*_0G\colon\ t\varrho\mathcal{R}i\mathcal{L}(g^{-1}t)\varrho\} \tag{4}$$

where $i\in E_{T/\varrho}\setminus 0$, and every subsemigroup without 0 in $T*_0G$ is contained in M_i for a unique $i\in E_{T/\varrho}\setminus 0$.

Proof. Since, in an $\mathcal{H}$-trivial Brandt semigroup, the only subsemigroups without 0 are the singletons containing idempotents, it suffices to find a 0-restricted homomorphism ψ of $T*_0G$ onto an $\mathcal{H}$-trivial Brandt semigroup such that the inverse images of the idempotents are just the M_i's. We shall use for this purpose an $\mathcal{H}$-trivial Brandt semigroup $B(I)$ which is the image of T under some 0-restricted homomorphism $\varphi\colon T\to B(I)$ with $\ker\varphi=\varrho$. Since φ is 0-restricted, $t\in T\setminus 0$ implies $t\varphi\neq 0$. Denote by φ_n $(n=1, 2)$ the mapping of $T\setminus 0$ into I assigning the nth component of $t\varphi$ to t for each $t\in T\setminus 0$. Define the mapping $\psi\colon T*_0G\to B(I)$ by $0\psi=0$ and $(t, g)\psi=[t\varphi_1, (g^{-1}t)\varphi_2]$. We prove that ψ is a 0-restricted homomorphism. By definition, ψ is 0-restricted. Now let $(t, g), (u, h)\in T*_0G$. Observe that $(t, g)(u, h)=0$ if and only if $t\cdot gu=0$, that is, if and only if $g^{-1}t\cdot u=0$. Since

φ is 0-restricted, the latter equality is equivalent to $(g^{-1}t\cdot u)\varphi=0$. This holds if and only if $(g^{-1}t)\varphi_2\neq u\varphi_1$. Thus we see that if $(t,g)(u,h)=0$ then $(t,g)\psi\cdot\cdot(u,h)\psi=0$. Moreover, if $(t,g)(u,h)\neq 0$ then $(g^{-1}t)\varphi_2=u\varphi_1$ and hence

$$((t,g)(u,h))\psi=(t\cdot gu, gh)\psi=\left[(t\cdot gu)\varphi_1, ((gh)^{-1}(t\cdot gu))\varphi_2\right]=$$
$$=[t\varphi_1, (h^{-1}u)\varphi_2]=[t\varphi_1, (g^{-1}t)\varphi_2]\cdot[u\varphi_1, (h^{-1}u)\varphi_2]=(t,g)\psi\cdot(u,h)\psi.$$

Here we have utilized that $(t\cdot gu)\varphi=t\varphi\cdot(gu)\varphi$ implies $(t\cdot gu)\varphi_1=t\varphi_1$ and, similarly, $((gh)^{-1}(t\cdot gu))\varphi=((gh)^{-1}t\cdot h^{-1}u)\varphi=((gh)^{-1}t)\varphi\cdot(h^{-1}u)\varphi$ imply

$$((gh)^{-1}(t\cdot gu))\varphi_2=(h^{-1}u)\varphi_2.$$

The proof is complete.

Proposition 3.7. (i) *Let G be a group acting on a strictly combinatorial semigroup T. Let $i\in E_{T/\varrho}\setminus 0$. Define $\bar{G}=\{g\in G\colon t\varrho\mathcal{R}i\mathcal{L}(g^{-1}t)\varrho$ for some $t\in T\setminus 0\}$, $\mathcal{Y}=\{t\in T\setminus 0\colon t\varrho\mathcal{R}i\mathcal{L}(g^{-1}t)\varrho$ for some $g\in G\}$ and $\mathcal{X}=\{ga\colon g\in\bar{G},\ a\in\mathcal{Y}\}$. Define a partial operation on $\mathcal{X}$ by restricting the operation of T to $\mathcal{X}$ and define an action of $\bar{G}$ on $\mathcal{X}$ by restricting the action of G on T to $\bar{G}$ and $\mathcal{X}$. Then $(\bar{G},\mathcal{X},\mathcal{Y})$ is a POM-triple and $PO(\bar{G},\mathcal{X},\mathcal{Y})=M_i$ (cf. (4)).*

(ii) *Conversely, for every POM-triple $(G,\mathcal{X},\mathcal{Y})$, the PO-semigroup $PO(G,\mathcal{X},\mathcal{Y})$ is a maximal subsemigroup without 0 in $\check{\mathcal{X}}*_0G$.*

Proof. (i) First we show that $\bar{G}$ is a subgroup in G. If $g\in G$, $t\in T\setminus 0$ with $t\varrho\mathcal{R}i\mathcal{L}(g^{-1}t)\varrho$ and $t'\in V_T(t)$, then we have $g^{-1}t'\in V_T(g^{-1}t)$ and $t'\varrho\mathcal{L}i\mathcal{R}(g^{-1}t')\varrho$. The latter relation can be written in the form $(g^{-1}t')\varrho\mathcal{R}i\mathcal{L}(g(g^{-1}t'))\varrho$. Since T is regular, this shows that $g\in\bar{G}$ implies $g^{-1}\in\bar{G}$. Assume that $g,h\in\bar{G}$. Then, by definition, there exist $t,u\in T\setminus 0$ with $t\varrho\mathcal{R}i\mathcal{L}(g^{-1}t)\varrho$ and $u\varrho\mathcal{R}i\mathcal{L}(h^{-1}u)\varrho$. Thus, by (4), we have $(t,g),(u,h)\in M_i$. Making use of the fact that, by Lemma 3.6, M_i is a subsemigroup in $T*_0G$, we obtain that $(t,g)(u,h)\in M_i$. Hence we infer that $(t\cdot gu)\varrho\mathcal{R}i\mathcal{L}((gh)^{-1}(t\cdot gu))\varrho$ which implies that $gh\in\bar{G}$. Thus $\bar{G}$ is, indeed, a subgroup in G.

Now we verify that $\mathcal{Y}$ is a right ideal in $\mathcal{X}$. Let $a\in\mathcal{Y}$ and $x\in\mathcal{X}$ such that $ax\neq 0$ in T. Then we have $(ax)\varrho\mathcal{R}a\varrho\mathcal{R}i$. On the other hand, by the definitions of $\mathcal{X}$ and $\mathcal{Y}$, there exist $g\in\bar{G}$ and $b\in\mathcal{Y}$ with $x=gb$, and $i\mathcal{L}(h^{-1}b)\varrho$ for some $h\in G$. Clearly, $h\in\bar{G}$ and $((gh)^{-1}(ax))\varrho=((gh)^{-1}a\cdot h^{-1}b)\varrho\mathcal{L}(h^{-1}b)\varrho\mathcal{L}i$. Thus, indeed, ax belongs to $\mathcal{Y}$ provided $a\in\mathcal{Y}$, $x\in\mathcal{X}$ and $ax\neq 0$ in T. This implies that $\check{\mathcal{X}}$ is a subsemigroup in T. For, let $x,y\in\mathcal{X}$ such that $xy\neq 0$ in T. Suppose that $x=gb$ where $g\in\bar{G}$ and $b\in\mathcal{Y}$. Since $\mathcal{Y}$ is a right ideal in $\mathcal{X}$, we have $b\cdot g^{-1}y\in\mathcal{Y}$ whence $xy=g(b\cdot g^{-1}y)\in\mathcal{X}$.

Now we show that $\check{\mathcal{X}}$ is regular. Let $ga\in\mathcal{X}$ where $g\in\bar{G}$ and $a\in\mathcal{Y}$. Then $a\varrho\mathcal{R}i\mathcal{L}(h^{-1}a)\varrho$ for some $h\in\bar{G}$. If $a'\in V_T(a)$ then we have $(h^{-1}a')\varrho\mathcal{R}i\mathcal{L}a'\varrho$

which implies that $h^{-1}a'\in\mathscr{Y}\cap V_T(h^{-1}a)$. Hence we deduce that $ga'=(gh)(h^{-1}a')$ is an inverse of ga in $\mathscr{X}$ for every $a'\in V_T(a)$. Consequently,

$$(5)\qquad V_T(x) = V_{\mathscr{X}}(x) \quad \text{for any} \quad x\in\mathscr{X}.$$

In particular, we obtain that $\check{\mathscr{X}}$ is regular. Finally, we verify that $\check{\mathscr{X}}/\varrho_{\check{\mathscr{X}}}$ is an $\mathscr{H}$-trivial Brandt semigroup. It is easy to see by definition that $\mathscr{X}=\{t\in T\colon (ge)\varrho\mathscr{R}t\varrho\mathscr{L}(he)\varrho$ for some $g, h\in\bar{G}\}$ where $e\in E_T$ with $e\varrho=i$. Hence $\mathscr{X}$ is a union of ϱ-classes in T and $\widecheck{\mathscr{X}\varrho}$, which is isomorphic to $\check{\mathscr{X}}/\varrho_{\check{\mathscr{X}}}$, is an $\mathscr{H}$-trivial Brandt semigroup since T/ϱ is an $\mathscr{H}$-trivial Brandt semigroup. Thus we have shown that $\check{\mathscr{X}}$ is a strictly combinatorial semigroup.

Returning to the properties of $\mathscr{Y}$ observe that $E_{\mathscr{Y}}=\{e\in E_T\setminus 0\colon e\varrho=i\}$. This clearly implies that $E_{\mathscr{Y}}$ is a subband in $\mathscr{Y}$. Property (PO3) follows by the definition of $\mathscr{X}$ and that of the action of $\bar{G}$ on $\mathscr{X}$ while (PO4) is a consequence of the definitions of $\bar{G}$ and $\mathscr{Y}$ and of the fact (easily deduced from (5)) that

$$(6)\qquad V_{\mathscr{X}}(\mathscr{Y}) = V_T(\mathscr{Y}) = \{t\in T\setminus 0\colon t\varrho\mathscr{L}i\mathscr{R}(g^{-1}t)\varrho \text{ for some } g\in\bar{G}\}.$$

Thus $(\bar{G}, \mathscr{X}, \mathscr{Y})$ is a *PO*-triple.

Now we show that (M) is satisfied. Suppose that $(\bar{G}, \mathscr{X}, \mathscr{Y}_1)$ is a *PO*-triple and $\mathscr{Y}\subseteq\mathscr{Y}_1$. Since $\mathscr{Y}_1$ is a right ideal and $\mathscr{X}$ is regular, $y\in\mathscr{Y}_1$ implies $yy'\in E_{\mathscr{Y}_1}$ for any $y'\in V_{\mathscr{X}}(y)$. Therefore we obtain that $y\varrho\mathscr{R}(yy')\varrho=i$ as $E_{\mathscr{Y}_1}$ is a subband in $\mathscr{X}\subseteq T\setminus 0$ containing $E_{\mathscr{Y}}$. Assume that $y=ga$ for some $g\in\bar{G}$ and $a\in\mathscr{Y}$. Then there exists $h\in\bar{G}$ with $(h^{-1}a)\varrho\mathscr{L}i$ whence we obtain that $((gh)^{-1}y)\varrho=(h^{-1}a)\varrho\mathscr{L}i$. Thus $y\in\mathscr{Y}$ is proved. Hence $\mathscr{Y}=\mathscr{Y}_1$ and therefore $(\bar{G}, \mathscr{X}, \mathscr{Y})$ is a *POM*-triple.

By the definitions of $(\bar{G}, \mathscr{X}, \mathscr{Y})$ and the *PO*-semigroup $PO(\bar{G}, \mathscr{X}, \mathscr{Y})$, Proposition 3.2 (i) implies that $PO(\bar{G}, \mathscr{X}, \mathscr{Y})$ is a subsemigroup without 0 in $T*_0G$. Then it follows from Lemma 3.6 that $PO(\bar{G}, \mathscr{X}, \mathscr{Y})\subseteq M_i$. The reverse inclusion follows if we observe that $t\varrho\mathscr{R}i\mathscr{L}(g^{-1}t)\varrho$ implies that $t\in\mathscr{Y}$, $g\in\bar{G}$ and, by (6), we have $g^{-1}t\in V_{\mathscr{X}}(\mathscr{Y})$. The proof of the direct part is complete.

Now we turn to the proof of the converse part.

(ii) By Proposition 3.2 (i), $S=PO(G, \mathscr{X}, \mathscr{Y})$ is clearly a subsemigroup in $\check{\mathscr{X}}*_0G$ and $0\notin S$. Then, by Lemma 3.6, we have $S\subseteq M_i$ for a unique $i\in E_{\check{\mathscr{X}}/\varrho}\setminus 0$ and thus, by Lemma 3.1, we infer $\mathscr{Y}\subseteq\{x\in\mathscr{X}\colon x\varrho\mathscr{R}i\}$. The latter subset which we will denote by $\mathscr{Y}_i$, is easily seen to be a right ideal in $\mathscr{X}$ where $E_{\mathscr{Y}_i}$ is a subband. Since $(G, \mathscr{X}, \mathscr{Y})$ is assumed to be a *POM*-triple we infer that $\mathscr{Y}=\mathscr{Y}_i$. Hence, if $(a, g)\in M_i$ then $a\in\mathscr{Y}$ and $(g^{-1}a)\varrho\mathscr{L}i$ in $\check{\mathscr{X}}/\varrho$. The latter relation implies $b\varrho\mathscr{R}i$ for any $b\in V(g^{-1}a)$, that is, we have $V(g^{-1}a)\subseteq\mathscr{Y}_i=\mathscr{Y}$. Hence it follows that $g^{-1}a\in V(\mathscr{Y})$ and we have $(a, g)\in S$. Thus the equality $S=M_i$ is proved.

We can summarize the results of Sections 2 and 3 as follows:

Theorem 3.8. *For a regular semigroup S the following conditions are equivalent to each other:*

(i) *S is E-unitary;*

(ii) *S is isomorphic to a PO-semigroup;*

(iii) *S is isomorphic to a PO-semigroup defined by a POM-triple;*

(iv) *S is a regular subsemigroup without 0 in a 0-semidirect product of a strictly combinatorial semigroup by a group;*

(v) *S is a maximal subsemigroup without 0 in a 0-semidirect product of a strictly combinatorial semigroup by a group.*

Proof. The implications (i)⇒(iii) and (ii)⇒(i) are stated in Propositions 3.3 and 3.2 (i), respectively. Since (iii)⇒(ii) is trivial, the conditions (i), (ii) and (iii) are equivalent to each other. Moreover, the implications (iii)⇒(v) and (iv)⇒(i) follow from Propositions 3.7 (ii) and 3.5, respectively. The proof is complete as (v)⇒(iv) is easily deduced from Propositions 3.7 (i) and 3.2 (i).

4. E-unitary regular semigroups and connected idempotent and regular categories

A theory generalizing McAlister's P-theorem for not necessarily regular semigroups with commuting idempotents was developed by Margolis and Pin in [4]. Although their terminology and methods are entirely different from ours, for inverse semigroups their main theorem says almost the same as our Theorem 3.8 (i), (v). After making a dictionary between the two terminologies we deduce the generalization of Theorem 4.1 [4] for E-unitary regular semigroups.

In this section the reader is assumed to be familiar with the paper [4]. The notions and notations of [4] are used without any reference.

Let C be a category. Then Mor (C) is a partial groupoid. Denote by $\check{\text{Mor}}\,(C)$ the groupoid obtained from Mor (C) by adjoining a new symbol 0 and extending the operation as in Section 1. Clearly, $\check{\text{Mor}}\,(C)$ is a semigroup which is categorical at 0.

The following proposition states that categories can be considered as certain semigroups with 0 together with a 0-restricted homomorphism into an $\mathcal{H}$-trivial Brandt semigroup.

Proposition 4.1. (i) *For every category C, the groupoid* $\check{\text{Mor}}\,(C)$ *is a semigroup and the mapping φ_C: $\check{\text{Mor}}\,(C)\to B(\text{Ob}\,(C))$ defined by $0\varphi_C=0$ and $p\varphi_C=[u,v]$ provided $p\in \text{Mor}\,(u,v)$ is a 0-restricted homomorphism onto a full subsemigroup of $B(\text{Ob}\,(C))$ such that, for every $e,f\in E_{B(\text{Ob}(C))}$, $e\varphi_C^{-1}$ is a monoid with identity 1_e and $1_ep=p1_f=p$ for any $p\in \text{Mor}\,(C)$ with $e\mathcal{R}p\varphi_C\mathcal{L}f$.*

(ii) *Let* S *be a semigroup with* 0 *and* $\varphi: S \to B(I)$ *a* 0-*restricted homomorphism onto a full subsemigroup of* $B(I)$ *such that, for every* $e, f \in E_{B(I)}$, $e\varphi^{-1}$ *is a monoid with identity* 1_e *and* $1_e s = s 1_f = s$ *for any* $s \in S$ *with* $e \mathcal{R} s\varphi \mathcal{L} f$. *Then we can define a category* $C_{(S,\varphi)}$ *as follows:* $\mathrm{Ob}(C_{(S,\varphi)}) = I$ *and* $\mathrm{Mor}\,(i,j) = \{s \in S \colon s\varphi = [i,j]\}$ *for any* $i, j \in I$.

(iii) *For every category* C, *we have* $C = C_{(\check{\mathrm{Mor}}(C), \varphi_C)}$.

(iv) *For every pair* (S, φ) *with properties required in* (ii), *we have* $S = \check{\mathrm{Mor}}\, C_{(S,\varphi)}$ *and* $\varphi = \varphi_{C_{(S,\varphi)}}$.

The proof is easy therefore it is left to the reader.

For brevity, a pair (S, φ) satisfying the properties required in (ii) is termed a *category pair.*

By the preceding proposition we immediately obtain

Proposition 4.2. (i) *The category* C *is connected if and only if* φ_C *is onto.*

(ii) *The category* C *is regular if and only if* $\check{\mathrm{Mor}}\,(C)$ *is regular.*

(iii) *The category* C *is idempotent if and only if* $\mathrm{Ker}\,\varphi_C \subseteq E_{\check{\mathrm{Mor}}(C)}$.

Assume that C is a connected, idempotent and regular category. Then the preceding proposition implies that $\check{\mathrm{Mor}}\,(C)$ is a regular semigroup and φ_C is a 0-restricted homomorphism of $\check{\mathrm{Mor}}\,(C)$ onto an $\mathcal{H}$-trivial Brandt semigroup with $\mathrm{Ker}\,\varphi_C \subseteq E_{\check{\mathrm{Mor}}(C)}$. Then Lemma 1.6 ensures that $\check{\mathrm{Mor}}\,(C)$ is $E\setminus 0$-unitary and $\ker \varphi_C = \varrho$. Thus $\check{\mathrm{Mor}}\,(C)$ is a strictly combinatorial semigroup in which every non-zero idempotent ϱ-class e is a monoid with identity 1_e and, for arbitrary non-zero idempotent ϱ-classes e, f and for any $p \in \mathrm{Mor}\,(C)$ with $e \mathcal{R} p\varrho \mathcal{L} f$, we have $1_e p = p 1_f = p$. Now let S be a strictly combinatorial semigroup in which every non-zero idempotent ϱ-class e is a monoid with identity 1_e and, for any non-zero idempotent ϱ-classes e, f and for any $s \in S$ with $e \mathcal{R} s\varrho \mathcal{L} f$, we have $1_e s = s 1_f = s$. Such an S will be termed a *strictly combinatorial semigroup with local identities.*

Let S be a strictly combinatorial semigroup with local identities. By definition, there exists an $\mathcal{H}$-trivial Brandt semigroup $B(I)$ and a surjective 0-restricted homomorphism $\varphi: S \to B(I)$ with $\ker \varphi = \varrho$. Clearly, (S, φ) is a category pair and, by Propositions 4.1 and 4.2, $C_{(S,\varphi)}$ is a connected, idempotent and regular category. Since Lemma 1.7 implies ϱ to be the only 0-restricted primitive inverse semigroup congruence on S, for each category pair (S, φ'), we have $\ker \varphi' = \varrho$. Consequently, for any category pairs $(S, \varphi\colon S \to B(I))$ and $(S, \varphi'\colon S \to B(I'))$, there exists an isomorphism $\psi\colon B(I) \to B(I')$ with $\varphi\psi = \varphi'$. Hence we can easily deduce

Proposition 4.3. *If* C *is a connected, idempotent and regular category then* $\check{\mathrm{Mor}}\,(C)$ *is a strictly combinatorial semigroup with local identities. Conversely, for every strictly combinatorial semigroup* S *with local identities, there exists an, up to* *isomorphisms, unique category* C_S *such that* $\check{\mathrm{M}}\mathrm{or}\,(C_S)$ *is isomorphic to* S. *The category* C_S *is connected, idempotent and regular.*

Recall that an *isomorphism of the category* C *onto the category* D is a functor $F\colon C\to D$ which induces a bijection of $\mathrm{Ob}\,(C)$ onto $\mathrm{Ob}\,(D)$ and a bijection of $\mathrm{Mor}\,(u, v)$ onto $\mathrm{Mor}\,(Fu, Fv)$ for every $u, v\in\mathrm{Ob}\,(C)$.

The connection between automorphisms of categories and automorphisms of the corresponding semigroups is easily described. Given a category C or a category pair (S, φ), denote by $\mathrm{Aut}\,C$ and $\mathrm{Aut}_\varphi\,S$, respectively, the group of automorphisms of C and the group of those automorphisms of S which possess the property that, for every $s, t\in S$, we have $s\ker\varphi\, t$ if and only if $s\alpha \ker\varphi\, t\alpha$.

Proposition 4.4. (i) *Let* $F\colon C\to C$ *be an automorphism of the category* C. *Then the mapping* F_m *induced by* F *on* $\mathrm{Mor}\,(C)$ *is an automorphism of the partial semigroup* $\mathrm{Mor}\,(C)$ *which can be extended to an automorphism of* $\check{\mathrm{M}}\mathrm{or}\,(C)$ *by setting* $0F_m=0$.

(ii) *Let* $(S, \varphi\colon S\to B(I))$ *be a category pair and* $\alpha\in\mathrm{Aut}_\varphi\,S$. *Then* α *induces an automorphism of* $B(I)$ *and, consequently, a permutation* π_α *of* I *in such a way that, if* $s\varphi=[i,j]$ *for some* $s\in S\setminus 0$ *and* $i,j\in I$, *then* $(s\alpha)\varphi=[i\pi_\alpha, j\pi_\alpha]$. *Define a functor* $F_\alpha\colon C_{(S,\varphi)}\to C_{(S,\varphi)}$ *as follows:* $F_\alpha i=i\pi_\alpha$ *for every* $i\in I=\mathrm{Ob}\,(C_{(S,\varphi)})$ *and* $F_\alpha s=s\alpha$ *for every* $s\in\hat{S}=\mathrm{Mor}\,(C_{(S,\varphi)})$. *Then* $F_\alpha\in\mathrm{Aut}\,C_{(S,\varphi)}$.

(iii) *The mappings* $(\mathrm{Aut}\,C)^d\to\mathrm{Aut}_{\varphi_C}\,\check{\mathrm{M}}\mathrm{or}\,(C)$, $F\mapsto F_m$ *and* $\mathrm{Aut}_{\varphi_C}\,\check{\mathrm{M}}\mathrm{or}\,(C)\to(\mathrm{Aut}\,C)^d$, $\alpha\mapsto F_\alpha$ *defined in* (i) *and* (ii) *are group-isomorphisms inverse to each other.*

(iv) *The mappings* $\mathrm{Aut}_\varphi\,S\to(\mathrm{Aut}\,C_{(S,\varphi)})^d$, $\alpha\mapsto F_\alpha$ *and* $(\mathrm{Aut}\,C_{(S,\varphi)})^d\to\mathrm{Aut}_\varphi\,S$, $F\mapsto F_m$ *are group-isomorphisms inverse to each other.*

By applying this description for connected, idempotent and regular categories the case becomes simpler. For, if S is a strictly combinatorial semigroup and (S, φ) is a category pair, then we have seen that $\ker\varphi=\varrho$. By Proposition 1.5, it is easy to check that, for any $s, t\in S\setminus 0$, we have $s\varrho t$ if and only if $s\alpha\varrho t\alpha$. Hence $\mathrm{Aut}_\varphi\,S=\mathrm{Aut}\,S$, the group of all automorphisms of S.

Proposition 4.5. (i) *If* C *is a connected, idempotent and regular category, then the mapping* $(\mathrm{Aut}\,C)^d\to\mathrm{Aut}\,\check{\mathrm{M}}\mathrm{or}\,(C)$, $F\mapsto F_m$ (cf. Proposition 4.4) *is a group-isomorphism.*

(ii) *If* S *is a strictly combinatorial semigroup with local identities and* C_S *is a connected, idempotent and regular category with* $\mathrm{Mor}\,(C_S)=\hat{S}$, *then, for every* $\alpha\in\mathrm{Aut}\,S$, *there exists a unique functor* F_α *which coincides with* α *on* $\mathrm{Mor}\,(C_S)$. *Moreover, the mapping* $\mathrm{Aut}\,S\to(\mathrm{Aut}\,C_S)^d$, $\alpha\mapsto F_\alpha$ (cf. Proposition 4.4) *is a group-isomorphism.*

Remark. Proposition 4.5 implies that if a group G acts on a connected, idempotent and regular category C then this action determines in a natural way an action of G on $\check{\mathrm{Mor}}\,(C)$. Conversely, if a group G acts on a strictly combinatorial semigroup S with local identities and C_S is a category with $\mathrm{Mor}\,(C_S)=\check{S}$, then this action can be extended to an action of G on C_S.

It remains to find connection between the monoids C_u (defined in [4]) and *PO*-semigroups.

Proposition 4.6. (i) *Let G be a group and C a connected, idempotent and regular category on which G acts transitively, and let $u\in\mathrm{Ob}\,(C)$. Then $(G, \mathrm{Mor}\,(C), \mathcal{Y}_u)$ with $\mathcal{Y}_u=\mathrm{Mor}\,(u, C)$ is a POM-triple and $C_u=PO(G, \mathrm{Mor}\,(C), \mathcal{Y}_u)$.*

(ii) *Let $(G, \mathcal{X}, \mathcal{Y})$ be a POM-triple where $\check{\mathcal{X}}$ is a strictly combinatorial semigroup with local identities. Let $C_{\mathcal{X}}$ be a category such that $\mathrm{Mor}\,(C_{\mathcal{X}})=\mathcal{X}$. Then $C_{\mathcal{X}}$ is a connected, idempotent and regular category on which G acts transitively, and $PO(G, \mathcal{X}, \mathcal{Y})=(C_{\mathcal{X}})_u$ for some $u\in\mathrm{Ob}\,(C_{\mathcal{X}})$.*

Proof. (i) By Proposition 4.3 and Remark, $\check{\mathrm{Mor}}\,(C)$ is a strictly combinatorial partial semigroup on which G acts. Let us define $\bar{G}$, $\mathcal{Y}$ and $\mathcal{X}$ by means of G, $T=\check{\mathrm{Mor}}\,(C)$ and $[u, u]\in E_{(\check{\mathrm{Mor}}(C))\varphi_C}$ as in Proposition 3.7 (i). We claim that $\bar{G}=G$, $\mathcal{Y}=\mathcal{Y}_u$ and $\mathcal{X}=\mathrm{Mor}\,(C)$.

First of all, observe that, for any $g\in G$ and $p\in\mathrm{Mor}\,(C)$, we have

$$p\in\mathrm{Mor}\,(u, gu) \quad \text{if and only if} \quad p\varphi_C\mathcal{R}[u, u]\mathcal{L}(g^{-1}p)\varphi_C. \tag{7}$$

On the one hand, hence it follows that $\bar{G}=G$ as C is connected and therefore $\mathrm{Mor}\,(u, gu)\neq\square$ for every $g\in G$. On the other hand, we obtain from (7) the equality $\mathcal{Y}=\mathcal{Y}_u$ by making use of the assumption that G acts transitively, and therefore any $v\in\mathrm{Ob}\,(C)$ is of the form gu for some $g\in G$. Now let $q\in\mathrm{Mor}\,(v, C)$. As we have just seen, there exists $g\in G$ with $gu=v$. Since g induces an automorphism on C, we infer that there exists $p\in\mathrm{Mor}\,(u, C)$ with $gp=q$. Hence $\mathcal{X}=G\mathcal{Y}_u=\mathrm{Mor}\,(C)$. Thus Proposition 3.7 (i) ensures that $(G, \mathrm{Mor}\,(C), \mathcal{Y}_u)$ is a *POM*-triple. The equality $C_u=PO(G, \mathrm{Mor}\,(C), \mathcal{Y}_u)$ immediately follows as $V(\mathcal{Y}_u)=\mathrm{Mor}\,(C, u)$.

(ii) Proposition 4.3 implies $C_{\mathcal{X}}$ to be a connected, idempotent and regular category. In the proof of Proposition 3.7 (ii) it is verified that $\mathcal{Y}=\{x\in\mathcal{X}: x\varrho\mathcal{R}i\}$ for some $i\in E_{\check{\mathcal{X}}/\varrho}$. Hence it follows that $\mathcal{Y}=\mathrm{Mor}\,(u, C_{\mathcal{X}})$ for some $u\in\mathrm{Ob}\,(C_{\mathcal{X}})$. Now let $v, w\in\mathrm{Ob}\,(C_{\mathcal{X}})$. Since $C_{\mathcal{X}}$ is connected, there exist $x\in\mathrm{Mor}\,(v, C_{\mathcal{X}})$ and $y\in\mathrm{Mor}\,(w, C_{\mathcal{X}})$. Since $G\mathcal{Y}=\mathcal{X}$ we have $g, h\in G$ and $a, b\in\mathcal{Y}$ with $ga=x$ and $hb=y$. Then the action of G on $C_{\mathcal{X}}$ has the property that $gu=v$ and $hu=w$. Hence we infer that $hg^{-1}v=w$, that is, G acts transitively on $C_{\mathcal{X}}$. Thus $C_{\mathcal{X}}$ satisfies the conditions required in (i) whence it follows that $(C_{\mathcal{X}})_u=PO(G, \mathrm{Mor}\,(C_{\mathcal{X}}), \mathcal{Y}_u)$ where $\mathrm{Mor}\,(C_{\mathcal{X}})=\mathcal{X}$ and $\mathcal{Y}_u=\mathcal{Y}$. This completes the proof.

Now we are ready to give a condition equivalent to each of (i)—(v) in Theorem 3.8 which is analogous to that in the main theorem of [4].

Theorem 4.7. *The following condition is equivalent to each of the conditions* (i)—(v) *in Theorem* 3.8.

(vi) S^1 *is isomorphic to a monoid* C/G *where* G *is a group acting transitively without fixpoints on a connected, idempotent and regular category* C.

Proof. Let S be an E-unitary regular semigroup. Then S^1 is also an E-unitary regular semigroup. By the method described in Section 2 we can construct a POM-triple $(G_{S^1}, \mathscr{X}, \mathscr{Y})$ (cf. Proposition 3.3). Consider the pair $(\check{\mathscr{X}}, \varphi)$ where φ is the homomorphism defined in (II) of Section 2. By (I) and (II), $\check{\mathscr{X}}$ is a strictly combinatorial semigroup and φ is a 0-restricted homomorphism of $\check{\mathscr{X}}$ onto $B(G_{S^1})$. It is easy to check by (1) that $(1, g)\circ(s, g)=(s, g)\circ(1, g\cdot s\sigma)=(s, g)$ for every $s\in S^1$ and $g\in G_{S^1}$. Since $(1, g)\varphi=[g, g]$, this implies that $\check{\mathscr{X}}$ is a strictly combinatorial semigroup with local identities. Thus $(\check{\mathscr{X}}, \varphi)$ is a category pair. Then, by Proposition 4.1 (ii), $C_{(\check{\mathscr{X}},\varphi)}$ is a category with $\mathrm{Mor}\,(C_{(\check{\mathscr{X}},\varphi)})=\mathscr{X}$. Moreover, Proposition 4.6 (ii) ensures that $C_{(\check{\mathscr{X}},\varphi)}$ is connected, idempotent and regular, and G_{S^1} acts on it transitively. Observe that the automorphism of $C_{(\check{\mathscr{X}},\varphi)}$ determined by an element $g\in G_{S^1}$ induces the regular left translation on $G_{S^1}=\mathrm{Ob}\,(C_{(\check{\mathscr{X}},\varphi)})$ corresponding to $g\in G_{S^1}$. Thus G_{S^1} acts on $C_{(\check{\mathscr{X}},\varphi)}$ without fixpoints. Property (VI) in Section 2 ensures S^1 to be isomorphic to $PO(G_{S^1}, \mathscr{X}, \mathscr{Y})$, and Proposition 4.6 (ii) implies that $PO(G_{S^1}, \mathscr{X}, \mathscr{Y})=(C_{(\check{\mathscr{X}},\varphi)})_u$ for some $u\in\mathrm{Ob}\,(C_{(\check{\mathscr{X}},\varphi)})$. Hence S^1 is isomorphic to $(C_{(\check{\mathscr{X}},\varphi)})_u$ for some $u\in\mathrm{Ob}\,(C_{(\check{\mathscr{X}},\varphi)})$. To complete the proof of the implication (i)$\Rightarrow$(vi) we refer to Proposition 3.11 [4] which states that if C is a category on which a group G acts transitively without fixpoints, then, for all $u\in\mathrm{Ob}\,(C)$, the monoid C_u is isomorphic to C/G.

Conversely, suppose G is a group acting transitively without fixpoints on a connected, idempotent and regular category C. Then Proposition 3.11 [4] just cited implies that C/G is isomorphic to C_u for all $u\in\mathrm{Ob}\,(C)$, while Proposition 4.6 (i) ensures C_u to be a PO-semigroup. Thus (vi) implies (iii), completing the proof of the theorem.

Finally, we show how one can reobtain McAlister's P-theorem from our results. Let $(G, \mathscr{X}, \mathscr{Y})$ be a PO-triple such that $PO(G, \mathscr{X}, \mathscr{Y})$ is an inverse semigroup or, equivalently, $E_{\mathscr{Y}}$ is a semilattice. It is not difficult to check that $(G, \widehat{\check{\mathscr{X}}/\gamma}, \mathscr{Y}\gamma)$ is also a PO-triple. We claim that the mapping $\eta\colon PO(G, \mathscr{X}, \mathscr{Y})\to PO(G, \widehat{\check{\mathscr{X}}/\gamma}, \mathscr{Y}\gamma)$, $(a, g)\eta=(a\gamma, g)$ is an isomorphism. It is immediate that η is a homomorphism. Let us verify that η is one-to-one. Assume that $(a, g), (b, h)\in PO(G, \mathscr{X}, \mathscr{Y})$ with $(a, g)\eta=(b, h)\eta$. Then $g=h$ and $a\gamma b$ in $\check{\mathscr{X}}$. The latter relation implies that $V(a)=$ $=V(b)$ and hence $V(g^{-1}a)=V(g^{-1}b)$. Since $(a, g), (b, g)\in PO(G, \mathscr{X}, \mathscr{Y})$ we have $g^{-1}a, g^{-1}b\in V(\mathscr{Y})$. Therefore there exists $c\in V(g^{-1}a)=V(g^{-1}b)$ with $c\in\mathscr{Y}$. Thus $(c, g^{-1})\in PO(G, \mathscr{X}, \mathscr{Y})$ and (c, g^{-1}) is an inverse of both (a, g) and (b, g). Since, by assumption, $PO(G, \mathscr{X}, \mathscr{Y})$ is an inverse semigroup we obtain that $(a, g)=(b, h)$.

Now we show that η is onto. Consider an arbitrary element $(x\gamma, g)$ in $PO(G, \widehat{\mathscr{X}/\gamma}, \mathscr{Y}\gamma)$ where $x\in\mathscr{X}$. Then $x\gamma\in\mathscr{Y}\gamma$ and $g^{-1}(x\gamma)\in V(\mathscr{Y}\gamma)$. The first relation implies the existence of an element a in $\mathscr{Y}$ with $a\gamma x$, and hence $g^{-1}x\gamma g^{-1}a$. Thus, by the second relation we see that $g^{-1}a\in V(\mathscr{Y})$ since $V(\mathscr{Y})=V(\{x\in\mathscr{X}\colon x\gamma b$ for some $b\in\mathscr{Y}\})$. Therefore $(x\gamma, g)=(a, g)\eta$ which completes the proof of the fact that η is an isomorphism.

The strictly combinatorial semigroup $\check{\mathscr{X}}/\gamma$ is an inverse semigroup. Thus, by slightly modifying the proof of Theorem 4.2 [4], we can deduce the following assertion. The triple $(G, \overline{\mathscr{X}}, \overline{\mathscr{Y}})$ where $\overline{\mathscr{X}}$ is the partially ordered set of $\mathscr{J}$-classes of $\widehat{\mathscr{X}/\gamma}$, $\overline{\mathscr{Y}}=\{J\in\overline{\mathscr{X}}\colon J\cap E_{\mathscr{Y}\gamma}\neq\square\}$ and the action of G on $\overline{\mathscr{X}}$ is defined by $g(x\mathscr{J})=(gx)\mathscr{J}$ $(g\in G,\ x\in\widehat{\mathscr{X}/\gamma})$ is a McAlister triple and $PO(G, \widehat{\mathscr{X}/\gamma}, \mathscr{Y}\gamma)$ is isomorphic to $P(G, \overline{\mathscr{X}}, \overline{\mathscr{Y}})$. By the preceding paragraph this implies that $PO(G, \mathscr{X}, \mathscr{Y})$ is isomorphic to $P(G, \overline{\mathscr{X}}, \overline{\mathscr{Y}})$.

Consider the partially ordered set of $\mathscr{J}$-classes of $\mathscr{X}$ and denote it by $\tilde{\mathscr{X}}$. Put $\tilde{\mathscr{Y}}=\{J\in\tilde{\mathscr{X}}\colon J\cap E_{\mathscr{Y}}\neq\square\}$ and define an action of G on $\tilde{\mathscr{X}}$ by $g(x\mathscr{J})=(gx)\mathscr{J}$ for every $g\in G$ and $x\in\mathscr{X}$. Since $\gamma\subseteq\mathscr{J}$ on $\check{\mathscr{X}}$, it is easily seen that the mapping $\nu\colon\tilde{\mathscr{X}}\to\overline{\mathscr{X}}$ defined by $(x\mathscr{J})\nu=(x\gamma)\mathscr{J}$ $(x\in\mathscr{X})$ is an order isomorphism with the properties that $\tilde{\mathscr{Y}}\nu=\overline{\mathscr{Y}}$ and $g(\tilde{x}\nu)=(g\tilde{x})\nu$ for every $g\in G$ and $\tilde{x}\in\tilde{\mathscr{X}}$. Thus the triple $(G, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ is equivalent to the triple $(G, \overline{\mathscr{X}}, \overline{\mathscr{Y}})$. Therefore $(G, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ is also a McAlister triple and $P(G, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ is isomorphic to $P(G, \overline{\mathscr{X}}, \overline{\mathscr{Y}})$. The following theorem sums up what we have just proved.

Theorem 4.8. *Let $(G, \mathscr{X}, \mathscr{Y})$ be a PO-triple such that $E_{\mathscr{Y}}$ is a semilattice. Let $\tilde{\mathscr{X}}$ be the partially ordered set of $\mathscr{J}$-classes of $\mathscr{X}$ and $\tilde{\mathscr{Y}}=\{J\in\tilde{\mathscr{X}}\colon J\cap E_{\mathscr{Y}}\neq\square\}$. Define an action of G on $\tilde{\mathscr{X}}$ by $g(x\mathscr{J})=(gx)\mathscr{J}$ for every $g\in G$ and $x\in\mathscr{X}$. Then $(G, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$ is a McAlister triple and $PO(G, \mathscr{X}, \mathscr{Y})$ is isomorphic to $P(G, \tilde{\mathscr{X}}, \tilde{\mathscr{Y}})$.*

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