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Representation of 2-distributive modular lattices of finite length

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Dedicated to the memory of Dr. András Huhn

1. Statement of the theorem. In this note we prove the following result.

Main Theorem. *Suppose L is a 2-distributive modular lattice of finite length n. If V is an n-dimensional vector space over a division ring D with $|D|>|L|$, then L can be embedded in $L(V)$.*

Here $L(V)$ is the lattice of all subspaces of the vector space V. To say that L is 2-distributive means that it satisfies the identity

$$a(x+y+z) = a(x+y)+a(x+z)+a(y+z).$$

A special case of this theorem, the case when L is of breadth 2, was proved in HERRMANN [3].

2. Preliminaries. A lattice is said to be n-distributive if it satisfies the identity

$$a \sum_{0\leqq i\leqq n} x_i = \sum_{0\leqq i\leqq n} a \sum_{j\neq i} x_j .$$

This concept was introduced by András Huhn and was investigated by him in a series of papers. His original definition, in HUHN [5], required the lattice to be modular, but this condition was dropped in [4]. We shall adhere to the revised terminology, although all the lattices under consideration here will be modular. The following two results will be needed.

Theorem A (HUHN [4], Theorem 3.1). *The dual of an n-distributive modular lattice is n-distributive.*

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Theorem B (Huhn [5], Theorem 1.1). *An algebraic modular lattice is n-distributive iff it does not contain as a sublattice (the lattice of all subspaces of) a non-degenerate projective n-space.*

In one direction, this result can be strengthened: If an algebraic modular lattice fails to be n-distributive, then it contains a non-degenerate projective n-space as an interval, not just as a sublattice. This shows that the result proved here is in a sense "best possible". For suppose that L is a modular lattice of finite length, and that L is not 2-distributive. Then L contains as an interval a non-degenerate projective plane P. If P is not Arguesian, then there does not exist any embedding $L \rightarrowtail L(V)$, with V a vector space over a division ring D, but if P is Arguesian, then such embeddings can at most exist for division rings having the same characteristic as the coordinate ring of P.

In order not to have to interrupt the argument later, we state and prove here two simple observations that will be used in the proof of the main theorem.

Lemma C. *Suppose V is a finite dimensional vector space over a division ring D. Then V is not the union of fewer than $|D|$ proper subspaces of V.*

Proof. Suppose $|I|<|D|$, and suppose V_i, $i \in I$, are proper subspaces of V. Let $K = \bigcup_{i \in I} V_i$. Assuming that A is a subspace of V, we prove by induction on the dimension of A that if $A \subseteq K$, then $A \subseteq V_i$ for some $i \in I$. This is certainly true when $\dim A \leqq 1$. Assuming that $n > 1$, and that the claim holds whenever $\dim A < n$, we consider the case $\dim A = n$. Then each proper subspace of A is contained in some V_i. Since A has $|D|$ subspaces of dimension $n-1$, it follows that at least two of these, say B and C, are contained in the same V_i, whence $A = B + C \subseteq V_i$.

Lemma D. *Suppose L and L' are modular lattices of the same finite lenght. If the mapping $f\colon L \rightarrow L'$ is one-to-one and preserves the covering relation, then f is an embedding of L into L'.*

Proof. By duality, it suffices to show that

$$f(a)f(b) = f(ab) \tag{1}$$

for all $a, b \in L$. The mapping is obviously monotone, so (1) holds whenever a and b are comparable. For the case when a and b are not comparable, we use induction on the lengths of the intervals a/ab and b/ab. If both intervals have length 1, then a and b are distinct covers of ab, and $f(a)$ and $f(b)$ are therefore distinct covers of $f(ab)$, so that (1) holds in this case. For the inductive step, we assume that $ab \prec c < a$ and let $d = b + c$. Then $ad = c$ and $bc = ab$, hence by the inductive hypothesis, $f(a)f(d) = f(c)$ and $f(b)f(c) = f(ab)$. Again using the monotonicity of f, we infer that (1) holds.

3. Proof of the Main Theorem. We are going to show that every embedding $F\colon a/0 \rightarrowtail A/0$, where a is a coatom in L and A is a coatom in $L(V)$, can be extended to an embedding $G\colon L \rightarrowtail L(V)$. From this the theorem follows by induction on n.

Let M be the set of all minimal elements of the set $\{x \in L\colon x \not\leqq a\}$. We need to look at some properties of the elements of M.

The elements of M are obviously join irreducible. For any $x \not\leqq a$, the element $a+x=1$ covers a, and ax is therefore covered by x. Consequently,

$$x = ax+p \quad \text{whenever} \quad x \geqq p \in M.$$

In particular,

$$p+q = p+a(p+q) \quad \text{for all} \quad p, q \in M.$$

The most important fact about the elements $p \in M$ is that the set

$$C_p = \{p+q\colon q \in M\}$$

is always a chain. In other words, any two joins $p+q$ and $p+r$, with $q, r \in M$, are comparable. Suppose this fails. Then $r \not\leqq p+q$, hence $r(p+q) \leqq a$, and similarly $q(p+r) \leqq a$. Consequently,

$$(p+q)(p+r)(q+r) = q(p+r)+r(p+q) \leqq a.$$

But using the dual of the 2-distributive law, we find that

$$\begin{aligned} a+(p+q)(p+r)(q+r) &= \\ = [a+(p+q)(p+r)][a+(p+q)(q+r)][a+(p+r)(q+r)] &\geqq \\ \geqq (a+p)(a+q)(a+r) &= 1. \end{aligned}$$

This contradiction proves that C_p is a chain.

Finally, we note that, for $p, q \in M$,

$$C_p \cap C_q = C_p \cap (1/(p+q)).$$

Certainly, the set on the left is included in the set on the right. To prove the opposite inclusion, consider any $c \in C_p \cap (1/(p+q))$. Then $c \geqq p+q$, and $c=p+r$ for some $r \in M$. Consequently $c=(q+p)+(q+r)$, and recalling that C_p is a chain, we infer that $c=q+p$ or $c=q+r$. In either case, $c \in C_q$, as was to be shown.

It will be convenient to have a fixed notation for the elements of the chain C_p, say

$$c_{p0} < c_{p1} < \ldots < c_{p\lambda_p}.$$

We also fix a one-to-one mapping $f\colon L \to D \setminus \{0\}$ and a vector $\xi \in V \setminus A$, and for each $c \in a/0$ we pick a vector $\alpha(c) \in F(c) \setminus \bigcup_{d<c} F(d)$. Such a vector always exists by Lemma C. Associating with each $p \in M$ the vector

$$\xi_p = \xi - \sum_{0 \leqq i < \lambda_p} f(c_{pi})\alpha(ac_{p(i+1)}),$$

we are now ready to describe the mapping $G\colon L\to L(V)$. For $x\leqq a$, we let $G(x)=F(x)$, but if $x\nleqq a$, then there exists $p\in M$ with $p\leqq x$. For each such p we let

$$G_p(x) = G(ax)+D\xi_p.$$

We claim that $G_p(x)$ is actually independent of p. To see this, consider another element $q\in M$ with $q\leqq x$. The element $p+q$ is in both C_p and C_q, say $p+q=c_{pk}=$ $=c_{qm}$. For $i<k$ we have $ac_{p(i+1)}\leqq a(p+q)\leqq ax$, and therefore $\alpha(ac_{p(i+1)})\in G(ax)$. Consequently,

$$G_p(x) = G(ax)+D\big(\xi-\sum_{k\leqq i<\lambda_p} f(c_{pi})\alpha(ac_{p(i+1)})\big).$$

From this, and the corresponding formula for $G_q(x)$, we infer that $G_p(x)=G_q(x)$. Dropping the subscript, we therefore have a well defined mapping from L to $L(V)$.

It is easy to check that $x\leqq y$ implies $G(x)\leqq G(y)$. We need to check that $x<y$ implies $G(x)<G(y)$. Since $\xi\notin A$, this is clear when $x\leqq a$. If $x\nleqq a$, then $y=ay+x$, which implies that $ay\nleqq x$, so that $ax<ay$. But then $G(ay)\nleqq G(ax)+D\eta$ whenever $\eta\notin A$. Hence $G(x)<G(y)$.

The mapping G preserves strict inclusion, and since the lattices L and $L(V)$ have the same length, it follows that G preserves the covering relation. To prove that G is an embedding, it suffices by Lemma D to show that G is one-to-one. We shall do this by showing that $G(x)\leqq G(y)$ implies $x\leqq y$.

Suppose $G(x)\leqq G(y)$. If $x\leqq a$ and $y\leqq a$, then it obviously follows that $x\leqq y$. The case $x\nleqq a$ and $y\leqq a$ is excluded, for we would then have $G(x)\nleqq A$ and $G(y)\leqq A$. Next suppose $x\leqq a$ and $y\nleqq a$. Choosing $q\in M$ with $q\leqq y$, we then have $G(x)\leqq G(ay)+D\xi_q$, hence $G(x)\leqq A\cap(G(ay)+D\xi_q)=G(ay)+(A\cap D\xi_q)=G(ay)$, and consequently $x\leqq ay\leqq y$. Finally suppose $x\nleqq a$ and $y\nleqq a$. If $xy\nleqq a$, then we can choose $p\in M$ with $p\leqq xy$, and we have $x=ax+p$ and $y=ay+p$. From the fact that $G(ax)\leqq G(y)$, we infer by the preceding case that $ax\leqq y$, and since $p\leqq y$, we conclude that $x\leqq y$.

To complete the proof, it suffices to show that it cannot happen that $x\nleqq a$, $y\nleqq a$ and $xy\leqq a$. Assuming that these three conditions are satisfied, we choose $p, q\in M$ with $p\leqq x$ and $q\leqq y$. Then $G(x)=G(ax)+D\xi_p$ and $G(y)=G(ay)+D\xi_q$, and the condition $G(x)\leqq G(y)$ therefore implies that $\xi_p\in G(ay)+D\xi_q$. Thus $\xi_p=\eta+s\xi_q$ for some $\eta\in G(ay)$ and $s\in D$. Actually $s=1$, because $\xi_p-s\xi_q\in A$. Let c_{pk} be the term in the chain C_p that precedes $p+q$, and let c_{qm} be the term in C_q that precedes $p+q$. Let

$$\beta_p = \sum_{0\leqq i<k} f(c_{pi})\alpha(ac_{p(i+1)}),\quad \gamma_p = \sum_{k<i<\lambda_p} f(c_{pi})\alpha(ac_{p(i+1)}),$$

and define β_q and γ_q similarly. Then $\gamma_p=\gamma_q$, and therefore

$$\big(f(c_{pk})-f(c_{qm})\big)\alpha\big(a(p+q)\big) = -\eta-\beta_p+\beta_q\in G(ay+ac_{pk}+ac_{qm}).$$

Since $\alpha(a(p+q))$ does not belong to $G(d)$ for any $d < a(p+q)$, it follows that

$$a(p+q) \leqq ay + ac_{pk} + ac_{qm}.$$

Recalling that $p+q = q+a(p+q)$, we infer that

$$p \leqq q + ay + ac_{pk} + ac_{qm}.$$

Since p is join irreducible and L is 2-distributive, it follows that p is included in the join of some two of the four elements q, ay, ac_{pk} and ac_{qm}. In fact, since $p \not\leqq a$, we have $p \leqq q+w$ for some $w \in \{ay, ac_{pk}, ac_{qm}\}$. But each of these three inclusions readily leads to a contradiction: $p \leqq q+ay$ would imply $p \leqq y$, hence $xy \not\leqq a$; $p \leqq q+ac_{pk}$ would imply $p \leqq (q+ac_{pk})c_{pk} = qc_{pk}+ac_{pk} \leqq a$; and $p \leqq q+ac_{qm}$ would imply $p \leqq c_{qm}$. This establishes the contradiction, and completes the proof of the theorem.

4. Connections with other representation problems. Modular lattices arise naturally in many contexts, and each source gives rise to a representation problem. The "most general" representation problem to receive extensive attention is the problem of representing a lattice as a lattice of permutable equivalence relations. The (modular) lattices for which such a representation exists are said to be of type 1. The discovery that there exist modular lattices that are not of type 1 led to the introduction of a six-variable identity, stronger than the modular law, that holds in every lattice of type 1. This identity holds in (the lattice of all subspaces of) a projective plane iff the plane is Arguesian, and the lattices in which the identity holds are therefore called Arguesian. Of course every modular lattice that contains a non-Arguesian projective plane as a sublattice is non-Arguesian, but as might be expected, the Arguesian identity can fail for other, more subtle reasons. However, the geometric flavor of these original examples carries over to a surprising extent to the general case. It is shown in Day and Jónsson [1] that if a modular lattice L fails to be Arguesian, then the ideal lattice of L contains a "non-Arguesian configuration" of ten points and ten lines. These twenty elements, however, may lie in up to twenty distinct non-degenerate planes that constitute intervals in the ideal lattice. In particular, therefore, every 2-distributive modular lattice L is Arguesian, for the ideal lattice of L is also 2-distributive and therefore does not contain a non-degenerate projective plane as a sublattice. We do not know whether every 2-distributive modular lattice is of type 1, although it seems likely that this is the case. In fact, we conjecture that a modified version of the result proved here is true without any restriction on the length of the lattice.

Conjecture. For any 2-distributive modular lattice L, and division ring D, L can be embedded in $L(V)$ for some vector space V over D.

After the research reported here was completed, we received a prepublication copy of a research announcement, HAIMAN [2], describing the construction of a lattice that is Arguesian but not of type 1. This important result confirms a conjecture of long standing, and it gives increased importance to various ongoing efforts to obtain positive representation results for special classes of modular lattices. Our result falls into that category, but it may have a special significance in this context. In DAY and JÓNSSON [1] it was shown that if a modular lattice fails to be Arguesian, then its ideal lattice either contains as an interval a bad plane, or else it contains two or more planes that are badly fitted together. It seems likely that a similar result holds for type 1. Our result gives some credence to this conjecture, for it shows that if a modular lattice of finite length is not of type 1, then it must contain as an interval a non-degenerate plane.

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