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The diagram invariant problem for planar lattices

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Dedicated to the memory of András P. Huhn

There are several graphical schemes in common use to represent a given finite ordered set. Of these the two that are best known are the 'comparability graph' and the 'diagram'.

The *comparability graph* of an ordered set P is the graph whose vertices are the elements of P and in which a pair of vertices x, y is adjacent if either $x<y$ or $x>y$. Much is known about the comparability graph: the characterization of comparability graphs (A. GHOUILA-HOURI [8], P. C. GILMORE and A. J. HOFFMAN [9]); the description of the order theoretical properties that are invariant among all ordered sets with the same comparability graph (M. HABIB [10]); the number of distinct orientations of a given comparability graph (L. N. ŠEVRIN and N. D. FILIPPOV [24]); a structure theory for comparability graphs (T. GALLAI [7]). (These and many further topics are treated closely in the recent survey articles of D. KELLY [13] and R. H. MÖHRING [17].)

The *diagram* of a finite ordered set P is that pictorial representation of P in the plane in which small circles, corresponding to the elements of P, are arranged in such a way that, for a and b in P, the circle corresponding to a is higher than the circle corresponding to b whenever $a>b$ and a straight line segment is drawn to connect the two circles whenever a covers b. Say that a *covers* b and write $a \succ b$ if $a>b$ and if $a>c\geqq b$ in P implies $c=b$. The diagram of P determines P up to isomorphism. Its economy of presentation accounts for the evident popularity of the diagram in the order literature today. Nevertheless, much less is known about it than about the comparability graph. (See I. RIVAL [23] for a recent survey of this theme.)

Closely related to the 'diagram' is the 'covering graph'. The *covering graph* of a finite ordered set P is the graph whose vertices are the elements of P and in which

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a pair x, y of vertices is adjacent if $x \succ y$ or $y \succ x$. Not every graph is a covering graph and even one that is may have numerous 'orientations', that is, there may be many ordered sets with the same underlying covering graph. This article is inspired by the question, still little explored, of the order theoretical properties, if any, common to all of the ordered sets with the same covering graph. There seem to be few such properties. As a matter of fact, besides the trivial properties, such as the number of vertices or the number of edges we do not know even of one single property which remains invariant among all of the orientations of a fixed, but arbitrary, covering graph. Unlike the comparability graph none of these often studied numerical properties of an ordered set are such invariants: *width, length, dimension, jump number*. Indeed, there is even the intriguing possibility that there is *no* nontrivial diagram invariant at all!

Consider for example these common integer-valued functions defined on a finite ordered set P: the *width*

$$w(P) = \max\{|A| \mid A \text{ antichain in } P\};$$

the *length*

$$l(p) = \max\{|C|-1 \mid C \text{ chain in } P\};$$

the *dimension*

$$\dim P = \min\Big\{m \mid L_1, L_2, \ldots, L_m \text{ linear extensions of } P \text{ and } \bigcap_{i=1}^{m} L_i = P\Big\};$$

the *jump number*

$$s(P) = \min\{s(P, L) \mid L \text{ linear extension of } P\},$$

where

$$s(P, L) = \big|\{(a, b) \in P \times P \mid a \succ b \text{ in } L \text{ and } a \not\geqq b \text{ in } P\}\big|.$$

Suppose that P and P' are ordered sets with graph isomorphic comparability graphs. Then $w(P)=w(P')$, $l(P)=l(P')$, $\dim P=\dim P'$ and $s(P)=s(P')$ (cf. D. Kelly and W. T. Trotter [14], M. Habib [10]). Quite different is the situation for the covering graph — even for simple ordered sets. For instance, if $P \cong \mathbf{3}$ (see Figure 1) the three-element chain and $P' \cong \mathbf{2} \oplus \mathbf{1}$ (see Figure 2) then the covering graphs of P and P' are, of course, graph isomorphic. However, $w(\mathbf{3})=1<2=w(\mathbf{2}\oplus\mathbf{1})$, $l(\mathbf{3})=2>1=l(\mathbf{2}\oplus\mathbf{1})$, $\dim \mathbf{3}=1<2=\dim(\mathbf{2}\oplus\mathbf{1})$, and $s(\mathbf{3})=0<1=s(\mathbf{2}\oplus\mathbf{1})$. Actually the

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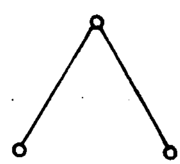
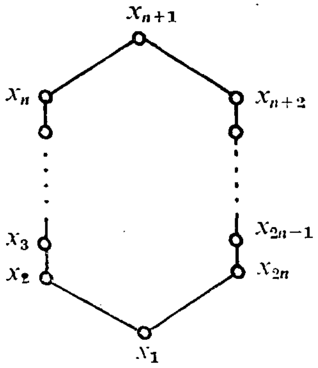
Figure 1



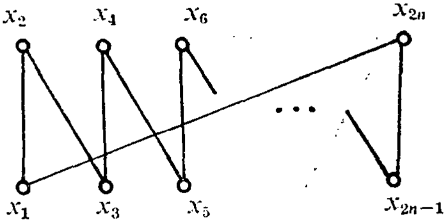
2⊕1

Figure 2



P_n

Figure 3



P_n'

Figure 4

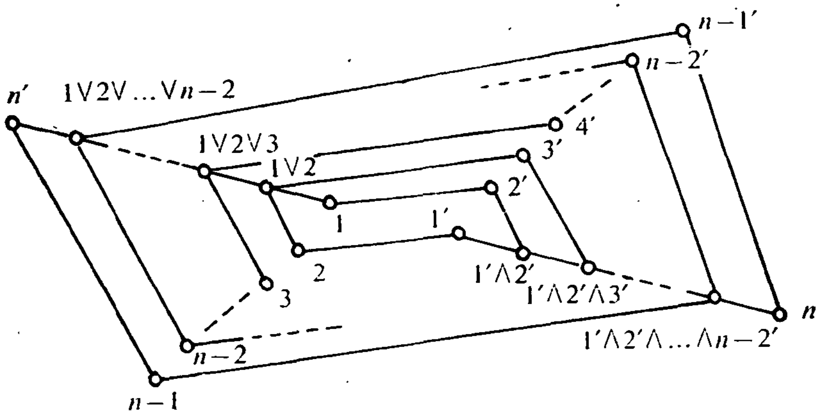
'deviation' can be much larger. For a positive integer n let P_n stand for the ordered set illustrated in Figure 3 and let P_n' stand for the $2n$-*cycle* illustrated in Figure 4. They have graph isomorphic covering graphs, yet $w(P_n)=2$, $w(P_n')=n$, $l(P_n)=n$, $l(P_n')=1$. $s(P_n)=1$, and $s(P_n')=n$. The dimension too differs, although for this pair of ordered sets the 'deviation' is only 1: $\dim P_n=2$ and $\dim P_n'=3$, provided that $n\geqq 3$. A more sophisticated example does show that the dimension can also 'deviate' considerably. A suitable pair of examples can be fashioned from an example constructed by D. KELLY [12] to show that planar ordered sets can have arbitrarily large dimension. Let $\mathbf{2}^n$ stand for the ordered set of all subsets of the n-element set. The subset $S_n=\{1, 2, \ldots, n\}\cup\{1', 2', \ldots, n'\}$ of the 'points' and 'copoints' $i'=\{1, 2, \ldots, i-1, i+1, \ldots, n\}$ of $\mathbf{2}^n$ has dimension n and so, in particular, also the subset of $\mathbf{2}^n$ consisting of

$$Q_n = S_n\cup\{1\vee 2\vee\ldots\vee i \mid 1 \leqq i \leqq n-1\}\cup\{1'\wedge 2'\wedge\ldots\wedge i' \mid 1 \leqq i \leqq n-1\}.$$

This ordered set Q_n of dimension n is illustrated in Figure 5 following the clever drawing of it proposed by D. KELLY [12]. A forty-five degree clockwise rotation of this illustration produces a planar lattice Q_n' of dimension two.

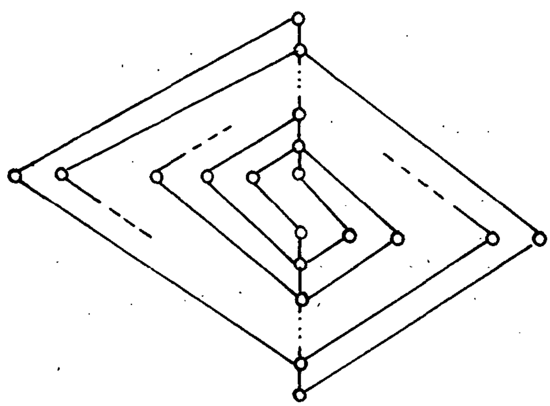
There is at least one residual positive fact. *Let P and P' be finite ordered sets with graph isomorphic covering graphs. If P is a chain then* $\dim P'-\dim P\leqq 1$. We must prove that $\dim P'\leqq 2$. To this end let $P=\{x_1\prec x_2\prec\ldots\prec x_n\}$, $n\geqq 3$, and suppose that P' is not a chain. There is no loss in generality if we assume too that x_1 is minimal in P'. We shall construct a chain decomposition $C_1, C_2, \ldots, C_m$, $m\geqq 2$, in this way. Let

$$C_1 = \{x_1 \prec x_2 \prec\ldots\prec x_i\},$$



Q_n

Figure 5



Q_n'

Figure 6

$1 \leqq i < n$, where x_{i+1} is maximal in P'. Thus, $x_{i+1} > x_i$. Suppose that $C_1, C_2, \ldots, C_t$ are already constructed and that $P - C_1 \cup C_2 \cup \ldots \cup C_t \neq \emptyset$. If

$$C_t = \{x_j \prec x_{j+1} \prec \ldots \prec x_k\}$$

where $j \leqq k$, then choose the least index $k+1 \leqq l \leqq n$ such that x_l is minimal in P' and set

$$C_{t+1} = \{x_l \prec \ldots \prec x_{k+2} \prec x_{k+1}\}.$$

If, on the other hand,

$$C_t = \{x_k \prec \ldots \prec x_{j+1} \prec x_j\},$$

with $j \leqq k$, then choose the least index $k+1 \leqq l \leqq n$ such that x_{l+1} is maximal in P' and, in this case, put

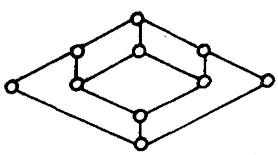
$$C_{t+1} = \{x_{k+1} \prec \ldots \prec x_{l-1} \prec x_l\}.$$

Finally, we can construct two linear extensions in two ways. First put

$$L_1 = C_1 \oplus C_2 \oplus \ldots \oplus C_m.$$

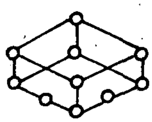
Now, construct L_2 in a similar way starting through with a 'dual' labelling beginning with x_n instead. It is easy to verify that $P' = L_1 \cap L_2$.

Our aim in this article is to consider a special case of this theme. Which order-theoretical properties are diagram invariants among all lattice orientations of a fixed, but arbitrary, covering graph of a planar lattice? Let P and P' be finite lattices with graph isomorphic covering graphs. If P is planar then must P' be planar too? The ordered sets illustrated in Figure 7 and Figure 8 show that this need not be the case at all. (Notice, moreover, that P' need not even be 'dismantlable' (cf. D. Kelly and I. Rival [14]).) It is a well known and useful fact that a planar lattice has, on either of its boundaries, elements which are both supremum irreducible and infimum irreducible, that is, *doubly irreducible* (cf. K. A. Baker, P. C. Fishburn and F. S. Roberts [2]). At least a fragment of this property is preserved by any lattice orientation.



P

Figure 7



P'

Figure 8

Theorem 1. *Let P and P' be finite lattices with graph isomorphic covering graphs. If P is planar then P' contains a doubly irreducible element too.*

Thus, the *existence* of a doubly irreducible element is a diagram invariant among all lattice orientations of a planar lattice. To prove this result we shall make extensive use of the geometric theory of planarity and planar embeddings for finite lattices established in D. Kelly and I. Rival [14] and in C. R. Platt [20].

Theorem 2. *Let P and P' be finite planar lattices with graph isomorphic covering graphs. If for some planar embedding $e(P)$ of P every doubly irreducible element of P lies on the boundary of $e(P)$ then, for any planar embedding $e(P')$ of P', the set of faces of $e(P')$ equals the set of faces of $e(P)$.*

Thus, under these hypotheses, the *set* of faces of any planar embedding of any planar lattice orientation of a planar lattice is a diagram invariant.

The problem to determine the lattice re-orientations of a fixed, but arbitrary, covering graph of a lattice has been more extensively studied, especially for distributive and modular lattices (cf. L. ALVAREZ [1], J. JAKUBÍK [11], D. DUFFUS and I. RIVAL [5], I. RIVAL [23]).

Planarity

The purpose of this section is to clarify the basic terms which we require to prove Theorem 1 and Theorem 2. The important references for our point of view are O. ORE [19], D. KELLY and I. RIVAL [15] and C. R. PLATT [20].

A finite graph G is *planar* if it can be embedded in the plane R^2 using a Jordan arc (that is, a homeomorphic image of the closed unit interval) for each edge such that different edges have, at most their endpoints in common. We denote by $e(G)$ such a *planar realization* of G. A *simple Jordan curve* or, for brevity, a *Jordan curve* in R^2 is a homeomorphic image of the unit circle. According to the well known Jordan Curve Theorem any Jordan curve C partitions the rest of the plane into two open sets, the *interior* $\mathrm{Int}\, C$ of C, and the *exterior* $\mathrm{Ext}\, C$ of C. Any Jordan arc connecting two vertices in $e(G)$ corresponds to an (elementary) *path* of G. Similarly, any Jordan curve in $e(G)$ corresponds to an (elementary) cycle of G.

We shall apply the Jordan Curve Theorem in this form. *Let C be a Jordan curve, let $x\in \mathrm{Ext}\, C$, and let $y\in \mathrm{Int}\, C$. Then any Jordan arc connecting x and y meets C in at least one point.*

To each planar realization $e(G)$ of G we associate a set of (connected) domains $\{F_0, F_1, \dots, F_k\}$ in R^2 called the *faces* of $e(G)$. (For a definition of a 'face' see, for example, O. ORE [19].) There is just one unbounded domain F_0, the exterior face of $e(G)$; the other domains defining the interior faces satisfy $\mathrm{Int}\, F_i \cap e(G) = \emptyset$, $1 \leqq i \leqq k$, where $\mathrm{Int}\, F_i$ stands for the topological interior of the domain F_i.

Let w and y be two distinct points of a Jordan curve C. There are exactly two Jordan arcs lying in C having only w and y in common, say $A(w, y)$ and $A(y, w)$. Four distinct points of C constitute a *quadrilateral* (w, x, y, z) on C if $x\in A(w, y)$ and $z\in A(y, w)$. This basic topological property is due to C. R. PLATT [20]. *Let C be a Jordan curve and let (w, x, y, z) be a quadrilateral on C. Let E (respectively F) be a Jordan arc with endpoints w and y (respectively x and z) and suppose that E and F are both outside or both inside C. Then $E\cap F\neq\emptyset$.*

We treat now some of the basic terminology concerning 'planar' ordered sets as developed in D. KELLY and I. RIVAL [15]. Let P be a finite ordered set. Let π_1 and π_2 stand for the first and the second projections of R^2 onto R. A *planar embedding* $e(P)$ of P consists of

(1) an injection $x \to \bar{x}$ from P to R^2 such that $\pi_2(\bar{x}) < \pi_2(\bar{y})$ whenever $x<y$ in P, and

(2) straight line segments $\bar{x}\bar{y}$ connecting $\bar{x}$ and $\bar{y}$ whenever $x \prec y$ in P, and which do not intersect except possibly at their endpoints.

For simplicity of notation we shall identify each point $\bar{x}$ in the plane with the corresponding point x in P and use x for both. P is *planar* if it has a planar embedding. A planar representation $e(P)$ of P is defined by (1) above and

(2)′ increasing Jordan arcs denoted by xy with endpoints x and y whenever $x \prec y$ in P, and which do not intersect except possibly at their endpoints.

An increasing Jordan arc is defined by $A = \{(f(t), t) \mid t \in [\alpha, \beta]\}$, where f is a continuous function from a closed interval $[\alpha, \beta]$ of R to R. A *decreasing* Jordan arc is defined similarly.

D. Kelly has proved that these two notions are equivalent, so we may speak by turns of planar embeddings and planar representations. A maximal chain from x to y in P, denoted by (x, y), is a sequence $x = x_0, x_1, \ldots, x_n = y$ of elements of P with $x_i \prec x_{i+1}$, $0 \leqq i \leqq n-1$. In a planar representation $e(P)$ of a planar ordered set P any increasing Jordan arc in $e(P)$ connecting two vertices x and y and denoted by $A^+(x, y)$, corresponds to a maximal chain (x, y) of P.

Let G be a finite graph. We shall denote by $\mathscr{P}(G)$, $\mathscr{L}(G)$ and $\mathscr{L}_P(G)$ the sets of all ordered sets, all lattices and all planar lattices, respectively, each of whose covering graphs is G. Obviously $\mathscr{L}_P(G) \subseteqq \mathscr{L}(G) \subseteqq \mathscr{P}(G)$. G is called *orientable* whenever $\mathscr{P}(G) \neq \emptyset$. If G is any connected orientable graph G having at least two edges then $\mathscr{L}(G) \subsetneqq \mathscr{P}(G)$. On the other hand, if $\mathscr{L}_P(G) \neq \emptyset$ the equality $\mathscr{L}(G) = \mathscr{L}_P(G)$ need not hold at all (cf. Figure 7 and Figure 8).

Let G be a graph and suppose that $\mathscr{L}_P(G) \neq \emptyset$. Let $L \in \mathscr{L}_P(G)$ and let $e(L)$ be a particular planar realization of G. Let

$$F(e(L)) = \{F_0, F_1, \ldots, F_k\}$$

denote the set of faces of $e(L)$. It is a trivial consequence of the familiar Euler formula relating the numbers of vertices, edges and faces that *the number of faces is an invariant of* $\mathscr{L}_P(G)$, that is, if $L, L' \in \mathscr{L}_P(G)$ and $e(L)$, $e(L')$ are corresponding planar representations then

$$|F(e(L))| = |F(e(L'))|.$$

The vertices corresponding to the (not necessarily elementary) cycle of G associated with the exterior face F_0 of a planar representation $e(L)$ of $L \in L_P(G)$ determine the *boundary* $B(e(L))$ of $e(L)$. We can define the *left boundary* and the *right boundary* as the maximal chains corresponding to the Jordan arcs $A_l^+(0, 1)$ and

$A_r^+(0, 1)$ of $B(e(L))$ connecting the images of the extremal elements of L such that for each $x \in A_l^+(0, 1)$ and for each $y \in A_r^+(0, 1)$ satisfying $\pi_2(x) = \pi_2(y)$ then

$$\pi_1(x) \leqq \pi_1(y).$$

A *region* of $e(L)$ is a subset of L consisting of all elements of L in the area of the plane bounded by the Jordan arcs corresponding to the maximal chains C and D having the same extremal elements. A subset S of an ordered set P is *cover-preserving* if $x \prec y$ in S implies $x \prec y$ in P. *A region is a planar cover-preserving sublattice of* L (D. Kelly and I. Rival [15]).

Call a region a *strict region* if it is defined by two maximal chains having only their extremal elements in common. Therefore, the geometric curve in $e(L)$ corresponding to such a region R is a Jordan curve consisting of two increasing Jordan arcs having only their extremal elements in common. These endpoints are the images in $e(L)$ of the least and greatest elements of R. *Any interior face of* $e(L)$ *is a strict region of* L *whose interior contains no vertices or edges of* L (D. Kelly and I. Rival [15]).

In what follows we assume that, for $L \in \mathscr{L}_P(G)$ and for one of its embeddings $e(L)$, $B(e(L))$ is a strict region. Otherwise, L can be decomposed into a linear sum $L_1 \oplus L_2 \oplus \ldots \oplus L_k$ in which the top element of L_i is the bottom element of L_{i+1}, for each $i = 1, 2, \ldots, k-1$. In that case we can apply this more general result.

Proposition 3. *If* $L \in \mathscr{L}(G)$ *is a linear sum*

$$L = L_1 \oplus \ldots \oplus L_k$$

then, for any $L' \in \mathscr{L}(G)$,

$$L' = L_1' \oplus \ldots \oplus L_k' \quad \text{or} \quad L'^d = L_1' \oplus \ldots \oplus L_k',$$

where each L_i *and* L_i' *have the same covering graph and the same extremal elements.*

Proof. It is enough to prove this property with $L = L_1 \oplus L_2$. Let us denote by a the greatest element of L_1; it is also the least element of L_2. Let $L' \in \mathscr{L}(G)$. The element a cannot be either the least element $0'$ of L' or the greatest element $1'$ of L'. For, if $a = 0'$, say, then we may consider the element $0 \vee 1$ in L', where 0 and 1 are the least and greatest elements of L. Using maximal chains $(0, 0 \vee 1)$ and $(1, 0 \vee 1)$ in L' we can construct a path from 0 to 1 in L' which does not contain a. This is a contradiction, since every maximal chain in L from 0 to 1 must contain a.

We can suppose that there exists x in L satisfying $x < a$, in L and $x < a$ in L', for otherwise we consider L'^d, the dual of L'.

Let y be any element of $L_2 - \{a\}$, that is, $y > a$. There is at least one maximal chain in L from x to y and all such maximal chains contain a. Consequently, any path from x to y in L' contains a. We claim that $y > a$ in L'. Otherwise, consider $x \wedge y$ in L'. There exist two maximal chains in L' not containing a, $(x \wedge y, x)$ and

$(x \wedge y, y)$. Then we can construct in L' a path from x to y not containing a either. That is a contradiction.

Similarly, if $z < a$ in L then $z < a$ in L'. This completes the proof.

Now, we consider $L \in \mathscr{L}_P(G)$, any one of its planar embeddings $e(L)$ and $F \in F(e(L))$. For any $L' \in \mathscr{L}(G)$, let $D'(F)$ denote the subdiagram of L' induced by the elementary cycle of G corresponding to F.

Proposition 4. *Let $L \in \mathscr{L}_P(G)$, let $e(L)$ be a planar embedding of L, let $F \in F(e(L))$ and let $L' \in \mathscr{L}(G)$. Then $D'(F)$ is a planar lattice.*

Proof. We must show that $D'(F)$ consists of two maximal chains of L'. For contradictions suppose not. Then $D'(F)$ has at least two maximal and at least two minimal elements. ($D'(F)$ is an elementary cycle.) Let w and y be two distinct minimal elements of $D'(F)$ such that $h(w) \leqq h(y)$ where h is the height function of L' and let x, z be the two elements on the cycle adjacent to y.

There exist four maximal chains $(0', w)$, $(0', y)$, $(x, 1')$ and $(z, 1')$ in L', where $0'$, $1'$ are, respectively, the bottom, top elements of L' satisfying

$$(0', w) \cap (x, 1') = \emptyset, \quad (0', w) \cap (z, 1') = \emptyset,$$

$$(0', y) \cap (x, 1') = \emptyset \quad \text{and} \quad (0', y) \cap (z, 1') = \emptyset.$$

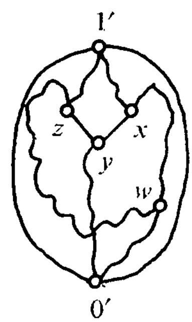


Figure 9

Using the Jordan arcs corresponding to these chains in $e(L)$ we can easily construct two Jordan arcs $A(w, y)$ and $A(x, z)$ of $e(L)$ which are inside C (the Jordan curve corresponding to F in $e(L)$) if F is the exterior face, or outside, otherwise (Fig. 10).

Now, (w, x, y, z) forms a quadrilateral on C and that is a contradiction since then $A(w, y)$ and $A(x, z)$ intersect in a point which is not a vertex of L. Then $D'(F)$ is formed by two maximal chains of L having in common only their endpoints.

In the case that $L' \in \mathscr{L}_P(G)$ we derive this consequence.

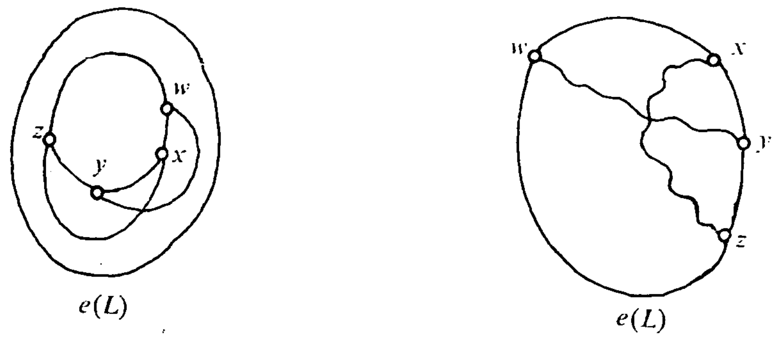


Figure 10

Corollary 5. *Let $L, L' \in \mathscr{L}_P(G)$ and let $e(L)$, $e(L')$ be corresponding planar representations. Then any face of $e(L)$ is a strict region of $e(L')$.*

The subset corresponding to a face of $e(L)$ is a planar, cover-preserving sublattice of L and is transformed to a planar, cover-preserving sublattice of L'. But this does not hold if we consider any planar, cover-preserving sublattice of L. The next figure, Figure 11, illustrates an example in which a sublattice of L is not transformed to a sublattice of L'.

Figure 11

Irreducible elements

An element of a lattice L is *doubly irreducible* in L if it has at most one lower cover and at most one upper cover. Let $\mathrm{Irr}(L)$ denote the set of all doubly irreducible elements of L. This fact was in a sense the start of the theory of planar lattices. *Any planar lattice has at least one doubly irreducible on the left boundary of any of its planar embeddings* (K. A. Baker, P. C. Fishburn, and F. S. Roberts [2]).

Proposition 6. *If L and L' are lattice orientations of the same covering graph G, then*

$$\big||\mathrm{Irr}(L)| - |\mathrm{Irr}(L')|\big| \leq 2.$$

Proof. Let us consider $T(G)=\{x\in V \mid \deg(x)=2\}$ where V is the vertex set of G and $\deg x$ is the *degree* of x. Then $\mathrm{Irr}\,(L)\subseteqq T(G)$ and

$$|T(G)|-2 \leqq |\mathrm{Irr}\,(L)| \leqq |T(G)|$$

because the greatest and the least elements of L can be in $T(G)$.

The number of doubly irreducible elements is not an invariant. In Figure 12 we illustrate three planar lattice orientations of the same covering graph having respectively 3, 4 and 5 doubly irreducible elements.

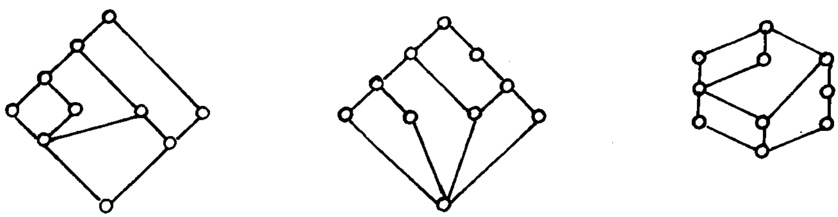


Figure 12

We are ready to prove our first principal result concerning doubly irreducible elements.

Theorem 1. *If* $\mathscr{L}_P(G)\neq\emptyset$ *then for each* $L'\in\mathscr{L}(G)$, $\mathrm{Irr}\,(L')\neq\emptyset$.

Proof. Consider $L\in\mathscr{L}_P(G)$ and let $e(L)$ be one of its planar embeddings. If $|\mathrm{Irr}\,(L)|\geqq 3$ the result is obvious according to the Proposition. Hence we can suppose that $\mathrm{Irr}\,(L)=\{a_1, a_2\}$ with a_1 on the left boundary of $e(L)$, say.

Now consider $L'\in\mathscr{L}(G)$ and suppose that $\mathrm{Irr}\,(L')=\emptyset$. Then L' cannot be planar and a_1 and a_2 must be the least and the greatest elements of L'. Our aim now will be to construct a planar embedding of L', which is a contradiction.

Let F_1 be the face containing a_1 in L and let us use F_1 too to denote the corresponding path in L'. According to Proposition 4, F_1 can have only one maximal element in L'.

Suppose that we have constructed in L' the subset corresponding to the faces $F_1, F_2, \ldots, F_k$, $k\geqq 1$, of L. $F_1\cup\ldots\cup F_k$ is a planar subset of L. Let us denote by $B_k(L)$ the path corresponding to its boundary, with respect to $e(L)$.

We shall show that the subset 'generated' by $\{F_1, \ldots, F_k\}$ in L' is planar and its boundary $B_k(L')$ is exactly the path $B_k(L)$.

The assertion is true for $k=1$. We proceed by induction on k. Let us denote by S the set of vertices of $B_k(L)$ which are contained in new faces of $e(L)$ in the sense that each vertex of S is adjacent to a vertex in $\mathrm{Ext}\,B_k(L)$. Let x_1 be a vertex of S of minimum height in L'. If F_{k+1} is a new face containing x_1 in L we can write

$$F_{k+1} = \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m\}$$

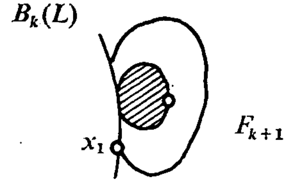


Figure 13

such that $n \geqq 3$, $m \geqq n$, $\{x_n, x_{n+1}, \ldots, x_m, x_1\} \subseteqq B_k(L)$ and $x_2, \ldots, x_{n-1} \notin B_k(L)$ (and possibly $x_n = x_m = x_1$). Notice that, since $\mathrm{Irr}(L') = \emptyset$, there can be no face within this new face F_{k+1} (cf. Figure 13).

Then the subset of L' generated by F_{k+1} has exactly one maximal element x_i and one minimal element x_j such that $1 \leqq i \leqq n$ and, $n \leqq j \leqq m$ or $j = 1$.

Now $x_2 > x_1$ in L'. Otherwise there exists a maximal chain $(a_1, \ldots, x_2, x_1)$ from a_1 to x_1 in L'. If y is the last point of intersection of this chain with $B_k(L') = B_k(L)$ then $y \in S$ and we have a contradiction because $h(y) < h(x_2) < h(x_1)$ in L', where h is the height function of L'.

On the other hand, if $x_n \neq x_1$ we have $x_{n+1} < x_n$ in L'. Indeed, if $x_{n+1} > x_n$ then consider three maximal chains (a_1, x_1), (a_1, x_n) and (x_{n+1}, a_2). Since $x_n \in S$, $h(x_1) \leqq h(x_n) < h(x_{n+1})$, $(a_1, x_1) \cap (x_{n+1}, a_2) = \emptyset$ and also $(a_1, x_n) \cap (x_{n+1}, a_2) = \emptyset$.

In L, using the Jordan arcs $A(a_1, x_1)$ and $A(a_1, x_n)$ corresponding to (a_1, x_1) and (a_1, x_n), respectively, we can construct a Jordan arc $A(x_1, x_n)$.

If $a_2 \notin F_{k+1}$ then the Jordan curve $C = A(x_1, x_n) \cup A(x_1, x_2, \ldots, x_n)$ (the latter part corresponding to the path $(x_1, x_2, \ldots, x_n)$ of F_{k+1}), has x_{n+1} in its inside which is a contradiction since $A(x_{n+1}, a_2)$, the Jordan arc associated with (x_{n+1}, a_2) would cut C in a point which is not a vertex of L (see Figure 14).

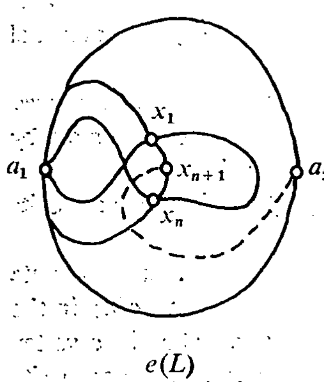


Figure 14

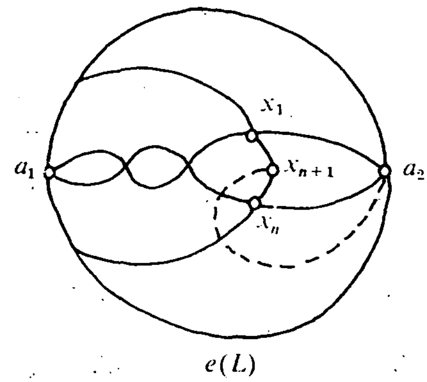


Figure 15

If $a_2 \in F_{k+1}$ then (x_1, a_2, x_n, x_{n+1}) is a quadrilateral on F_{k+1} since $x_1 \neq a_2$ and $x_n \neq a_2$. The previous Jordan arcs $A(x_1, x_n)$ and $A(x_{n+1}, a_2)$ must cut outside F_{k+1}, which is contradiction (cf. Figure 15).

We conclude by verifying that $B_{k+1}(L) = (B_k(L) - \{x_{n+1}, \ldots, x_m\}) \cup \cup \{x_2, \ldots, x_{n-1}\} = B_{k+1}(L')$.

So, if $F_1 \cup \ldots \cup F_k$ has a planar embedding in L', then $F_1 \cup \ldots \cup F_{k+1}$ also has a planar embedding in L'. Thus we obtain a planar embedding of L'.

The converse of Theorem 1 is false. There are finite graphs G such that, for each $L' \in \mathscr{L}(G)$, $\mathrm{Irr}\,(L') \neq \emptyset$, and yet there is no planar lattice orientation of G at all (see Figure 16).

Figure 16

Faces

Let G be a graph such that $\mathscr{L}_P(G) \neq \emptyset$ and let $L, L' \in \mathscr{L}_P(G)$. We denote by 0, 1 and $0'$, $1'$ the extremal elements of L and L', respectively. In this section we consider relations that exist between the faces of planar representations $e(L)$ and $e(L')$, of L and L'.

We shall require this.

Lemma 7. *If there exists L in $\mathscr{L}_P(G)$ such that $\mathrm{Irr}\,(L) \subseteq B(e(L))$ for a planar embedding $e(L)$ of L then $B(e(L)) = B(e(L'))$ for any planar embedding $e(L')$ of any L' in $\mathscr{L}_P(G)$.*

Proof. According to the Proposition 3 we can suppose that the exterior face F_0 of $e(L)$ is a strict region of L. Hence this property is also true for L'. Using the Corollary 5 the Jordan curve C_0' corresponding to the image of F_0 in L' defines a strict region, so it is a planar, cover-preserving sublattice of L'.

Let us assume that $B(e(L')) \neq B(e(L))$. By hypothesis, there exist at least two elements $a, b \in B(e(L)) \cap B(e(L'))$ such that $a, b \in \mathrm{Irr}\,(L')$, a is on the left boundary of L' and b is on the right boundary of L'. Let us denote by x, y, z, t respectively the four elements on the path of L' corresponding to $A(a, b) \subseteq C_0'$ such that $1 \in A(a, b)$, y is the first element not in $B(e(L'))$, x is its predecessor, t is the first element after y on $B(e(L'))$ and z is its predecessor, if the path is directed from a

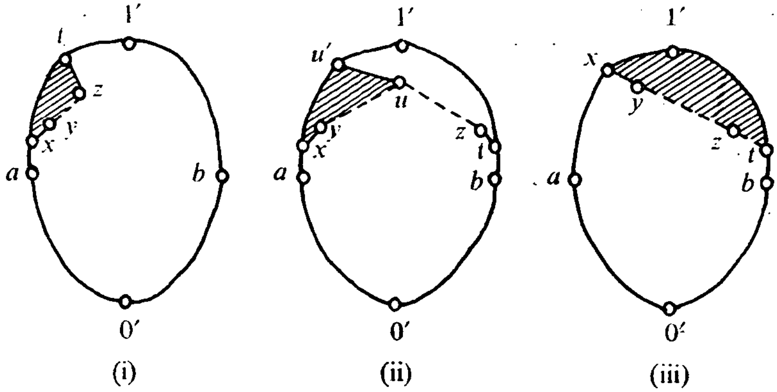


Figure 17

to b. In this way we get three different configurations as illustrated schematically in Figure 17. In the case (ii) u is the greatest element of the sublattice defined by C_0' in L' and u' is the first common element of a maximal chain from u to $1'$ in L'.

In each of these three cases the shaded strict regions must be planar, cover-preserving sublattices of L' and thus have at least one doubly irreducible element on their left boundaries, for cases (i) and (ii), and on the right boundary too. Hence we obtain a contradiction because the doubly irreducible elements of L' are necessarily on C_0'. Therefore $B(e(L'))=B(e(L))$.

Theorem 2. *If there exists L in $\mathcal{L}_P(G)$ such that* $\operatorname{Irr}(L)\subseteqq B(e(L))$ *for some planar embedding $e(L)$ of L then any planar embedding $e(L')$ of any $L'\in\mathcal{L}_P(G)$ satisfies $F(e(L'))=F(e(L))$.*

Proof. The previous lemma implies the invariance of the exterior face and thus $\operatorname{Irr}(L')\subseteqq B(e(L'))$.

Let F be any interior face of L. In L' the image of F defines a planar cover-preserving sublattice, say $L'(F)$. Let $0_F'$ and $1_F'$ denote its least and greatest elements, respectively.

Let us suppose that $L'(F)$ is not a face of $e(L')$.

If x and y are, respectively, elements on the left and on the right boundaries of $L'(F)$ then we claim that every path in $L'(F)$ from x to y contains either $0_F'$ or $1_F'$. Indeed if such a path $p(x, y)$ does not exist, let $(1_F', 1')$ and $(0', 0_F')$ be two maximal chains from $1_F'$ to $1'$ and from $0'$ to $0_F'$, respectively, in L'. We have $(0', 0_F')\cap$ $\cap p(x,y)=\emptyset$ and $(1_F', 1')\cap p(x,y)=\emptyset$.

Now consider $A(x,y)$, $A(1_F', 1')$ and $A(0', 0_F')$, the Jordan arcs in $e(L)$ corresponding to these paths. We know that $0'$ and $1'$ lie on the boundary of L (because $B(e(L))=B(e(L'))$. Thus we can consider $A(0', 1')$, a Jordan arc connecting $0'$ and $1'$ and having only these two points in common with $e(L)$. Therefore

$$A(0'_F, 1'_F) = A(0'_F, 0') \cup A(0', 1') \cup A(1', 1'_F) \quad \text{and} \quad A(x, y)$$

are two Jordan arcs lying outside of the Jordan curve C corresponding to F in $e(L)$.

Now, using the quadrilateral $(0'_F, y, 1'_F, x)$ on C these two Jordan arcs must cross at a point which is not a vertex of L. That is a contradiction (see Figure 18).

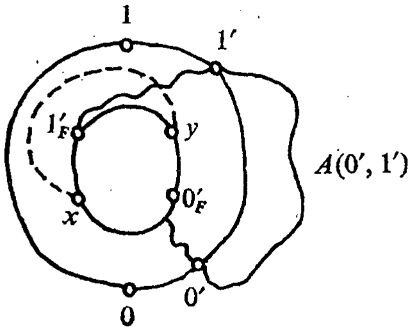


Figure 18

Now consider the upper covers $x_1, x_2, \ldots, x_k$, $k \geqq 2$, of $0'_F$ in $L'(F)$. In $L'(F)$, $\{x_i, x_{i+1}\}$, $1 \leqq i < k$, are in the same face, say F_i. Then there is $1 \leqq i \leqq k-1$ such that $1'_F \in F_i$. For otherwise, for each $1 \leqq i \leqq k-1$, $1'_F \notin F_i$ and there is a path from x_1 to x_k containing neither $0'_F$ nor $1'_F$, a contradiction according to the previous property.

The region R defined by the left boundaries of $L'(F)$ and F_i (see Figure 19) is a planar sublattice of L'. Then it must have at least one doubly irreducible element on its right boundary, which is impossible because $\mathrm{Irr}\,(L') \subseteq B(e(L'))$.

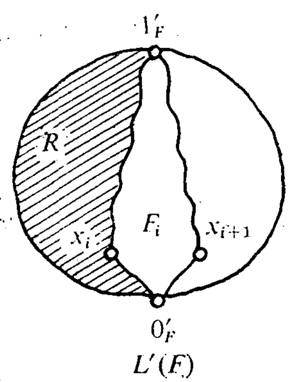


Figure 19

Then $L'(F)$ cannot contain an element in its inside and it must be a face of $e(L')$.

The converse of Theorem 2 is false. The planar lattice illustrated in Figure 20 has essentially just one planar lattice orientation and, in particular, the set of faces is invariant. Nevertheless, not all of its doubly irreducible elements lie on the boundary.

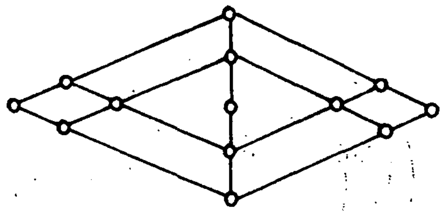
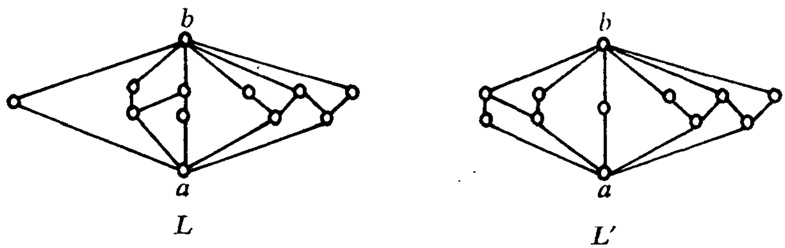


Figure 20

A conjecture

An understanding of the re-orientations of planar lattices may well advance our knowledge of the orientations of covering graphs. Are there 'canonical operations' which 'transform' one planar lattice orientation to another?

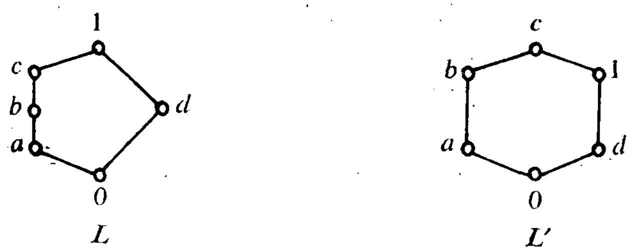
D. Kelly and I. Rival [15] have described a procedure, call it *permutation-reflection*, which can be applied to produce all planar embeddings from any fixed planar embedding of a planar lattice. Loosely speaking the idea is to consider $a, b \in L$ such that $a < b$ and all regions $R_1, R_2, \ldots$ with a and b as extremal elements. If $R_i \cap R_j = \{a, b\}$, we permute $R_i - \{a, b\}$ with $R_j - \{a, b\}$ (according to the linear order defined by considering the projections on the x-axis), without affecting the planarity itself (cf. Figure 21). Every planar embedding of L



L' is obtained from L by permutation-reflections
Figure 21

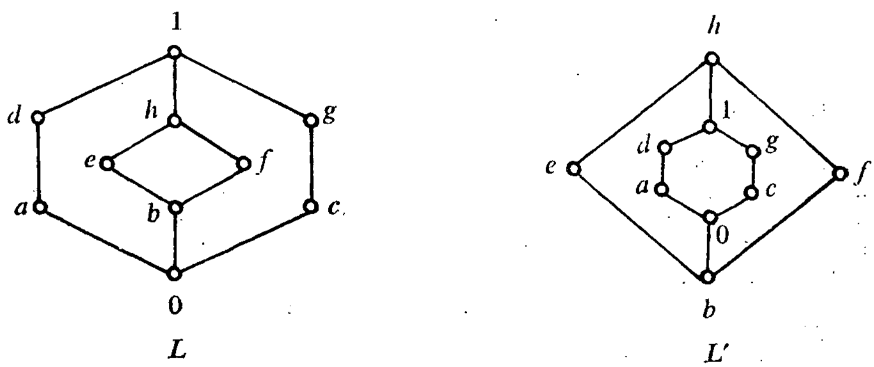
is produced from any fixed one by a sequence of permutation-reflection transformations.

For $L, L' \in \mathscr{L}_P(G)$ we say that L' is obtained from L by a *rotation* of L provided there are planar embeddings $e(L)$, $e(L')$ of L, L', respectively, such that $F(e(L)) = F(e(L'))$. For instance, under the hypotheses of Theorem 2, every planar lattice orientation L' is obtained from L by a rotation (cf. Figure 22).



L' is obtained from L by a rotation
Figure 22

The stereographic projection from the sphere to the plane and its inverse obtained by selecting the north pole in some face F produces a different planar embedding of a planar graph with F as its exterior face (cf. O. Ore [19], C. R. Platt [20]). This transformation which we shall call *inversion* leaves fixed the set of faces (see Figure 23). This transformation applied to an arbitrary face of a planar embedding of a planar lattice will not necessarily produce another planar lattice embedding. We do not at this time yet know which faces of a planar lattice embedding can be the exterior faces of a planar lattice embedding, obtained by inversion.



L' is obtained from L by inversion
Figure 23

Here is our conjecture. *Any planar embedding* $e(L')$ *of any* $L'\in\mathscr{L}_P(G)$ *is obtained from any* $L\in\mathscr{L}_P(G)$ *by a sequence of transformations each either a permutation-reflection, or a rotation, or an inversion.*

Added in proof. Theorem 1 has an important extension to *dismantlable lattices* (cf. D. Kelly and I. Rival [14], [15]).

Corollary. *Let* P *and* P' *be finite lattices with graph isomorphic covering graphs. If* P *is dismantlable then* P' *contains a doubly irreducible.*

Proof. If P is planar then the assertion is precisely Theorem 1. If P is nonplanar then the dimension of P is at least three (cf. [15]). In this case P contains at least three doubly irreducible elements (cf. Theorem 6.11, [15]). Then, as in Proposition 6, P' must contain a doubly irreducible element as well.

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