Acta Sci. Math., 50 (1986), 335—336

A note on minimal clones

ZS. LENGVÁRSZKY

For $3<k<n$ the existence of k-ary minimal clones of operations on the n-element set was proved by P. P. PÁLFY [1] who exhibited concrete examples. Here we give a very simple nonconstructive proof of this theorem.

First we prove the following claim:

Let $(A, \leq)$ be a partially ordered set and k a natural number such that the cardinality of an arbitrary antichain in $(A, \leq)$ is at most $k-1$. Then the arity of any nontrivial monotone semiprojection on $(A, \leq)$ does not exceed k.

Indeed, let f be a monotone semiprojection on $(A, \leq)$ with arity $m \geqq k+1$. Without loss of generality we may assume that f is a semiprojection to the first variable x_1. Let $a_1, \dots, a_m$ be arbitrary elements of A. By assumptions for some i and j, $2 \leqq i \neq j \leqq m$, $a_i \leq a_j$ holds. For $1 \leqq l \leqq m$ we define the elements b_l and c_l by

$$b_l = \begin{cases} a_i & \text{if } l = i, j, \\ a_l & \text{if } l \neq i, j, \end{cases} \quad c_l = \begin{cases} a_j & \text{if } l = i, j, \\ a_l & \text{if } l \neq i, j. \end{cases}$$

Since f is monotone,

$$a_1 = b_1 = f(b_1, \dots, b_m) \leq f(a_1, \dots, a_m) \leq f(c_1, \dots, c_m) = c_1 = a_1,$$

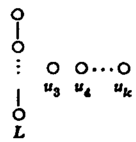
i.e. f is trivial.

In order to prove Pálfy's theorem, let k and n be natural numbers such that $3 \leqq k \leqq n$. Suppose we have an n-element poset $(A, \leq)$ such that the cardinality of an arbitrary antichain is at most $k-1$ and f is a nontrivial, monotone, k-ary semiprojection on $(A, \leq)$. Since A is finite, there exists a minimal clone C contained in the clone generated by f. Let g be a nontrivial function of minimal arity from C. Then g is a semiprojection (see [2]) with arity m. Clearly $m \geqq k$; on the other hand g is also monotone, hence $m \leqq k$ by our previous claim, i.e. $m=k$.

We will be done if for all k and n, $3 \leqq k \leqq n$, we give an n-element poset $(A, \leq)$ and a nontrivial, monotone, k-ary semiprojection f on it. For this aim let A be an

Received August 29, 1985.

n-element set and $u_3, \ldots, u_k$ distinct fixed elements in A, $L=A\setminus\{u_3, \ldots, u_k\}$. Let $\preceq$ be a parial order on A such that the distinct elements u and v are comparable if and only if $u, v \in L$ (see the diagram).



Then any antichain of $(A, \preceq)$ has at most $k-1$ elements. Define a k-ary operation f by

$$f(a_1, \ldots, a_k) = \begin{cases} a_2 & \text{if } a_1, a_2 \in L \text{ and } a_3 = u_3, \ldots, a_k = u_k \\ a_1 & \text{otherwise.} \end{cases}$$

Since $|L| \geqq 2$, f is a nontrivial semiprojection to the first variable. Suppose that $a_1 \preceq b_1, \ldots, a_k \preceq b_k$. If $a_1, a_2 \in L$ and $a_3=u_3, \ldots, a_k=u_k$, then $b_1, b_2 \in L$ and $b_3=u_3, \ldots, b_k=u_k$, hence $f(a_1, \ldots, a_k)=a_2 \preceq b_2=f(b_1, \ldots, b_k)$. If either $a_1, a_2 \in L$ or $a_3=u_3, \ldots, a_k=u_k$ does not hold then either $b_1, b_2 \in L$ or $b_3=u_3, \ldots, b_k=u_k$ does not hold. Thus, $f(a_1, \ldots, a_k)=a_1 \preceq b_1=f(b_1, \ldots, b_k)$, i.e. f is monotone, completing the proof.

References

[1] P. P. Pálfy, The arity of minimal clones, *Acta Sci. Math.*, 50 (1986), 000—000.

[2] I. G. Rosenberg, Minimal clones I: The five types, in: *Lectures in Universal Algebra* (Proc. Conf. Szeged, 1983), Coll. Math. Soc. János Bolyai, Vol. 43, North-Holland (Amsterdam, 1986); pp. 405—427.

BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY