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Abstract Galois theory and endotheory. I

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Introduction

Let us consider the situation in the classical Galois theory. Then we have a field extension K/k which is either normal and algebraic or algebraically closed. A permutation $\sigma: K \to K$ is called an *automorphism* of K/k if, for each $a, b \in K$ and $\alpha \in k$, we have

(1) $$\sigma \cdot (a+b) = \sigma \cdot a + \sigma \cdot b,$$

(2) $$\sigma \cdot ab = (\sigma \cdot a)(\sigma \cdot b),$$

and

(3) $$\sigma \cdot \alpha = \alpha.$$

Observe that the assumption "bijective" can be replaced by the weaker one "*surjective*", i.e., the automorphisms of K/k are just the surjective mappings $\sigma: K \to K$ satisfying (1), (2) and (3). This observation follows readily from the fact that fields have no non-trivial ideals.

The automorphisms of K/k are known to form a group under the composition of mappings. This group, denoted by $G(K/k)$, is called the *Galois group* of K/k. Let A be a subset of K and let $G(K/k; A)$ denote $\{\sigma \in G(K/k); (\forall a \in A)(\sigma \cdot a = a)\}$, which is a subgroup of $G(K/k)$. Then $\bar{A}_k = \{\bar{a} \in K; (\forall \sigma \in G(K/k; A))(\sigma \cdot \bar{a} = \bar{a})\}$, the set of all elements in K preserved by each $\sigma \in G(K/k; A)$, is called the *Galois closure* of A. The classical Galois theory asks and answers the following two questions:

(a) How to characterize $\bar{A}_k$ in terms of A and the field extension structure $(K; x+y, xy, k)$ of K/k? Answer: $\bar{A}_k$ is the closure of $A \cup k$ with respect to the operations $x+y$, xy (defined on $K \times K$), x^{-1} (defined on $K \setminus \{0\}$) and, if the characteristic p of k is not zero, $\sqrt[p]{x}$ (defined on $K^p = \{a^p; a \in K\}$).

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(b) Which subgroups of $G(K/k)$ are "Galoisian", i.e. of the form $G(K/k; A)$ for some $A \subseteq K$? When the degree $[K:k]$ is finite then the answer is: all subgroups.

Conditions (1), (2) and (3) in defining automorphisms seem heterogeneous at the first sight since (1) and (2) concern the preservation of some binary operations while (3) concerns the preservation of some elements. Yet, these conditions turn out to be of the same nature when formulated in terms of relations. There are two manners of defining the automorphisms of K/k in this way:

I. An automorphism of K/k is a permutation σ of K such that

(α) $a+b=c \Leftrightarrow \sigma\cdot a+\sigma\cdot b=\sigma\cdot c$,

(β) $ab=c \Leftrightarrow (\sigma\cdot a)(\sigma\cdot b)=\sigma\cdot c$, and

(γ) for each $\alpha\in k$, $a=\alpha \Leftrightarrow \sigma\cdot a=\alpha$.

We can formulate Conditions (α), (β) and (γ) by saying that σ preserves the relations $x+y=z$, $xy=z$ and, for each $\alpha\in k$, $x=\alpha$, where σ is said to preserve a, say ternary, relation ϱ if for any triple $(a,b,c)\in K^3$ we have $(a,b,c)\in\varrho \Leftrightarrow (\sigma\cdot a,\sigma\cdot b,\sigma\cdot c)\in\varrho$.

II. An automorphism of K/k is a self-surjection μ of K such that

(α') $a+b=c \Rightarrow \mu\cdot a+\mu\cdot b=\mu\cdot c$,

(β') $ab=c \Rightarrow (\mu\cdot a)(\mu\cdot b)=\mu\cdot c$, and

(γ') for each $\alpha\in k$, $a=\alpha \Rightarrow \mu\cdot a=\alpha$.

If we drop the condition that μ is surjective and consider self-mappings of K satisfying (α'), (β') and (γ') then we obtain the notion of *endomorphisms* of K/k. It is not hard to show that all the endomorphisms of K/k are automorphisms, provided K/k is normal and algebraic. But in the general case the endomorphisms of K/k form only a monoid, i.e. a unitary semigroup, $D(K/k)$ with respect to the composition of mappings. This monoid always contains 1_K, the identical mapping of K. As previously, $D(K/k, A)$ will denote the set $\{\delta\in D(K/k);\ (\forall a\in A)(\delta\cdot a=a)\}$, which is a submonoid of $D(K/k)$. Further, it can be proved that $\{\bar{a}\in K;\ (\forall\delta\in D(K/k))(\delta\cdot\bar{a}=\bar{a})\}$ is the closure of $A\cup k$ with respect to the operations $x+y$, xy, x^{-1} and, if $p\neq 0$, $\sqrt[p]{x}$, as before.

We can formulate Conditions (α'), (β') and (γ') by saying that δ *stabilizes* the relations $x+y=z$, $xy=z$ and, for each $\alpha\in k$, $x=\alpha$, i.e. δ transforms every system of values satisfying any of these relations into a system of values satisfying the same relation, but nothing is required for systems of values not satisfying these relations.

In both of these manners we have a particular case of the following general situation: given a relational system, i.e. a base set A (here $A=K$) and some relations on it (here $x+y=z$, $xy=z$ and $x=\alpha$ for $\alpha \in k$). (In the sequel relational systems will be referred to as *structures* though the term "structure" has a more general meaning in the literature.) We consider the group G of all permutations of the base set preserving the given relations and the monoid D of self-mappings of this set stabilizing these relations. Then in Question (a) we asked what was the set of relations of a certain form (in our case $x=\bar{a}$ for $\bar{a} \in K$) preserved by all $\sigma \in G$ and stabilized by all $\delta \in D$? The answer was that it is the set of relations $x=\bar{a}$ where $\bar{a}$ belongs to the closure of $A \cup k$ with respect to some operations arising from the relations $x+y=z$ and $xy=z$.

The above considerations naturally raise the idea of considering any first order structure E/R where R is a set of (not necessarily finitary) relations on a base set E, the *automorphism group* alias *Galois group* G of E/R (consisting of all permutations of E that preserve each $r \in R$), and the *endomorphism monoid* alias *stability monoid* of E/R (consisting of all self-mappings of E stabilizing each $r \in R$). Then the question analogous to (a) is how to characterize the relations on E that are preserved by all $\sigma \in G$ or that are stabilized by each $\delta \in D$, respectively. Of course, we want the answer be somewhat similar to that in the classical Galois theory. Therefore the answer should be (and, in fact, will be) that they are the relations belonging to the closure of R with respect to some appropriate operations. But, first of all, all such relations do not form a set because any set occurs among their argument sets. So, at least in the first study, we have to limit the argument sets of the considered relations so that we fix a set X^0 (of sufficiently large cardinality) and consider the relations whose argument sets are subsets of X^0. On the other hand, we cannot hope in this general situation that the sets

$$\bar{R}(X^0) = \{r;\ r \subseteqq E^X \text{ for some } X \subseteqq X^0 \text{ and each } \sigma \in G \text{ preserves } r\}$$

and

$$\bar{\bar{R}}(X^0) = \{r;\ r \subseteqq E^X \text{ for some } X \subseteqq X^0 \text{ and each } \delta \in D \text{ stabilizes } r\}$$

are the closures of R with respect to some operations on the base set E arising from the relations $r \in R$. A priori, it may be hoped only that $\bar{R}(X^0)$ and $\bar{\bar{R}}(X^0)$ are closures of R with respect to some set theoretical operations on relations and these operations do not depend on the particular choice of R. Such is the case, indeed: there exists a family of such operations, called *fundamental operations*, so that $\bar{R}(X^0)$ is the closure (within the set of relations with argument sets included in X^0) of R with respect to these operations while $\bar{\bar{R}}(X^0)$ is the closure of R with respect to a part of this family, called the family of *direct fundamental operations*. The theory concerning the preservation of relations by permutations of the base set and the Galois group is called *abstract Galois theory* while that dealing with the stability of relations by arbitrary self-mappings of E and with the stability monoid is called *abstract Galois endotheory*. Although Question (a) has the same answer for both theories in the classical Galois theory, it has quite different answers in the general case. Question (b) in the general case asks: which permutation groups on E are the Galois groups of appropriate structures E/R and which monoids of self-mappings of E are endomorphism monoids of some structures E/R? The answer is that *any* group and *any* monoid on E are such.

Once this study with a fixed X^0 has been done we can introduce the classes $\bar{R}$ and $\bar{\bar{R}}$ of *all* relations preserved by each $\sigma \in G$ and stabilized by each $\delta \in D$, respectively, and we can study them in a non-axiomatic way. In the present paper we will do it roughly within the frame of Bernays—Gödel axiomatic set theory albeit this is not the only possibility. On the deep analogy of the terminology of classical Galois theory, the classes $\bar{R}$ and $\bar{\bar{R}}$ will be called *abstract fields* and *abstract endofields*, respectively. With some precaution, it is possible to consider and to study certain mappings between them. In particular, a bijection of an abstract field or endofield onto another one (with a different base set in general) which commutes with all fundamental operations is called an isomorphism. It will be proved that any isomorphism between abstract fields or, under certain conditions, between abstract endofields is of a special form called transportation of structures. After these so-called isomorphism theorems the notion of abstract endofield homomorphisms is also introduced and a much more difficult homomorphism theorem is proved in order to characterize these homomorphisms.

Let k be an abstract endofield (which may be, in particular, an abstract field), and let A be a subset of the corresponding base set E. Denote by $k(A)$ the endo-extension of k by the set of relations $\{x=a;\ a \in A\}$, i.e., the abstract endofield generated by $k \cup \{x=a;\ a \in A\}$. Then $k(A)$ does not depend on the particular choice of x. Such extensions $k(A)$ of k are called its *set extensions*, and their study is called *abstract Galois set theory*. A theorem is proved, which describes the relations in $k(A)$ in terms of A and the relations in k. As a consequence of this theorem, a family of partial operations on E is defined from k such that $\bar{A}_k = \{e; (x=e) \in k(A)\}$, the so-called *rationality domain* of $k(A)$, is just the closure of A with respect to these partial operations. This result can be considered as the first step of deducing the answer to (a) in the classical Galois theory from that given in the general Galois theory. The second step is the theory of "eliminating structures", which will not be exposed in this paper. The third step starts from the main result of the second one and allows us to understand and to foreseee the deep reasons why $\bar{A}_k$ is the closure of $A \cup k$ with respect to $x+y$, xy, x^{-1} and $\sqrt[p]{x}$ in the classical Galois theory.

While dealing with the theory concerning $\bar{R}(X^0)$ and $\bar{\bar{R}}(X^0)$ then X^0 is of sufficiently large cardinality means that card $X^0 \geqq$ card E. In particular, when the base set E is finite, we can take a finite X^0. Then the fundamental operations become the realisations of operations in the (first order) predicate calculus with equality for

X^0 as the set of object variables on the model E/R. Similarly, the direct fundamental operations are just the realizations of a "strongly positive" part of this calculus, which is generated by $\vee$, &, $\exists x$ for $x \in X^0$, adjunctions of variables belonging to X^0, equalities $x_i = x_j$ $(x_i, x_j \in X^0, x_i \neq x_j)$, and the identity. So in the general case the fundamental operations can be considered as infinitary generalizations of the previous finitary operations.

Except for certain points our approach is independent from the axiom of choice, but this is not the case for the afore-mentioned theory of "eliminating structures".

In Section 1 we give the precise notions of points, relations, structures, etc., define the action of mappings of the base set on them, and introduce the Galois group, the stability monoid and their "invariants". The fundamental operations are defined and studied in Section 2. In Section 3 we study the interaction between mappings (and, in particular, self-mappings) of base sets and fundamental operations. In Section 4 we prove the main theorems, i.e. the equivalence and existence theorems, of the abstract Galois theory and endotheory, but considering only relations whose argument sets are included in some fixed set X^0 such that card $X^0 \geqq$ card E. In Section 5 we introduce the abstract fields and endofields, study their mappings and, in particular, prove isomorphism and homomorphism theorems. Finally, Section 6 is devoted to the abstract Galois set theory.

Historical remarks. I found the abstract Galois theory during the summer vacation of 1935, submitted it to the jubelee volume of Journal des mathématiques pures at appliquées dedicated to J. Hadamard in 1936, and this first exposition [1] of the theory appeared in 1938. It is the following question that was the intuitive origin of this research: Let $z_1, \ldots, z_n$ be the roots of a polynomial $f(X)$ of degree n over some base field k, and let G be the Galois group of $f(X)$; how can the system of equations satisfied by the n-tuples $(\sigma \cdot z_1, \ldots, \sigma \cdot z_n)$ $(\sigma \in G)$, and only by these n-tuples, be obtained from the rational relations among the roots $z_1, \ldots, z_n$? (This situation is somewhat obscured in the usual treatments of the classical Galois theory because of the use of the so-called "Galois resolvent", i.e. the replacement of the n-tuple $(z_1, \ldots, z_n)$ by a convenient linear combination $\xi = \Sigma \lambda_i z_i$ $(\lambda_i \in k)$ of z_i, and, in more modern treatments, because of the emphasis put on the field structure and the normal case.) This question led me to the fundamental operations and gave me the key idea to the proof of the equivalence theorem, i.e., the idea of considering the relation r^* (cf. Section 4, later).

This theory, as elaborated in a set theoretical frame, had certainly no precursors. Yet, it might be connected with some vague ideas or projects expressed before in much narrower contexts. A rather enigmatic phrase in the last letter by Galois to his friend Auguste Chevalier ("Tu sais, mon cher Auguste que ces sujets ne sont pas les seuls que j'aie explorés. Mes principales méditations depuis quelque temps étaient dirigées sur l'application à l'analyse transcendante de la théorie de l'ambiguïté. Il s'agissait de voir à priori dans une relation entre les quantités ou fonctions transcendantes quelles échanges on pouvait faire, quelles quantités on pouvait substituer aux quantités données sans que la relation pût cesser d'avoir lieu. Cela fait reconnaître tout de suite l'impossibilité de beaucoup d'expressions que l'on pourrait chercher. Mais je n'ai pais le temps et mes idées ne sont pas bien développées sur ce terrain, qui est immense") suggests that he had a vague feeling that his theory for roots of polynomial equations is, maybe, only a particular case of a much wider theory, where the polynomial relations among roots are replaced by some more or less arbitrary relations on convenient domains. Clearly, these domains and relations could be only as general as conceivable for him and his contemporaries. However, the mathematics of that time was not set theory but a science of magnitudes, i.e., of real and complex numbers and sufficiently smooth functions (in the quoted phrase Galois speaks, among others, of the "quantités ou fonctions transcendantes"), and the relations used in this mathematics were differential, functional, etc. equations and systems of equations. Some people see, in this phrase, an allusion to what he knew about abelian functions. But if it had been so, he would not have spoken of his "méditations" and he certainly would not have written "cela faît reconnaitre tout de suite l'impossibilité de *beaucoup* d'expressions ..."; in fact, he would have spoken rather of the results known by him.

Cayley and Silvester's theory of invariants may be considered as a vague precursor of the abstract Galois theory in some very particular case. Indeed, if A is an invariant of some "generic" form (i.e., its coefficients are independent variables), a is a value of A, and the set of all points (in some field) of all projective varieties for which a is the value of A is assigned to a, then this set of points is preserved by all elements of the projective group or some of its subgroups. So, an invariant A can be interpreted by the set $\{r_{A,a}\}$ consisting of relations of the form $r_{A,a}$ which are invariant with respect to some (very particular) groups. Remarkably enough, Cayley raised the problem of determining *all* the invariants. Yet, he could not consider the definition of these groups as that of Galois groups with respect to a system of invariants.

The "Galois type theories" which appeared for differential equations at the end of the nineteenth century (S. Lie, Picard—Vessiot, etc.) can hardly be considered as the beginning of the abstract Galois theory because they were developed, like the classical Galois theory, by using quite particular methods, specific for each case and without any more general spirit. Nevertheless they created a vague feeling that Galois' ideas were, in some unknown manner, applicable in a wider framework.

In some very faint sense the abstract Galois theory for arbitrary (first order) structures can be considered analogous to F. Klein's "Erlangener Programm" for geometries (1872), though the former goes far beyond this program. Klein's main idea is that a geometry is a system of "all invariants" of an "arbitrary" permutation group of a "Mannigfaltigkeit" ("variety"). As no set theory existed that time (the first results on abstract sets were obtained by Cantor in 1873), Klein could not express his ideas in a precise manner. In particular, he did not give any precise meaning of "all invariants" of "Mannigfaltigkeit" (which seems to be a riemannian variety or something stronger), and his "arbitrary" groups are far from being arbitrary as their elements must be automorphisms of the structure of "Mannigfaltigkeit". Anyhow, though the abstract Galois theory deals with the duality between permutation groups and classes of their invariant relations, it is much more than a simple idea of this duality.

In my first paper [1] on abstract Galois theory the fundamental operations are defined almost in the same way as in this paper, but in a rather misleading manner: *instead of arbitrary argument sets canonical argument sets, indexed by ordinals*, are used, whence the axiom of choice is widely used without real necessity. In later publications ([2] and other papers, which I do note quote) I only formulated the results of the theory in a clearer form by using canonical identifications (cf. Section 3 later), but without giving new proofs because of those in [1] being easily adaptable to the new manner. There the essential part of the theory is developed for relations with a fixed, sufficiently large (i.e., of cardinality not smaller than card E) argument set X. The defect of this approach is that certain rather complicated and not very intuitive fundamental operations (called "mutations") have to be considered as well. That is why I return to the first manner, but without its past imperfections.

I found the abstract Galois endotheory, if my memory is good, in 1964, and formulated it in printed form first in the paper [3] of the International Congress of Mathematicians, Moscow, 1966. I found the first but not completely true version of the homomorphism theorem of this theory in 1966, before the above-mentioned congress. The gap in its proof was remarked by some my 3rd year students in 1973, After having tried to fill this gap unsuccessfully I found the necessary modification which made the theorem correct, i.e., I replaced norms by regular pseudo-norms, and published it in [5]. Yet, until the present paper, the abstract Galois endotheory (even without its homomorphism theorem) has never been published with complete proofs. Only some rough ideas of certain proofs were indicated in [4].

The idea of abstract Galois set theory (in relation with eliminative structures and the passage from the abstract Galois theory to the classical one) goes back as early as to the end of thirties. However, the proof of its main theorem was put in a clear form only in the first years after the second world was (1946? 1947?). This theorem and the characterization of rationality domains (in case of abstract fields) were formulated in several papers, but never with proofs. I exposed the abstract

Galois theory and, after it had come to existence, the abstract Galois endotheory in my Seminar (1953—1959) and in my courses for 3rd year students (Clermont-F^d, 1960—1965, and Paris, 1965—1980).

Terminology and notations. We shall use the ordinary notations of set theory and mathematical logic. The set difference $\{x\in A;\ x\notin B\}$ will be denoted (in Russian manner) by $A\setminus B$, though personally I prefer the notations $A\cdot\cdot B$ or $A\urcorner B$ to $A\setminus B$ as they cannot interfere with algebraic notations. When $a\in A$, $\bar{A}\subseteq A$ and $\varphi\colon A\to B$ is a mapping, the φ-image of a and $\bar{A}$ will be denoted by $\varphi\cdot a$ and $\varphi\cdot\bar{A}$, respectively. This will be the only mathematical use of the dot "$\cdot$". The product of two objects, say x and y, will be denoted by xy. Similarly, the composition of mappings ψ and φ will be denoted by $\varphi\psi$ or sometimes, following Bourbaki's notation, by $\varphi\circ\psi$. The mappings $\psi\colon C\to D$ and $\varphi\colon A\to B$ are considered composable iff $\varphi\cdot A\subseteq C$; in this case $\psi\varphi$ (or $\psi\circ\varphi$) is the mapping $A\to D$ such that $\psi\varphi\cdot a= =\psi\cdot(\varphi\cdot a)$ for every $a\in A$. If $\varphi\colon A\to B$ is a mapping, $a\in A$ and $b=\varphi\cdot a$, then we write $\varphi\colon a\to b$ if a is an arbitrary element of A while $\varphi\colon a\mapsto b$ is reserved for the case of fixed a. The equivalence relation $\{(x,y);\ \varphi\cdot x=\varphi\cdot y\}$ on A, referred to as the kernel of φ in the literature, will be called the *type* of φ and will be denoted by $T(\varphi)$. If $\varphi\colon A\to B$ is a mapping and C is an equivalence relation on A then φ is said to be *compatible* with C iff $T(\varphi)$ is thicker than C, i.e., iff $x\equiv y \pmod C$ always implies $x\equiv y \pmod{T(\varphi)}$. Similarly, a mapping $\gamma\colon A\to C$ is called compatible with a mapping $\beta\colon A\to B$ iff it is compatible with $T(\beta)$, i.e., if for every $a\in A$ the image $\gamma\cdot a$ depends only on $\beta\cdot a$.

Let F and G be two families of subsets of A and B, respectively, and suppose F and G are complete semilattices with respect to the union $\cup$. Consider a mapping $\Phi\colon F\to G$. This mapping is said to be *additive* if $\Phi\cdot\bigcup_{X\in\bar{F}} X=\bigcup_{X\in\bar{F}}\Phi\cdot X$ holds for every subfamily $\bar{F}$ of F, i.e., if Φ commutes with the operation $\cup$. If $\Phi\colon F\to G$ is additive and, in addition, F is the family of all subsets of some subset $\bar{A}$ of A, then Φ is called *$\bar{A}$-hyperpunctual.* Then we have $\Phi\cdot X=\bigcup_{x\in X}\Phi\cdot\{x\}$ for $X\subseteq\bar{A}$, and we shall write $\Phi\cdot x$ instead of $\Phi\cdot\{x\}$. If $\Phi\colon F\to G$ is an $\bar{A}$-hyperpunctual mapping and the image $\Phi\cdot\{x\}$ of any singleton in F is a singleton (denoted by $\{\varphi\cdot x\}$) in G, then Φ is called *punctual* and the mapping $\bar{A}\to B$, $x\to\varphi\cdot x$ is called the *point-mapping* of Φ. If φ is injective, Φ is called an *injectively punctual* mapping. Then φ induces a bijection φ^0 of $\bar{A}$ onto $\varphi\cdot\bar{A}\subseteq B$, and if Φ^0 is the mapping of the family $P(\bar{A})$ of all subsets of $\bar{A}$ onto $P(\varphi\cdot\bar{A})$ which prolongs φ^0 (i.e., induced by Φ), then Φ^0 clearly commutes with all boolean operations. So, if $\bar{P}$ is a subfamily of $P(\bar{A})$, we have

$$\Phi\cdot\bigcup_{X\in\bar{P}} X=\Phi^0\cdot\bigcup_{X\in\bar{P}} X=\bigcup_{X\in\bar{P}}\Phi^0\cdot X=\bigcup_{X\in\bar{P}}\Phi\cdot X,$$

$$\Phi\cdot\bigcap_{X\in\bar{P}} X=\Phi^0\cdot\bigcap_{X\in\bar{P}} X=\bigcap_{X\in\bar{P}}\Phi^0\cdot X=\bigcap_{X\in\bar{P}}\Phi\cdot X$$

and, for $X \subseteq \bar{A}$,

$$\Phi \cdot (\bar{A} \setminus X) = \Phi^0 \cdot (\bar{A} \setminus X) = \varphi \cdot \bar{A} \setminus \varphi \cdot X = \Phi \cdot \bar{A} \setminus \Phi \cdot X.$$

Sometimes, in order to define certain notions or to formulate certain results with elegance and well, we shall have to deal not only with sets but with classes as well. This will be done only for the sake of convenience and better understanding, but not for raising any question of Foundations. In fact, all we do could be done in the language of sets, though in a longer and more complicated way. In principle, the word "class" will be understood in Bernays' sense*) but in a freer and more naive manner of speaking. However, as it can be shown, this manner of speaking is only an "abuse of the language" from the point of view of Bernays' theory, since the existence of classes and their mappings occurring in this paper can be proved based on Bernays—Gödel axioms.

1. Relations, structures and mappings

Let E and X be two sets called *base set* and *argument set*, respectively. Generally, for the sake of some of the proofs, we assume that E consists of at least two elements, though our results are trivially valid for a one-element base set E, too. The elements of E^X are called *X-points* while subsets of E^X are called *X-relations*. I.e., X-points are mappings of X into E (denoted by, e.g., $P: X \to E$ or $X \xrightarrow{P} E$) and X-relations are sets of such points. When there is no danger of ambiguity, we often do not indicate the argument set X. An (ordered) pair $S=(E, R)$ is called a (first order) *structure* on E (or *with base set* E) provided R is a non-empty set of relations on E. (The argument set X_r of $r \in R$ may depend on r.) In particular, when all the relations in R have the same argument set, say X, then $S=(E, R)$ is called an *X-structure*. By the *arity* of an X-point, X-relation or X-structure we mean the cardinal $\operatorname{card} X$ of X. We say that a structure $S=(E, R)$ is *under* a set X^0 if $X_r \subseteq X^0$ for all $r \in R$.

Let $d: E \to E'$ be a mapping between the sets E and E'. This mapping can be extended to points, relations, sets of relations and structures on E in the following evident way: put $d \cdot P = d \circ P$ (Bourbaki's notation!), $d \cdot r = \{d \cdot P;\ P \in r\}$, $d \cdot R = \{d \cdot r;\ r \in R\}$, $d \cdot (E, R) = (E', d \cdot R)$. In particular, when δ is a self-mapping of E (i.e. $E'=E$), we say that δ *stabilizes* a relation r on E (or, in other words, r is *stable* by δ) iff $\delta \cdot r \subseteq r$. We say that a permutation σ of E *preserves* r (in other

*) But this does not mean that I consider the foundation of mathematics based on the Bernays—Gödel axioms (and, generally, based on any predicative calculus formalism) as an adequate one.

words, r is *preserved* by σ or *invariant* by σ) iff $\sigma \cdot r = r$. More generally, a self-mapping δ of E is said to preserve r iff it stabilizes both r and its complement $E^{X_r} \setminus r$. We say that δ is *stabilizing* or *preserving on* a set R of relations if it stabilizes or preserves each $r \in R$, respectively. For a structure $S=(E, R)$ the set of all self-mappings of E that are stabilizing on R constitutes a monoid with respect to the composition of mappings. This monoid is called the *stability monoid* or *endomorphism monoid* of E/S (or S), and it is denoted by $D(E/S)$ or, sometimes, by $D(E/R)$. The endomorphism monoid is never empty for it always contains the identical mapping of E. The set of all permutations of E that are preserving on R is called the *Galois group* or *automorphism group* of E/S (or S) and is denoted by $G(E/S)$ or $G(E/R)$.

Remark 1. $G(E/S)$ is the largest permutation group contained in $D(E/S)$. Indeed, a permutation σ belongs to $G(E/S)$ iff σ and σ^{-1} belong to $D(E/S)$, whence the assertion follows.

Remark 2. Suppose R is a set of relations on E such that $r \in R$ implies $\neg r = E^{X_r} \setminus r \in R$. Then $G(E/S)$ consists of the permutations that belong to $D(E/S)$. This remark is a straightforward consequence of the definitions.

Particular relations. Firstly, we mention the *empty relation* $\emptyset$, which is the only relation without a unique argument set and base set. I.e., $\emptyset$ can be considered an X-relation on E for any X and E. The *X-identity* on E is E^X and is also denoted by $I(X, E)$. Let C be an equivalence relation on X. Then

$$I_C(E) = \{P \in E^X;\ (\forall x \in X)(\forall x' \in X)(x \equiv x' \pmod C \Rightarrow P \cdot x = P \cdot x')\}$$

is called the *C-multidiagonal* on E. It consists of all X-points that are compatible with C. In particular, if $\bar{X} \subseteq X$ is a C-class and all C-classes but $\bar{X}$ are singletons then

$$D_{\bar{X}}(E) = I_C(E) = \{P \cdot E^X;\ (\forall x \in \bar{X})(\forall x' \in \bar{X})(P \cdot x = P \cdot x')\}$$

is called the $\bar{X}$-diagonal of E. When $\bar{X} = \{x, y\}$,

$$D_{x,y}(E) = D_{\{x,y\}}(E) = \{P \in E^X;\ P \cdot x = P \cdot y\}$$

is called the (x, y)-diagonal on E, and such diagonals are called *simple*. The relation

$$I_C^0(E) = \{P \in I_C(E);\ T(P) = C\}$$

is called the *strict C-multidiagonal* on E, while $E^X \setminus I_C(E)$ is referred to as the C-antidiagonal on E.

A relation r will be called *semi-regular* iff there exists an equivalence relation C on its argument set X such that $r \subseteq I_C(E)$ and $C = T(P)$ for some $P \in r$. When r is semi-regular then this C is unique, is denoted by $T(r)$, and is called the type of r. Further, $t(r) = \{P \in r;\ T(P) = T(r)\}$ is called the *head* of the semi-regular relation r. If $r \subseteq I^0_C(E)$ for some C, i.e. r is semi-regular and $r = t(r)$, then r is said to be *regular*.

An X-point $P\colon X \to E$ is said to be *surjective, injective* and *bijective* if it is such as a mapping, while a relation r is said to be such if every $P \in r$ is such. In particular, r is injective iff it is regular and $T(r)$ is the discrete equivalence relation on X.

In case $X = \emptyset$ there is only one $\emptyset$-point $P_\emptyset(E)\colon \emptyset \to E$ (indeed, no two mappings can differ at any argument belonging to $\emptyset$). Hence there are only two $\emptyset$-relations: $\emptyset$ and $I(\emptyset, E) = \{P_\emptyset(E)\}$.

Let $D(E)$ denote the monoid consisting of all self-mapping of E, called the *symmetric monoid* of E, and let $S(E)$ stand for the (full) *symmetric group* of E, consisting of all permutations of E. For a subset Δ of $D(E)$ the *class* of all relations on E that are stabilized resp. preserved by each $\delta \in \Delta$ will be denoted by s-Inv Δ resp. p-Inv Δ, and will be called the *stability invariant* resp. *preservation invariant* of Δ. (Note that these classes are never sets.) When the context shows clearly what kind(s) of invariants is considered, the letters s or p before Inv may be omitted. Clearly, the mappings $\Delta \mapsto \text{s-Inv}\,\Delta$ and $\Delta \mapsto \text{p-Inv}\,\Delta$ are decreasing. Further, if Θ is a family of subsets of $D(E)$ and Inv stands for any of our two invariants, we have $\text{Inv}\big(\bigcup_{\Delta \in \Theta} \Delta\big) = \bigcap_{\Delta \in \Theta} \text{Inv}\,\Delta$. For a set X^0 let $R(E; X^0)$ denote the set of all relations on E under X^0. Now the *sets*

$$\text{s-Inv}^{(X^0)}\Delta = \text{s-Inv}\,\Delta \cap R(E; X^0) \quad \text{and} \quad \text{p-Inv}^{(X^0)}\Delta = \text{p-Inv}\,\Delta \cap R(E; X^0)$$

are called the stability and preservation invariants of Δ *under* X^0, respectively.

If R is a set of relations on E then $\bar{\bar{R}} = \text{s-Inv}\,D(E/R)$ and $\bar{R} = \text{p-Inv}\,G(E/R)$ are called the *stability* and *preservation closures* of R, respectively, while

$$\bar{\bar{R}}^{(X^0)} = \text{s-Inv}^{(X^0)}D(E/R) \quad \text{and} \quad \bar{R}^{(X^0)} = \text{p-Inv}^{(X^0)}G(E/R)$$

are called the stability and preservation closures of R *under* X^0, respectively. The main problem of the next two paragraphs is to characterize these closures in terms of R but without any intervention of self-mappings of E.

2. Fundamental operations

In order to characterize the above-mentioned closures of R in terms of relations we have to introduce certain operations acting on relations. Some of these operations act on sets of relations while others on single relations, but any of these operations results in single relations. Some of these operations are only partial, i.e. they are defined for (sets of) relations satisfying some prescribed conditions.

While defining our fundamental operations in the sequel, all relations are assumed to have a fixed base set E.

I. *Infinitary boolean operations:*

Ia. *Infinitary union,* Ib. *Infinitary intersection.* Both of these operations act on non-empty sets R of relations, and are defined iff all $r\in R$ have the same argument set, say X. These operations are denoted by $\bigcup\cdot R=\bigcup_{r\in R} r$ and $\bigcap\cdot R=\bigcap_{r\in R} r$, and their results are relations with the same argument set X.

Ic. *Negation.* This fundamental operation acts on any single relation r, and is denoted by $\neg$. If X denotes the argument set of r then $\neg\cdot r=E^X\setminus r$, the complement of r in E^X, has the same argument set X.

Remark 1. The above three fundamental operations are not independent. Indeed, if $\neg\cdot R$ denotes $\{\neg\cdot r;\ r\in R\}$, we have $\bigcup\cdot R=\neg(\bigcap\cdot\neg R)$ and $\bigcap\cdot R=$ $=\neg(\bigcup\cdot\neg R)$. However, $\bigcup$ and $\bigcap$ are independent.

Remark 2. Two well-known properties of these operations, namely $r\cap(\neg\cdot r)=\emptyset$ and $r\cup(\neg\cdot r)=E^X$, where X is the argument set of r, will be of relevance later.

Remark 3. When E and X are finite then there are only a finite number of X-relations, whence the infinitary boolean operations are in fact the ordinary (finitary) ones.

Remark 4. We define the following preorder for sets R and R' of relations. Put $R\leqq R'$ iff there exists a surjective mapping $\varphi\colon R'\to R$ such that for every $r'\in R'$ we have $r'\supseteqq\varphi\cdot r'$. A (possibly partial) operation ω, acting on sets of relations, will be said *increasing* if for arbitrary sets R and R' of relations $R\leqq R'$ implies $\omega(R)\subseteqq\omega(R')$, provided that ω is defined for R and R'. Further, an operation ω (possibly partial) that acts on relations is said to be *increasing* if for any two relations r and r' belonging to the domain of ω, $r\subseteqq r'$ implies $\omega\cdot r\subseteqq\omega\cdot r'$. It is easy to see that the infinitary union and intersection are increasing, while the negation is not.

II. *Projective operations,* which act on relations:

IIa. *Projections* (or *restrictions*) $\mathrm{pr}_{\bar X}$. This operation is defined for a relation r iff the argument set X of r contains $\bar X$ as a subset. For an X-point $P\colon X\to E$ let $(P|\bar X)$ denote the restriction of P onto $\bar X\subseteqq X$, and define $\mathrm{pr}_{\bar X}\cdot r$ as $\{(P|\bar X);\ P\in r\}$. This relation will also be denoted by $\mathrm{pr}^X_{\bar X}\cdot r$, $r^X_{\bar X}$ and, abusing the scripture, even by $(r|\bar X)$. This fundamental operation transforms a relation with argument set $X\supseteqq\bar X$ into a relation with argument set $\bar X$.

IIb. *Antiprojections* (or *extensions*) $\mathrm{ext}_{X'}$. This operation is defined for relations r with argument set $X\subseteqq X'$, and $\mathrm{ext}_{X'}\cdot r$ is the cartesian product $r\times E^{X'\setminus X}$.

With the usual identification in cartesian products, $\mathrm{ext}_{X'} \cdot r$ is the set of all points (P, P') such that $P \in r$ and P' is an arbitrary $(X' \setminus X)$-point. This relation will also be denoted by $\mathrm{ext}^{X}_{X'} \cdot r$ and ${}^{X}_{X'}(r)$.

Remark 5. *We can extend the operations* $\mathrm{pr}_{\bar{X}}$ *and* $\mathrm{ext}_{X'}$ *to any relation* r with an arbitrary argument set X via defining $\mathrm{pr}_{\bar{X}} \cdot r$ as $\mathrm{pr}^{X}_{X \cap \bar{X}} \cdot r$ and $\mathrm{ext}_{X'} \cdot r$ as $\mathrm{ext}^{X}_{X \cup X'} \cdot r$. Then we have $\mathrm{pr}_{\bar{X}}\, \mathrm{pr}_{\bar{X}'} = \mathrm{pr}_{\bar{X} \cap \bar{X}'}$ and $\mathrm{ext}_{X'}\, \mathrm{ext}_{X''} = \mathrm{ext}_{X' \cup X''}$.

Remark 6. We have $\mathrm{pr}_{\emptyset} \cdot r = \{P_{\emptyset}\}$ for $r \neq \emptyset$ and $\mathrm{pr}_{\emptyset} \cdot r = \emptyset$ for $r = \emptyset$.

Remark 7. Both projections and extensions are increasing.

Remark 8. When relations are considered as sets of points, extensions are hyperpunctual mappings and projections are even punctual, if relations with a fixed argument set are considered. Therefore, the projections commute with $\cup$, and it is easy to see that the extensions commute with all boolean operations.

Remark 9. While $\mathrm{pr}^{X'}_{X}\, \mathrm{ext}^{X}_{X'} \cdot r = r$ is always true, r is only a subset of $\mathrm{ext}^{\bar{X}}_{X}\, \mathrm{pr}^{X}_{\bar{X}} \cdot r$. If $\mathrm{ext}^{\bar{X}}_{X}\, \mathrm{pr}^{X}_{\bar{X}} \cdot r = r$ then r is said to be *identical* on $X \setminus \bar{X}$ or outside $\bar{X}$, and the arguments belonging to $X \setminus \bar{X}$ are called *fictitious*. The operation $c_{\bar{X}} = \mathrm{ext}^{\bar{X}}_{X}\, \mathrm{pr}^{X}_{\bar{X}}$, which preserves the argument set, is called the $\bar{X}$-cylindrification. For $X \subseteq X'$ we have $E^{X'} = \mathrm{ext}^{X}_{X'} \cdot E^{X}$ and, in particular, $I(X, E) = E^{X} = \mathrm{ext}_{X} \cdot E^{\emptyset} = \mathrm{ext}_{X} \cdot \{P_{\emptyset}\}$.

Canonical identification. Let r and r' be relations on E with argument sets X and X'. Let $r \sim r'$ mean that there exists a set $X'' \supseteq X \cup X'$ such that $\mathrm{ext}_{X''} \cdot r = \mathrm{ext}_{X''} \cdot r'$. If this equality holds for some set $X'' \supseteq X \cup X'$ then it holds for every $X'' \supseteq X \cup X'$. Consider $\sim$ as a relation on the class of all relations on E; then $\sim$ is easily seen to be an equivalence relations, i.e., $\sim$ is reflexive, symmetric and transitive. As infinitary boolean operations commute with extensions, they are compatible with this equivalence. For $X'' \supseteq X \supseteq \bar{X}$ we have $\mathrm{pr}^{X}_{\bar{X}} = \mathrm{pr}^{X}_{\bar{X}}(\mathrm{pr}^{X''}_{X}\, \mathrm{ext}^{X}_{X''}) = (\mathrm{pr}^{X}_{\bar{X}}\, \mathrm{pr}^{X''}_{X})\, \mathrm{ext}^{X}_{X''} = \mathrm{pr}^{X''}_{\bar{X}}\, \mathrm{ext}^{X}_{X''}$, whence $\mathrm{pr}_{\bar{X}} \cdot r = \mathrm{pr}_{\bar{X}}(\mathrm{ext}_{X''} \cdot r)$. Hence it is easy to conclude that if an X-relation r and an X'-relation r' are equivalent modulo $\sim$ and $\bar{X} \subseteq X \cap X'$ then $\mathrm{pr}_{\bar{X}} \cdot r = \mathrm{pr}_{\bar{X}} \cdot r'$. So projections are also compatible with this equivalence. Further, we have $\mathrm{ext}_{X'} \cdot r \sim r$. It is also obvious that if r and r' have the same argument set X and $r \sim r'$ then r and r' must coincide. Therefore, for any X-relation r and X', there is at most one X'-relation equivalent to r, and there is certainly such an X'-relation if $X' \supseteq X$.

Relations on E can be considered modulo $\sim$, i.e., we may identify relations that are equivalent. This means that if a relation can be obtained from another one via omitting and adding some fictitious arguments then these two relations are considered the same. This identification will be called *canonical*. Since infinitary boolean operations and projections are compatible with $\sim$, they are meaningful after canonical identification. It is trivial that $\text{ext}_{X'}$ becomes the identical operation when relations are canonically identified. Further, the infinite union and intersection become defined for every set R of relations in this case. Indeed, take a sufficiently large set X that includes the argument set of any $r \in R$. Then for each $r \in R$ there is exactly one X-relation r' such that $r \sim r'$, so we can and must define $\bigcup \cdot R$ and $\bigcap \cdot R$ as $\bigcup_{r \in R} r'$ and $\bigcap_{r \in R} r'$, respectively.

III. *Contractive operations.*

IIIa. *Contraction* (φ). Given a surjection $\varphi\colon X \to Y$ and an X-point $P\colon X \to E$ which is compatible with φ, we can define a Y-point $Q\colon Y \to E$ by the condition $Q \cdot y = P \cdot x$ where $x \in \varphi^{-1} \cdot y$. The mapping that sends P to Q and maps the multidiagonal $I_{T(\varphi)}(E)$ onto $E^Y = I(Y, E)$ will be denoted by (φ). It is easy to check that (φ) is injective. An X-relation r is said to be *compatible* with φ iff every $P \in r$ (as a mapping of X into E) is compatible with φ. In this case $(\varphi) \cdot r = \{(\varphi) \cdot P;\ P \in r\}$ is a Y-relation. This mapping (φ), which maps the set of X-relations compatible with φ into the set of Y-relations, is called the *contraction* (φ). This mapping is punctual and injective, so it commutes with the infinitary union and intersection, and we have $(\varphi) \cdot \big(I_{T(\varphi)}(E) \cap (\neg r)\big) = (\varphi) \cdot \big(I_{T(\varphi)}(E) \setminus r\big) = \neg\big((\varphi) \cdot r\big)$. Clearly, φ is increasing. If $\varphi\colon X \to Y$ and $\varphi'\colon Y \to Z$ are surjections then $(\varphi') \cdot \big((\varphi) \cdot P\big)$ is defined iff P is compatible with $\varphi'\varphi$, and in this case $(\varphi'\varphi) \cdot P = (\varphi') \cdot \big((\varphi) \cdot P\big)$. The same formula holds for relations.

IIIb. *Dilatations* $[\psi]$. Let $\psi\colon Y \to X$ be a surjection. Then for any X-point $P\colon X \to E$ the mapping $[\psi] \cdot P = P \cdot \psi\colon y \to P \cdot (\psi \cdot y)$ is a Y-point compatible with ψ. The mapping $P \to [\psi] \cdot P$ is injective and maps E^X onto the multidiagonal $I_{T(\psi)}(E)$. For an X-relation r let $[\psi] \cdot r$ stand for $\{[\psi] \cdot P;\ P \in r\}$. Then $[\psi]$, called the *dilatation* $[\psi]$, is a mapping of $P(E^X)$, the set of all X-relations, onto $P\big(I_{T(\psi)}(E)\big)$. Further, this mapping is punctual and injective. Hence it commutes with the infinitary intersection and union, and we have $[\psi] \cdot (\neg r) = I_{T(\psi)}(E) \setminus [\psi] \cdot r$. If $\psi\colon Y \to X$ and $\psi'\colon Z \to Y$ are surjections then $[\psi'\psi] \cdot P = [\psi'] \cdot ([\psi] \cdot P)$ and $[\psi'\psi] \cdot r = [\psi'] \cdot ([\psi] \cdot r)$. Obviously, $(\psi)[\psi] \cdot r = r$ for every X-relation r, and $[\psi](\psi) \cdot r = r$ for every X-relation r compatible with ψ.

Remark 10. Every multidiagonal can be obtained by an appropriate dilatation from some identity.

Note that any multidiagonal can also be obtained as an intersection of (extensions of) simple diagonals. Indeed, $I_C(E) = \bigcap_{x \equiv y(C)} \text{ext}_X \cdot D_{x,y}(E)$ or, up to canonical identification, $I_C(E) = \bigcap_{x \equiv y(C)} D_{x,y}(E)$. The strict multidiagonal $I_C^0(E)$ can be obtained by an intersection of simple diagonals and antidiagonals.

Contractive operations and canonical identification. Let $\varphi\colon X \to Y$ be a surjection and let r be an X-relation compatible with φ. For $x \in X$ the $T(\varphi)$-class of x

is a singleton, provided x is a fictitious argument of r. If $y=\varphi\cdot x$, $\bar{X}=X\setminus\{x\}$, $\bar{Y}=Y\setminus\{y\}$ and $\bar{\varphi}=(\varphi|\bar{X})$, then $\bar{\varphi}$ maps $\bar{X}$ onto $\bar{Y}$, $(r|\bar{X})$ is compatible with $\bar{\varphi}$, and $(\varphi)\cdot r=(\bar{\varphi})\cdot(r|\bar{X})\times E^{\{y\}}$. Thus y is also a fictitious argument of $(\varphi)\cdot r$. More generally, if r is identical outside $\bar{X}\subseteq X$, then φ is injective on $X\setminus\bar{X}$, $\bar{\varphi}=(\varphi|\bar{X})$ maps $\bar{X}$ onto $\bar{Y}=Y\setminus\varphi\cdot(X\setminus\bar{X})$, $(r|\bar{X})$ is compatible with $\bar{\varphi}$, $(\bar{\varphi})\cdot(r|\bar{X})=((\varphi)\cdot r|\bar{Y})$ and $(\varphi)\cdot r$ is identical on $Y\setminus\bar{Y}$. So, $(\bar{\varphi})\cdot(r|\bar{X})\sim(\varphi)\cdot r$.

Two relations, say r and r' with respective argument sets X and X', are equivalent if and only if $(r|X\cap X')=(r'|X\cap X')$ and both are identical outside $X\cap X'$. Indeed, $r\sim r'$ iff $\mathrm{ext}_{X\cup X'}\cdot r=\mathrm{ext}_{X\cup X'}\cdot r'$, and we can use the equalities $(r|X\cap X')=(\mathrm{ext}_{X\cup X'}\cdot r|X\cap X')$ and $(r'|X\cap X')=(\mathrm{ext}_{X\cup X'}\cdot r'|X\cap X')$. Furthermore, if r and r' are equivalent, $r=\mathrm{ext}_X\cdot(r|X\cap X')$.

Let r and r' be relations with respective argument sets X and X', and let $\varphi\colon X\to Y$ and $\varphi'\colon X'\to Y$ be mappings. The pairs (r,φ) and (r',φ') will be said *equivalent* iff $r\sim r'$ and, furthermore, there exists a subset $\bar{X}$ of $X\cap X'$ such that $(\varphi|\bar{X})=(\varphi'|\bar{X})$ and r is identical on $X\setminus\bar{X}$. (Note that in this case $(r|\bar{X})=(r'|\bar{X})$ and r' is identical on $X'\setminus\bar{X}$.) Let $(r,\varphi)\sim(r',\varphi')$ denote that (r,φ) and (r',φ') are equivalent. It is easy to see that this binary relation is in fact an equivalence.

Similarly, let r be an X-relation, r' be an X'-relation, further let $\psi\colon Y\to X$ and $\psi'\colon Y\to X'$ be surjections. Then the pairs (r,ψ) and (r',ψ') are said to be *equivalent* iff $r\sim r'$ and there exists a subset $\bar{X}$ of $X\cap X'$ such that (α) $\psi^{-1}\cdot\bar{X}=\psi'^{-1}\cdot\bar{X}$ (this set will be denoted by $\bar{Y}$) and $(\psi|\bar{Y})=(\psi'|\bar{Y})$, (β) $(\psi|Y\setminus\bar{Y})\colon Y\setminus\bar{Y}\to X\setminus\bar{X}$ and $(\psi'|Y'\setminus\bar{Y})\colon Y'\setminus\bar{Y}\to X'\setminus\bar{X}$ are bijections, and (γ) r and r' are identical outside $\bar{X}$. Note that if (r,ψ) and (r',ψ') are equivalent then $[\psi]\cdot r\sim[\psi']\cdot r'$.

Floatage. Floating equivalences. Free and semi-free intersections. For a bijection $\varphi\colon X\to Y$, every X-relation r is compatible with φ. Then the contraction (φ), which coincides with the dilatation $[\varphi^{-1}]$, is called a *flotage* (of arguments). A subset $\bar{X}$ of X such that $(\varphi|\bar{X})$ is the identity mapping is called an *anchor set* of the floatage (φ), while the maximal anchor set of (φ) will be called its *anchor* and is denoted by $A(\varphi)=\{x\in X;\ \varphi\cdot x=x\}$. The elements of $A(\varphi)$ are referred to as *anchor arguments.* When considering a floatage with an anchor set $\bar{X}$, we say that we *let* the arguments outside $\bar{X}$ *float.*

Two relations, say r and r' with respective argument sets X and X', are called *floatingly equivalent* (in notation $r\underset{f}{\sim}r'$) if there exists a bijection $\varphi\colon X\to Y$ such that $(\varphi)\cdot r\sim r'$. It is easy to verify that this binary relation is really an equivalence, called *floating equivalence.* The existence of a floatage (φ) such that $(\varphi)\cdot r=r'$ is called *restricted floating equivalence* and is denoted by $r\underset{rf}{\sim}r'$. When we allow floatages with a fixed anchor set $\bar{X}$ only then we obtain the analogous notions of *semi-*

floating and *restricted semifloating equivalences of anchor* $\bar{X}$ (in notation $r \underset{\bar{X},f}{\sim} r'$ and $r \underset{\bar{X},r,f}{\sim} r'$, resp.).

Let r be an X-relation and let $\bar{X} \subseteq X$. It is clear that the $\bar{X}$-projection of r does not change when we let the "dumb" arguments $x \in X \setminus \bar{X}$ float. We have also seen that $(r|\bar{X})$ is invariant under canonical equivalence. Therefore $\mathrm{pr}_{\bar{X}} \cdot r = \mathrm{pr}_{\bar{X}} \cdot r'$, provided $r \underset{\bar{X},f}{\sim} r'$. In particular, it is without changing $\mathrm{pr}_{\bar{X}} \cdot r$ that we can let the "dumb" arguments $x \notin \bar{X}$ float so that their images avoid some prescribed set $\hat{X} \supseteq \bar{X}$. If it is so then the image of r by this floatage is said to be regularized for $\hat{X}$.

Let R be a set of relations on E, and let X_r denote the argument set of $r \in R$. For each $r \in R$ take a floatage (φ_r) so that the sets $Y_r = \varphi_r \cdot X_r$ $(r \in R)$ be pairwise disjoint. If Y is a set including $\bigcup_{r \in R} Y_r$ then $\bigcap_{r \in R} \mathrm{ext}_Y (\varphi_r) \cdot r$ does not depend, modulo floating equivalence, on the choice of floatages (φ_r) and of Y. Hence $\bigcap_{r \in R} \mathrm{ext}_Y (\varphi_r) \cdot r$ can be called the *free intersection* of $r \in R$; it is determined up to floating equivalence. When we take $Y = \bigcup_{r \in R} Y_r$ then $\bigcap_{r \in R} \mathrm{ext}_Y (\varphi_r) \cdot r$ is clearly the cartesian product $\prod_{r \in R} (\varphi_r) \cdot r$, which is equivalent to $\prod_{r \in R} r$. So the free intersection is, up to floating equivalence, the (complete) cartesian product. The free intersection of all $r \in R$ will be denoted by $\bigcap_f \cdot R$.

Given a set $\bar{X}$ and a set R of relations r on E and with argument sets X_r, the *semi-free intersection of anchor* $\bar{X}$ of R is defined as follows. Put $\bar{X}_r = \bar{X} \cap X_r$ and choose floatages (φ_r) with anchor sets $\bar{X}_r$ so that $\bar{X} \cap \varphi_r \cdot (X_r \setminus \bar{X}_r) = \emptyset$ and the sets $\varphi_r \cdot (X_r \setminus \bar{X}_r)$, $r \in R$, be pairwise disjoint. Further, take a set Y including all $\varphi_r \cdot X_r$. Then $\bigcap_{r \in R} \mathrm{ext}_Y (\varphi_r) \cdot r$, called the semi-free intersection of anchor $\bar{X}$ of R and denoted by $\bigcap_f^{(\bar{X})} \cdot R$, does not depend, up to semi-floating equivalence of anchor $\bar{X}$, on the choice of Y and that of φ_r $(r \in R)$.

Lemma 1. *Let R be a set of relations r with argument sets X_r, let $\bar{X}_r \subseteq X_r$, and put $\bar{X} = \bigcup_{r \in R} \bar{X}_r$. Then*

$$\bigcap_{r \in R} \mathrm{ext}_{\bar{X}} \mathrm{pr}_{\bar{X}_r} \cdot r = \mathrm{pr}_{\bar{X}} \cdot \left(\bigcap_f^{(\bar{X})} \cdot R\right).$$

Proof. Put $\varrho = \bigcap_{r \in R} \mathrm{ext}_{\bar{X}} \mathrm{pr}_{\bar{X}_r} \cdot r$ and $\varrho^* = \mathrm{pr}_{\bar{X}} \cdot (\bigcap_f^{(\bar{X})} \cdot R)$. Let $\bigcap_f^{(\bar{X})}$ be represented by $\bigcap_{r \in R} \mathrm{ext}_{\bar{X} \cup Y'} (\varphi_r) \cdot r$ where $\varphi_r \colon X_r \to \bar{X}_r \cup Y_r$ are bijections subject to the previous conditions. I.e., the Y_r are pairwise disjoint and $Y' = \bigcup_{r \in R} Y_r$ is disjoint from $\bar{X}$. An $\bar{X}$-point $\bar{P} \colon \bar{X} \to E$ is in ϱ iff for every $r \in R$ $(\bar{P}|\bar{X}_r)$ is a point such that there exists an $(X_r \setminus \bar{X}_r)$-point P_r' with $((\bar{P}|\bar{X}_r), P_r')$ belonging to r. But this is equivalent to $(\varphi_r) \cdot ((\bar{P}|\bar{X}_r), P_r') \in (\varphi_r) \cdot r$ and, as $(\varphi_r|\bar{X}_r)$ is the identity map, also to the existence of a Y_r-point $P_r''=(\varphi_r|X_r\setminus\bar{X}_r)\cdot P_r'$ such that $(\bar{P}, P_r'')\in \mathrm{ext}_{X\cup Y_r}(\varphi_r)\cdot r$. Since the sets Y_r are pairwise disjoint, for any set $\{P_r'': Y_r\to E;\ r\in R\}$ of points there exists exactly one point $P'': Y'\to E$ such that $(P''|Y_r)=P_r''$ holds for each $r\in R$. Clearly, $(\bar{P}, P'')\in \mathrm{ext}_{X\cup Y_r}(\varphi_r)\cdot r$ is equivalent to

$$(\bar{P}, P'')\in \mathrm{ext}_{X\cup Y'}\,\mathrm{ext}_{X\cup Y_r}(\varphi_r)\cdot r = \mathrm{ext}_{X\cup Y'}(\varphi_r)\cdot r.$$

Thus the simultaneous existence of all $P_r'': Y_r\to E$ with $(\bar{P}, P_r'')$ belonging to $\mathrm{ext}_{X\cup Y_r}(\varphi_r)\cdot r$ is equivalent to the existence of a single $P'': Y'\to E$ such that $(\bar{P}, P'')\in \mathrm{ext}_{X\cup Y'}(\varphi_r)\cdot r$ for every $r\in R$, i.e., $(\bar{P}, P'')$ belongs to $\bigcap_{r\in R} \mathrm{ext}_{X\cup Y'}(\varphi_r)\cdot r= =\bigcap_{r\in R} \mathrm{ext}_Y(\varphi_r)\cdot r$, which is equivalent to $\bar{P}\in\varrho^*$. I.e., $\bar{P}\in\varrho$ is equivalent to $\bar{P}\in\varrho^*$, which completes the proof.

The operations Ia, Ib, Ic $(\cup, \cap, \neg)$, IIa, IIb $(\mathrm{pr}_X, \mathrm{ext}_{X'})$, IIIa and IIIb $((\varphi), [\psi])$ are called *fundamental operations.* The increasing fundamental operations, i.e. all but the negation $\neg$, are called *direct fundamental operations.* Two nullary operations, namely IVa: (adding to any set of relation) the empty relation $\emptyset$ and IVb: (adding) the $\emptyset$-identity $I(\emptyset, E)=\{P_\emptyset\}$, are also considered as direct fundamental operations. These two nullary operations are combinations of the rest of the fundamental operations when we start from a nonempty set R of relations. Indeed, take an $r\in R$, then $\emptyset=r\cap(\neg r)$ and $I(\emptyset, E)=\mathrm{pr}_\emptyset\cdot(r\cup(\neg r))$. Yet, they are not combinations of direct fundamental operations in general.

If $r\sim r'$ or $r\underset{f}{\sim} r'$ (or, in particular, $r\underset{rf}{\sim} r'$, $r\underset{X,rf}{\sim} r'$ or $r\underset{X,f}{\sim} r'$) then each of r and r' can be obtained from the other by a suitable combination of direct fundamental operations. On the other hand if we identify the canonically equivalent relations, we can drop all extensions from (direct) fundamental operations. When passing from relations to floatingly (and even restricted floatingly) equivalent ones is permitted, we may fix a representative set $X(c)$ of cardinality c for each cardinal c, e.g., we may put $X(c)=\{\alpha;\ \alpha$ is an ordinal and $\alpha<\omega(c)\}$ where $\omega(c)$ denotes the smallest ordinal with cardinality c. (I follow Cantor's point of view rather than that of von Neumann. In fact, the second point of view has been adopted in my first paper [1] on abstract Galois theory, while the first one in all of my other papers.)

When the axiom of choice is admitted, contractions become combinations of projections and floatages, while dilatations become combinations of extensions, floatages and intersections with simple diagonals. Indeed, if $\varphi: X\to Y$ is a surjection, for each $y\in Y$ we can choose an $x(y)\in X$ such that $\varphi\cdot x(y)=y$. Put $\bar{X}=\{x(y);\ y\in Y\}$. As $(\varphi|\bar{X}): \bar{X}\to Y$ is a bijection, $((\varphi|\bar{X}))$ is a floatage, and $(\varphi)\cdot r=(\varphi|\bar{X})\,\mathrm{pr}_{\bar{X}}\cdot r$, provided r is compatible with φ. Similarly, if $\psi: Y\to X$ is a surjection, $x\to y(x)$ is a mapping of X into Y such that $\psi\cdot y(x)=x$, and $\bar{Y}=\{y(x);\ x\in X\}$ then $\bar{\psi}=(\psi|\bar{Y}): y(x)\to x$ is a bijection of $\bar{Y}$ onto X. Hence $[\bar{\psi}]=(\bar{\psi}^{-1})$ is a floatage, and for any X-relation r we have

$$[\psi]\cdot r=\mathrm{ext}_Y[\bar{\psi}]\cdot r\cap\Big(\bigcap_{\bar{y}\in\bar{Y}}\ \bigcap_{y=\bar{y}(T(\psi))}D_{\bar{y},y}\Big).$$

When the image of φ or ψ is finite, the axiom of choice is not necessary for the above results.

We say that fundamental or direct fundamental operations are used below X^0, if these operations are used only for relations with argument sets included in X^0 and only when these operations result in relations whose argument sets are also included in X^0. I.e., in case of $\mathrm{pr}_{\bar{X}}$, $\mathrm{ext}_{X'}$, $(\varphi\colon X\to Y)$ and $[\psi\colon Y\to X]$ the inclusions $\bar{X}\subseteq X^0$, $X'\subseteq X^0$ and $Y\subseteq X^0$ are also required. In particular, if E is finite and these operations are used below some finite X^0 then they are equivalent to the realizations of operations of the predicate calculus with equality on the finite model E, the set of object variables being X^0. Indeed, any X-relation r on E (where $X\subseteq X^0$) can be considered as the realization of some predicate $P_r=P_r(X)$ on E. Further, $P_{r\cup r'}=P_r\vee P_{r'}$, $P_{r\cap r'}=P_r\,\&\,P_{r'}$, $P_{\neg r}=\neg P_r$, $P_{\mathrm{pr}_{\bar{X}}\cdot r}=(\exists x_1)\dots(\exists x_s)P_r$ where $\{x_1,\dots,x_s\}=X\setminus\bar{X}$, $P_{\mathrm{ext}_{X'}\cdot r}=P_r(X')$ where $P_r(X')$ is the predicate obtained from $P_r=P_r(X)$ by adding the set $X'\setminus X$ of fictitious variables, if $\varphi\colon X\to Y$ is a bijection and $X=\{x_1,\dots,x_n\}$ then $P_{(\varphi)\cdot r}=P_r^{(\varphi)}$ is a predicate such that $P_r^{(\varphi)}(\varphi\cdot x_1,\dots,\varphi\cdot x_n)=$ $=P_r(x_1,\dots,x_n)$, and we have $P_{D_{x,y}}=(x=y)$. The direct fundamental operations are equivalent to the part of predicate calculus generated by $\vee$, $\&$, existential quantifiers, addition of fictitious variables, substitutions of object variables, the inequality $x_i\neq x_i$ and equalities $x_i=x_j$ (in particular, $x_i=x_i$). In the general case we may consider the fundamental operations as realizations on models of an (unlimited) infinitary generalization of predicate calculus, and direct fundamental operations as that of certain "positive" part of it.

Lemma 2. *For any relation r there exists a set R_r of relations with the same argument set such that*

(1) *every relation in R_r can be obtained from r by a combination of direct fundamental operations;*

(2) *all the relations in R_r are semi-regular, and for each point P of an arbitrary relation $s\in R_r$ there exists a relation $\bar{s}\in R_r$ such that $\bar{s}\subseteq s$ and $P\in t(\bar{s})$; and*

(3) $r=\bigcup\cdot R_r$.

Proof. Firstly, every multidiagonal is obtained by a successive use of direct fundamental operations (starting from the empty set!). Indeed, it is obtained by dilatation from some identity $I(X,E)=\mathrm{ext}_X\cdot I(\emptyset,E)$. Let P be a point of r with type $T(P)$ and let $r_P=r\cap I_{T(P)}(E)$. Put $R_r=\{r_P;\ P\in r\}$. Then (1) is clearly satisfied. It is easy to see that r_P is semiregular and of type $T(P)$, and $P\in t(r_P)$. For $P\in s\in R_r$ we have $s=r_Q\subseteq r$ where Q is a point of r such that P is compatible with

$T(Q)$. Then $T(P)$ is thicker than $T(Q)$; so $I_{T(P)}(E) \subseteqq I_{T(Q)}(E)$ and $r_P \subseteqq r_Q = s$. Put $\bar{s} = r_P$, and (2) is clearly satisfied. Finally, $P \in r_P \subseteqq r$ implies

$$r = \{P;\ P \in r\} = \bigcup_{P \in r} \{P\} \subseteqq \bigcup_{P \in r} r_P \subseteqq r,$$

which yields $\bigcup \cdot R_r = \bigcup_{P \in r} r_P = r$, i.e. (3) holds.

The set R_r constructed in the previous proof is called the *semi-regular decomposition* of r.

Formalism of fundamental operations. If we apply a (generally infinite) combination of fundamental operations to relations, we obtain an operation, which can be represented by a (generally infinite) formula of the "fundamental operation calculus". The best way of denoting these formulas is to use (generally infinite) trees with finite branches. By a tree we mean an unoriented, connected graph without loops, without circles (i.e. closed paths), and with a distinguished vertex, called its root. Let T be a tree with root r. Then for any vertex v in T (in notation, $v \in T$) there is exactly one path in T that connects v and r, provided $v \neq r$. Let w be the vertex next to v on this path. Now w is called the *father* of v, while v is called the *son* of w. A vertex $v \neq r$ has one and only one father, while the set $S(v)$ of sons of v can be of arbitrary cardinality. Vertices without sons are called extremities of T. We shall write $v < v'$ if $v \neq v'$ and v is on the path connecting r and v'. For a vertex v the set $\{v';\ v \leqq v'\}$ of vertices spans a subgraph, called the *subtree of T of root v*. The number of edges in the path connecting v and r is denoted by $h(v)$ and is called the *height* of v. In particular, $h(r) = 0$. Let $h(T) = \sup_{v \in T} h(v)$ be called the *height* of T, which is a non-negative integer or ∞. T is called of *finite* or *infinite height* according to $h(T) \neq \infty$ or $h(T) = \infty$.

A *branch* of T is a maximal linearly ordered set of vertices together with the edges connecting them. We shall deal only with trees without infinite branches. I.e., we always assume that our trees have no infinite, linearly ordered sequence of vertices $v_1 < v_2 < v_3 < \dots$. For such trees there is another invariant, the so-called depth, which is more important than the height. Given a tree T, a mapping ν from its vertex set into the class of ordinals is called a *depth function* of T if for any vertex $v \in T$ we have $\nu(v) = \sup_{u \in S(v)} (\nu(u) + 1)$ (in particular, $\nu(v) = 0$ for any extremity of T). If a tree has a depth function then it cannot have infinite branches. Indeed, if $r = v_0 < v_1 < v_2 < v_3 < \dots$ were an infinite branch then $\nu(v_0) > \nu(v_1) > \nu(v_2) > \nu(v_3) > \dots$ would be an infinite decreasing sequence of ordinals, which is impossible. As to the converse, admitting the denumerable axiom of choice, we can prove the following

Lemma 3. *Any tree without infinite branches has one and only one depth function.*

Proof. Assume that a vertex $v \in T$ is irregular in the sense that the lemma is not true for the subtree T_v. I.e., T_v has no depth function or has more than one depth functions. Then v must have at least one irregular son u. Really, if for all $u \in S(v)$ the subtree T_u has a unique depth function ν_u, then the mapping ν_v defined by $\nu_v(w) = \nu_u(w)$ if $w \in T_u$ and $u \in S(v)$ and, further, $\nu_v(v) = \sup_{u \in S(v)} (\nu_u(u)+1)$ is a depth function of T_v. As the restiction of this ν_v to any T_u, $u \in S(v)$, is unique by the assumption, ν_v is the only depth function of T_v, which is a contradiction. Thus we have seen that any irregular vertex has an irregular son. Now, if the lemma is not true for a tree T, then its root r is irregular as $T = T_r$. Hence an irregular son r_1 of r, then an irregular son r_2 of r_1, etc. can be chosen. I.e., the dènumerable axiom of choice yields the existence of a denumerable sequence $r = r_0, r_1, r_2, \dots$ of irregular vertices such that each r_i, $i > 0$, is a son of r_{i-1}. But then $r_0 < r_1 < r_2 < \dots$ contradicts the fact that T has no infinite branch.

When T is a tree without infinite branches and ν is its unique depth function, $d(T) = \nu(r)$ is called the *depth* of T. Any ordinal can be the depth of some tree. The depth of T is finite iff $h(T)$ is finite; in this case $d(T) = h(T)$.

A *formula* is a mapping F from the vertex set of a tree T without infinite branches such that

(1) $F(v)$ is a fundamental operation provided v is not an extremity, and $F(v)$ is either $\cup$ or $\cap$ if $S(v)$ is not a singleton;

(2) for any extremity v with father w, $F(v)$ is a *relation set variable* denoted by $X(v)$ (with capital X) provided $S(w) = \{v\}$ and $F(w)$ is $\cup$ or $\cap$, and $F(v)$ is a *relation variable* $x(v)$ in all other cases.

Note that $F(u)$ and $F(v)$ may coincide even for distinct extremities u and v.

Let E be a base set. A map F' from T is called an *E-formula* if it is obtained from some formula F via replacing certain relation set variables $X(v)$ and certain relation variables $x(v)$ at all of their occurrences by some relation sets $R(v)$ and some relations $r(v)$, resp., on E. Clearly, $X(u) = X(v)$ or $x(u) = x(v)$ must imply $R(u) = R(v)$ or $r(u) = r(v)$, respectively.

Given a base set E and a formula F, let $\Phi(F) = \{F(v);\ v \text{ is an extremity of } T\}$. A mapping ϱ of $\Phi(F)$ is called the *system of values* of variables of F if $\varrho \cdot F(v)$ is a set of relations on E with the same argument set when $F(v) = X(v)$ is a relation set variable while $\varrho \cdot F(v)$ is a single relation on E when $F(v) = x(v)$ is a relation variable. A mapping $t(\varrho)$ from the vertex set of T, $t(\varrho)\colon v \to t(\varrho; v)$, will be called a *coherent valuation* of F for ϱ if:

(1) $t(\varrho; v) = \varrho \cdot F(v)$ for every extremity v of T;

(2) if v is not an extremity then $t(\varrho; v)$ is a relation on E;

(3) if $F(v)$ is $\cup$ or $\cap$, $S(v) = \{v'\}$, v' is an extremity, and $F(v') = X(v')$ then $\varrho \cdot F(v) = F(v) \cdot (\varrho \cdot F(v'))$;

(4) in any other case when $F(v)$ is $\cup$ or $\cap$, all the $\varrho \cdot F(v')$, $v' \in S(v)$, have the same argument set and $\varrho \cdot F(v) = \bigcup_{v' \in S(v)} (\varrho \cdot F(v'))$ or $\varrho \cdot F(v) = \bigcap_{v' \in S(v)} (\varrho \cdot F(v'))$, respectively;

(5) if $F(v)$ is $\neg$, pr_Y^X, $\mathrm{ext}_{X'}^X$, $(\varphi\colon X \to Y)$ or $[\psi\colon Y \to X]$ then $S(v)$ is a singleton $\{u\}$ and $\varrho \cdot F(u)$ is an X-relation such that, in case $F(v) = (\varphi)$, $\varrho \cdot F(u)$ is compatible with φ, and, in all cases, $\varrho \cdot F(v) = F(v) \cdot (\varrho \cdot F(u))$.

Given ϱ, an easy induction on $\nu(v)$, $v \in T$, shows that there is at most one such coherent valuation $t(\varrho)$.

It is possible to define the fundamental operations for formulas. Let U be a set of formulas (or E-formulas), and let r_F denote the root of the formula $F \in U$. We construct a new formula $F^0 = \bigcup U$ or $F^0 = \bigcap U$, resp., via taking a new root r^0, adding a new edge (r^0, r_F) to every $F \in U$, and putting $F^0(r^0) = \cup$ or $F^0(r^0) = \cap$, respectively. Note that $S(r^0) = \{r_F;\ F \in U\}$ and each F becomes $F^0_{r_F}$. If F is a formula or E-formula and ω is one of the operations $\neg$, pr_Y^X, $\mathrm{ext}_{X'}^X$, $(\varphi\colon X \to Y)$, and $[\psi\colon Y \to X]$ then the formula (or E-formula) $\omega \cdot F$ is constructed so that we join a new root r^0 with the root r of F by a new edge (then r becomes the only son of r^0) and we put $(\omega \cdot F)_r = F$ and $(\omega \cdot F)(r^0) = \omega$. It is easy to verify that if ϱ_F is the value of $\Phi(F)$, $F \in U$, and ϱ is the value of $\Phi(\bigcup U)$ or $\Phi(\bigcap U)$, respectively, such that we have $(\varrho | \Phi(F)) = \varrho_F$ for every $F \in U$, then $(\bigcup U)(\varrho)$ resp. $(\bigcap U)(\varrho)$ is defined iff all the $F(\varrho_F)$, $F \in U$, are defined and $\cup$ resp. $\cap$ is applicable to $\{F(\varrho_F);\ F \in U\}$. If this is so then we have $(\bigcup U)(\varrho) = \bigcup_{F \in U} F(\varrho_F)$ and $(\bigcap U)(\varrho) = \bigcap_{F \in U} F(\varrho_F)$. If ω is one of the operations $\neg$, pr_Y^X, $\mathrm{ext}_{X'}^X$, (φ), and $[\psi]$ and ϱ is a value of $\Phi(F)$ then $(\omega \cdot F)(\varrho)$ is defined iff $F(\varrho)$ is defined and ω is applicable to it. In this case we have $(\omega \cdot F)(\varrho) = \omega \cdot F(\varrho)$.

The formalism can be defined modulo canonical identification, too. Then ext disappears and the results of the rest of fundamental operations are always defined for arbitrary relations and, for $\cup$ and $\cap$, for arbitrary sets of relations. Indeed, we only have to consider relations and operations modulo canonical identification and then to replace $(\varphi) \cdot r$, where $\varphi\colon X \to Y$ is a surjection, by $(\varphi)(\mathrm{pr}_X^{X'}\, \mathrm{ext}_{X_r}^{X'} \cdot r \cap I_{T(\varphi)}(E))$ where X_r is the argument set of r and X' includes $X_r \cup X$.

3. Fundamental operations and mappings

Let $\omega(\ldots)$ be one of the considered fundamental operations. We denote by ξ an arbitrary value of its argument, which may be a set of relations (for Ia, Ib) or a relation (for I3, IIa, IIb, IIIa, IIIb) or nothing (for IVa, IVb). Let $d\colon E \to E'$ be a mapping from the base set E into another set E'. We say that ω *commutes* with d if for any ξ (with base set E) ω is defined for $d \cdot \xi$ provided it is defined for ξ, and $d \cdot \omega(\xi) = \omega(d \cdot \xi)$. We say that ω *semi-commutes* with d if $\omega(d \cdot \xi)$ is defined when $\omega(\xi)$ is, and $d \cdot \omega(\xi) \subseteqq \omega(d \cdot \xi)$.

Proposition 1. (1) *Every fundamental operation commutes with any bijection.*

(2) *Every direct fundamental operation semi-commutes with any mapping.*

(3) *More precisely, the infinitary union, projections, contractions, dilatations, and the addition of* $\emptyset$ *and* $I_0(E)$ *commute with all mappings, the infinitary intersection commutes only with injections, and extensions commute only with surjections.*

Proof. As every mapping d of the base set preserves the argument sets of relations, it is clear that if any of the operations Ia, Ib, Ic, IIa, IIb, IIIb, IVa, and IVb is defined for some value ξ of its argument then it is also defined for $d \cdot \xi$. If $\varphi: X \to Y$ is a surjection and $P: X \to E$ is compatible with φ then $d \cdot P$ is also compatible with φ, where $d: E \to E'$ is an arbitrary mapping from E. Indeed, for $x \in X$, $(d \cdot P) \cdot x = d \cdot (P \cdot x)$ depends only on $P \cdot x$, which depends only on $\varphi \cdot x$. Therefore, if an X-relation r is compatible with φ then so is $d \cdot r$, i.e., $(\varphi) \cdot r$ being defined implies that $(\varphi) \cdot (d \cdot r)$ is also defined. So the preliminary condition on commutation and semi-commutation is always fulfilled.

Let us compute:

$$d \cdot (\cup R) = d \cdot \{P; (\exists r \in R)(P \in r)\} = \{d \cdot P; (\exists r \in R)(P \in r)\} =$$
$$= \{Q; (\exists s \in d \cdot R)(Q \in s)\} = \cup \cdot (d \cdot R).$$

For $\bar{X} \subseteqq X$ and an X-relation r we have

$$d \cdot (r | \bar{X}) = d \cdot \{(P | \bar{X}); P \in r\} = \{d \cdot (P | \bar{X}); P \in r\} = \{(d \cdot P | \bar{X}); P \in r\} =$$
$$= \{(Q | \bar{X}); Q \in d \cdot r\} = (d \cdot r | \bar{X}), \quad \text{i.e.,} \quad d \cdot \mathrm{pr}_{\bar{X}}\, r = \mathrm{pr}_{\bar{X}} (d \cdot r).$$

If $P: X \to E$ is an X-point compatible with the surjection $\varphi: X \to Y$ then $(\varphi) \cdot P$ is the mapping $Y \to E$, $y \to P \cdot x$ where $\varphi \cdot x = y$. So, $d \cdot ((\varphi) \cdot P)$ is the mapping $y \to d \cdot (((\varphi) \cdot P) \cdot y) = d \cdot (P \cdot x) = (d \cdot P) \cdot x$, where $\varphi \cdot x = y$. Thus $d \cdot ((\varphi) \cdot P) = (\varphi) \cdot (d \cdot P)$, and, if r is an X-relation compatible with φ, we have

$$d \cdot ((\varphi) \cdot r) = d \cdot \{(\varphi) \cdot P; P \in r\} = \{(\varphi) \cdot (d \cdot P); P \in r\} =$$
$$= \{(\varphi) \cdot Q; Q \in d \cdot r\} = (\varphi) \cdot (d \cdot r).$$

When $\psi: Y \to X$ is a surjection and r is an X-relation,

$$d \cdot ([\psi] \cdot r) = d \cdot \{[\psi] \cdot P; P \in r\} = \{d \circ (P \circ \psi); P \in r\} = \{(d \circ P) \circ \psi; P \in r\} =$$
$$= \{Q \circ \psi; Q \in d \cdot r\} = [\psi] \cdot (d \cdot r].$$

It is clear that $\emptyset$ and $I(\emptyset, E)$ depend on no argument, $d \cdot \emptyset = \emptyset$, and $d \cdot I(\emptyset, E) = I(\emptyset, E)$.

If R is a set of X-relations, we have

$$d\cdot(\bigcap R) = d\cdot\{P;\ (\forall r\in R)(P\in r)\} = \{d\cdot P;\ (\forall r\in R)(P\in r)\}.$$

Since $P\in r$ implies $d\cdot P\in d\cdot r$, $Q\in d\cdot(\bigcap R)$ implies $Q\in d\cdot r$, for each $r\in R$, i.e., $d\cdot(\bigcap R)\subseteqq \bigcap\cdot(d\cdot R)$. The converse implication $d\cdot P\in d\cdot r\Rightarrow P\in r$ holds for all $r\subseteqq E^X$ iff d is injective. Therefore the equality $d\cdot(\bigcap R)=\bigcap\cdot(d\cdot R)$ holds for any set R of X-relations only if d is injective.

Let r be an X-relation and let $X'\supset X$, i.e., $X'\setminus X\neq\emptyset$. Then $d\cdot\mathrm{ext}_{X'}r= =d\cdot(r\times E^{X'\setminus X})=(d\cdot r)\times(d\cdot E)^{X'\setminus X}\subseteqq(d\cdot r)\times(E')^{X'\setminus X}=\mathrm{ext}_{X'}(d\cdot r)$ and we have the equality $d\cdot\mathrm{ext}_{X'}r=\mathrm{ext}_{X'}(d\cdot r)$ (even for only one arbitrary $r\neq\emptyset$) iff d is surjective. This proves (3). Now (1) and (2), except the case of $\neg$, are consequences of (3). But if $s\colon E\to E'$ is a bijection then it commutes with all the Boolean operations, so, in particular, with the negation $\neg\colon r\to E^X\setminus r$. The proof of Proposition 1 is complete.

Applying Proposition 1 to the particular case $E=E'$ we obtain

Corollary 1. (1) *Every fundamental operation commutes with any permutation of the base set.*

(2) *Every direct fundamental operation semi-commutes with any self-mapping of E.*

(3) *The infinitary union, projections, contractions, dilatations, and the addition of $\emptyset$ and $I_\emptyset(E)$ commute with all self-mappings of E, the infinitary intersection commutes only with its self-injections, while the extensions commute only with its self-surjections.*

Proposition 2. *Let σ be a permutation of E, let R be a set of relations on E, and assume that a fundamental operation ω is applied to a subset or element ξ of R. (It is a subset when $\omega=\bigcup$ or $\omega=\bigcap$, and it is an element otherwise.) If σ is preserving on R then σ preserves $\omega(\xi)$.*

Proof. As $\sigma\cdot\xi=\xi$ and ω commutes with σ, we have $\sigma\cdot\omega(\xi)=\omega(\sigma\cdot\xi)= =\omega(\xi)$, indeed.

Proposition 3. *If δ is a self-mapping of E stabilizing on a set R of relations and if ω is an increasing fundamental operation semi-commuting with δ which is applicable to a subset or element ξ of R then δ stabilizes $\omega(\xi)$.*

Proof. Indeed, we have $\delta\cdot\omega(\xi)\subseteqq\omega(\delta\cdot\xi)$. Further, if ξ is a set of relations, $\varphi\colon r\to\delta\cdot r$ $(r\in\xi)$ is a surjection of ξ onto $\delta\cdot\xi$ such that $\varphi\cdot r=\delta\cdot r\subseteqq r$ for all $r\in\xi$. Hence $\delta\cdot\xi\leqq\xi$. When ξ is a single relation, $\delta\cdot\xi\subseteqq\xi$. Therefore, as ω is increasing, we have $\omega(\delta\cdot\xi)\subseteqq\omega(\xi)$ and $\delta\cdot\omega(\xi)\subseteqq\omega(\delta\cdot\xi)$, whence $\delta\cdot\omega(\xi)\subseteqq\omega(\xi)$.

Corollary 2. *Let ω be a fundamental operation which is applicable to a subset or element ξ of a set R of relations on E. If a self-mapping δ of E is stabilizing on R then it stabilizes $\omega(\xi)$.*

It follows from the preceding results that for any set $\Delta \subseteqq D(E)$ of self-mappings of E, s-Inv Δ is closed with respect to all direct fundamental operations. I.e., if a direct fundamental operation ω is applied to an element or a subset ξ of s-Inv Δ then $\omega(\xi)$ belongs to s-Inv Δ. Similarly, s-Inv$^{(X^0)}\Delta$ is closed with respect to these operations below X^0. In particular, the same closedness is true for $\bar{R}$ and $\bar{R}^{(X^0)}$, where R is a set of relations on E. If $\Delta \subseteqq S(E)$ is a set of permutations of E then p-Inv Δ is closed with respect to all fundamental operations, and so is the preservation closure $\bar{R}$ of a set R of relations on E. Similarly, p-Inv$^{(X^0)}\Delta$ and $\bar{R}^{(X^0)}$ are closed with respect to all fundamental operations below X^0.

4. Equivalence and existence theorems of abstract Galois theory and endotheory

Let X^0 be a set and let R be a set of relations under X^0. I.e., the argument sets of relations in R are subsets of X^0. The set R is said to be *logically* resp. *directly closed below* X^0 if it is closed with respect to all fundamental resp. all direct fundamental operations below X^0. If F is a logically or directly closed family of sets of relations on E then the intersection $\bigcap_{R \in F} R$ of this family is also logically or directly closed, respectively. If R is a non-empty set of relations on E then the family of all relation sets that include R, are under X^0 and are logically resp. directly closed is not empty as it contains $R^{(X^0)}$, the set of all relations on E under X^0. The intersection $R_f^{(X^0)}$ resp. $R_{df}^{(X^0)}$ of this family is called the *logical* resp. *direct logical* closure of R below X^0. $R_f^{(X^0)}$ and $R_{df}^{(X^0)}$ are the smallest relation sets (on E) under X^0 that are logically and directly closed, respectively. Let $S=(E, R)$ and $S'=(E, R')$ be two structures on E so that both R and R' be under X^0. (In this case S and S' are said to be structures under X^0.) We say that S and S' are *equivalent* resp. *directly equivalent below* X^0 if $R_f^{(X^0)}=R_f'^{(X^0)}$ resp. $R_{df}^{(X^0)}=R_{df}'^{(X^0)}$. Generally, these equivalences depend on X^0. Yet, as it will be shown, they do not depend on X^0 when card $X^0 \geqq$ card E is assumed. Indeed, our main purpose in this paragraph is to prove the following four theorems, in which card $X^0 \geqq$ card E is always supposed.

Equivalence theorem of abstract Galois endotheory. *Let S and S' be structures under X^0 and assume that* card $X^0 \geqq$ card E, *where E is the common base set of these structures. Then S and S' are directly equivalent iff $D_{E/S}=D_{E/S'}$.*

Equivalence theorem of abstract Galois theory. *Let S and S' be two structures under X^0 on E where* card $X^0 \geqq$ card E. *Then S and S' are equivalent iff $G_{E/S}=G_{E/S'}$.*

Existence theorem of abstract Galois endotheory. *For any semigroup D of self-mappings of E, if D contains the identical mapping 1_E and X^0 is a set with* card $X^0 \geqq$ card E *then there exists a structure S under X^0 on E such that* $D=D_{E/S}$.

Existence theory of abstract Galois theory. *Let G be an arbitrary permutation group on E and let X^0 be a set with* card $X^0 \geqq$ card E. *Then there exists a structure S under X^0 on E such that* $=G_{S/E}$.

Proof of equivalence and existence theorems of abstract Galois endotheory. (a) Consider $\bar{R}^{(X^0)}=\text{s-Inv}^{(X^0)} D_{E/S}$. This set is directly closed below X^0, so it contains $R_{df}^{(X^0)}$. Hence $D_{E/Q} \supseteqq D_{E/S}$ where Q stands for $R_{df}^{(X^0)}$. As $Q \supseteqq R$, we also have $D_{E/S}=D_{E/R} \supseteqq D_{E/Q}$, i.e. $D_{E/Q}=D_{E/S}$. Therefore if S and S' are directly equivalent below X^0, i.e. $R_{df}^{(X^0)}=R_{df}^{\prime(X^0)}$, then $D_{E/S}=D_{E/S'}$.

(b) Let D be a submonoid of $D(E)$. For an X-point P on E the set $D\cdot P= =\{\delta\cdot P;\ \delta\in D\}$ is called the *D-orbit* of P. Every D-orbit is stabilized by all $\delta\in D$, as $\delta\cdot(D\cdot P)=\delta D\cdot P\subseteqq D\cdot P$. It is also clear that any $\delta\in D$ stabilizes every union of D-orbits. Conversely, assume that each $\delta\in D$ stabilizes a relation r. Then, for any $P\in r$, we have

$$\{P\}=\{1_E\cdot P\}\subseteqq D\cdot P=\{\delta\cdot P;\ \delta\in D\}=\bigcup_{\delta\in D}\{\delta\cdot P\}\subseteqq\bigcup_{\delta\in D}\delta\cdot r\subseteqq\bigcup_{\delta\in D} r=r,$$

i.e., $\{P\}\subseteqq D\cdot P\subseteqq r$. By forming unions we infer that

$$r=\bigcup_{P\in r}\{P\}\subseteqq\bigcup_{P\in r} D\cdot P\subseteqq\bigcup_{P\in r} r=r,$$

i.e., $r=\bigcup_{P\in r} D\cdot P$. Hence every relation stabilized by D is a union of certain D-orbits.

The set of all relations under X^0 that are stabilized by D will be denoted by $R_D^{(X^0)}$. It is clear that the endomorphism monoid of $E/R_D^{(X^0)}$ includes D.

If $P\colon X\to E$ is a surjective point then the mapping $D(E)\to D(E)$, $\delta\to\delta\cdot P$ is injective. If card $X^0\geqq$ card E then there exist surjective $\tilde{X}$-points $\tilde{P}$ with $\tilde{X}\subseteqq X^0$. If δ stabilizes the D-orbit of such a point $\tilde{P}$ then δ must belong to D. Indeed, $\delta\cdot(D\cdot P)=\delta D\cdot P$, $\delta\cdot P=\delta\cdot(1_E\cdot P)\in\delta\cdot(D\cdot P)$, so $\delta\notin D$ would imply $\delta\cdot P\notin D\cdot P$, i.e., δ would not stabilize $D\cdot P$. This means that, denoting $R_D^{(X^0)}$ by Q, $\delta\notin D_{E/Q}$, i.e. $D_{E/Q}=D$, which proves the existence theorem of abstract Galois endotheory.

(c) As card $X^0\geqq$ card E, there is a bijective point $\tilde{P}\colon\tilde{X}\to E$ under X^0, i.e. $\tilde{X}\subseteqq X^0$. Let us fix such a point $\tilde{P}$ arbitrarily. Let $P\colon X\to E$ be another arbitrary point under X^0, and put $E_P=P\cdot X$ and $\tilde{X}_P=\tilde{P}^{-1}\cdot E_P$. Then $\tilde{P}^{-1}P\colon X\to\tilde{X}$ is a mapping, which induces a surjection $\varepsilon_{P,\tilde{P}}\colon X\to\tilde{X}_P$. Since $\tilde{P}$ is bijective, the type $T(\varepsilon_{P,\tilde{P}})$ of this mapping is equal to that of P. Hence P is compatible with $\varepsilon_{P,\tilde{P}}$. For any $x\in X$ and $\tilde{x}=\varepsilon_{P,\tilde{P}}\cdot x\in\tilde{X}_P$ we have

$$\big((\varepsilon_{P,\tilde{P}})\cdot P\big)\cdot\tilde{x}=P\cdot x=(\tilde{P}\tilde{P}^{-1})P\cdot x=\tilde{P}\cdot(\tilde{P}^{-1}P\cdot x)=\tilde{P}\cdot(\varepsilon_{P,\tilde{P}}\cdot x)=\tilde{P}\cdot\tilde{x}.$$

Note that $(\varepsilon_{P,\mathfrak{p}})\cdot P$ denotes the image of P by the contraction $(\varepsilon_{P,\mathfrak{p}})$ induced by the surjection $\varepsilon_{P,\mathfrak{p}}\colon X\to\tilde{X}_P$, not the composite $\varepsilon_{P,\mathfrak{p}}\circ P$, which even does not exist in general. So, we have $(\varepsilon_{P,\mathfrak{p}})\cdot P=(\tilde{P}|\tilde{X}_P)$ and, conversely, $P=[\varepsilon_{P,\mathfrak{p}}]\cdot(\tilde{P}|\tilde{X}_P)$. As every self-mapping δ of E commutes with projections, contractions and dilatations, we also have $\delta\cdot(\tilde{P}|\tilde{X}_P)=(\varepsilon_{P,\mathfrak{p}})\cdot(\delta\cdot P)$ and $\delta\cdot P=[\varepsilon_{P,\mathfrak{p}}]\cdot(\delta\cdot\tilde{P}|\tilde{X}_P)$. Further, if P is compatible with some surjection $\varphi\colon X\to Y$ then, clearly, so is $\delta\cdot P$. Finally, if D is a semigroup of self-mappings of E, we have $D\cdot(\tilde{P}|\tilde{X}_P)=(\varepsilon_{P,\mathfrak{p}})\cdot(D\cdot P)$ and $D\cdot P=[\varepsilon_{P,\mathfrak{p}}]\cdot(D\cdot\tilde{P}|\tilde{X}_P)$. If, in particular, D is a monoid containing 1_E, then the D-orbit $D\cdot P$ of an arbitrary point $P\colon X\to E$ can be obtained from the D-orbit of the fixed bijective point $\tilde{P}$ by a combination of direct fundamental operations. If P is under X^0 then these operations are below X^0. Therefore $R_D^{(X^0)}\subseteqq\{D\cdot\tilde{P}\}_{df}^{(X^0)}$. Indeed, every $r\in R_D^{(X^0)}$, which is a union of D-orbits by (b), is obtainable from $D\cdot\tilde{P}$ by means of direct fundamental operations (more precisely, by infinitary union, projections, and dilatations) below X^0. Suppose $D=D_{E/S}$. Then, if $S=(E,R)$, we have $R\subseteqq R_D^{(X^0)}$, whence $R_{df}^{(X^0)}\subseteqq(R_D^{(X^0)})_{df}^{(X^0)}=R_D^{(X^0)}$. On the other hand, $\{D\cdot\tilde{P}\}_{df}^{(X^0)}\subseteqq R_D^{(X^0)}$ is trivial and $\{D\cdot\tilde{P}\}_{df}^{(X^0)}\supseteqq R_D^{(X^0)}$ has already been proved, whence $\{D\cdot\tilde{P}\}_{df}^{(X^0)}=R_D^{(X^0)}$. Therefore if we prove $D\cdot\tilde{P}\subseteqq R_{df}^{(X^0)}$ then we also have $R_D^{(X^0)}=\{D\cdot\tilde{P}\}_{df}^{(X^0)}\subseteqq R_{df}^{(X^0)}$ and $R_{df}^{(X^0)}=R_D^{(X^0)}$, from which the equivalence theorem follows. Indeed, if $S=(E,R)$ and $S'=(E,R')$ are two structures under X^0 with the same stability monoid $D=D_{E/S}=D_{E/S'}$ then $R_{df}^{(X^0)}=R_D^{(X^0)}=R_{df}'^{(X^0)}$ would mean the equivalence of S and S' below X^0.

(d) Now we prove $D\cdot\tilde{P}\in R_{df}^{(X^0)}$ via obtaining this orbit explicitly from R by direct fundamental operations. Firstly, we replace every $r\in R$ by the set R_r of semi-regular relations having the same argument sets as r, which has been defined in Lemma 2, Section 2. Then R is replaced by the set of relations $\hat{R}=\bigcup_{r\in R}R_r$, which is under X^0 provided so is R, and which has the following properties implied by the quoted lemma:

(1) every $\hat{r}\in\hat{R}$ can be derived from some $r\in R$ by direct fundamental operations;

(2) for any $P\in\hat{r}\in\hat{R}$ there exists an $\hat{r}'\in\hat{R}$ such that $\hat{r}'\subseteqq\hat{r}$ and $P\in t(\hat{r}')$; and

(3) every $r\in R$ is the union $\bigcup\cdot\hat{R}'$ of some subset $\hat{R}'$ of $\hat{R}$.

These three properties show that $\hat{R}\subseteqq R_{df}^{(X^0)}$ and $R\subseteqq\hat{R}_{df}^{(X^0)}$, whence $R_{df}^{(X^0)}=\hat{R}_{df}^{(X^0)}$ and it is sufficient to prove $D\cdot\tilde{P}\in\hat{R}_{df}^{(X^0)}$. Therefore it suffices to prove $D\cdot\tilde{P}\in R_{df}^{(X^0)}$ only for sets R of *semi-regular* relations under X^0 that have property (2).

Let R be such a set of semi-regular relations and consider the relation

$$r^*=\bigcap_{r\in R}\bigcap_{P\in t(r)}\operatorname{ext}_{\tilde{X}}(\varepsilon_{P,\mathfrak{p}})\cdot r.$$

Clearly, r^* is obtained from R by direct fundamental operations, whence it belongs to $R_{df}^{(X^0)}$. We shall prove that r^* is precisely $D \cdot \tilde{P}$ where $D = D_{E/S}$.

First, as $P \in t(r)$, $(\varepsilon_{P,\mathfrak{p}}) \cdot r$ is defined and $(\tilde{P} | \tilde{X}_P) = (\varepsilon_{P,\mathfrak{p}}) \cdot P \in (\varepsilon_{P,\mathfrak{p}}) \cdot r$. Thus $\tilde{P} \in \text{ext}_{\mathfrak{X}} \cdot ((\varepsilon_{P,\mathfrak{p}}) \cdot r) = \text{ext}_{\mathfrak{X}} (\varepsilon_{P,\mathfrak{p}}) \cdot r$ and $\tilde{P} \in r^*$. But, as r^* is obtained from R by a combination of direct fundamental operations and all $\delta \in D$ are stabilizing on R, every $\delta \in D$ stabilizes r^* and $D \cdot \tilde{P} \subseteq r^*$.

For an arbitrary $\tilde{X}$-point $\tilde{Q}$ we have $\tilde{Q} = \tilde{Q} \circ 1_{\mathfrak{X}} = \tilde{Q} \circ (\tilde{P}^{-1} \circ \tilde{P}) = (\tilde{Q} \circ \tilde{P}^{-1}) \circ \tilde{P}$, for $\tilde{P}$ is injective. Hence $\tilde{Q} = \delta \cdot \tilde{P}$ where $\delta = \tilde{Q} \circ \tilde{P}^{-1}$ is a self-mapping of E. Therefore every $\tilde{X}$-point of E is the transform of $\tilde{P}$ by some self-mapping δ of E.

Now assume that $\delta \notin D$, and let us prove that $\delta \cdot P \notin r^*$. Since $\delta \notin D = D_{E/R}$, there exists some $r \in R$ not stabilized by δ, i.e. $\delta \cdot r \nsubseteq r$. Therefore there is a point $P \in r$ such that $\delta \cdot P \notin r$. Since (2) holds for R, there exists an $r' \in R$ such that $P \in t(r')$ and $r' \subseteq r$. Then $\delta \cdot \tilde{P} \notin r'$. As the contraction $(\varepsilon_{P,\mathfrak{p}})$, which is defined also for $\delta \cdot P$, is injective (for points), we have

$$(\delta \cdot \tilde{P} | \tilde{X}_P) = \delta \cdot (\tilde{P} | \tilde{X}_P) = \delta \cdot \big((\varepsilon_{P,\mathfrak{p}}) \cdot P\big) = (\varepsilon_{P,\mathfrak{p}}) \cdot (\delta \cdot P) \notin (\varepsilon_{P,\mathfrak{p}}) \cdot r'$$

and that $\{\delta \cdot \tilde{P} | \tilde{X}_P\} \times E^{\mathfrak{X} \setminus \mathfrak{X}_P}$ is disjoint from $\big((\varepsilon_{P,\mathfrak{p}}) \cdot r'\big) \times E^{\mathfrak{X} \setminus \mathfrak{X}_P} = \text{ext}_{\mathfrak{X}} (\varepsilon_{P,\mathfrak{p}}) \cdot r'$. Thus $\delta \cdot \tilde{P} \in (\delta \cdot \tilde{P} | \tilde{X}_P) \times E^{\mathfrak{X} \setminus \mathfrak{X}_P}$ does not belong to $\text{ext}_{\mathfrak{X}} (\varepsilon_{P,\mathfrak{p}}) \cdot r' \supseteq r^*$ and $\delta \cdot \tilde{P} \notin r^*$. Therefore, $\delta \cdot \tilde{P} \in r^*$ iff $\delta \in D$. Thus the equivalence theorem of of abstract Galois endotheory is proved.

Proof of the equivalence and existence theorems of abstract Galois theory. Since $\bar{R}^{(X^0)}$ is closed with respect to all fundamental operations below X^0, an argument analogous to (a) shows that if $S = (E, R)$ and $S' = (E, R')$ are equivalent below X^0 then $G_{E/S} = G_{E/S'}$. We have already seen (cf. Remark 1 in Section 1) that if σ is a permutation and both σ and σ^{-1} stabilize a relation r then σ preserves r. Consequently, if a monoid G consisting of some self-mappings of E happens to be a group, i.e., a permutation group on E, then every $\sigma \in G$ even preserves and not only stabilizes all $r \in R_G^{(X^0)}$. Therefore $R_G^{(X^0)} = s\text{-Inv}^{(X^0)} G = p\text{-Inv}^{(X^0)} G$ and, if card $X^0 \geqq \text{card}\, E$, G is the stability monoid and also the preservation monoid of $E/R_G^{(X^0)}$. Hence for any permutation group G on E there exists a structure $S = (E, R)$ such that $G = G_{E/S}$, which proves the existence theorem of abstract Galois theory.

Considering the particular case $D = G$ and keeping the notations of (c) of the preceding proof we have $G \cdot (\tilde{P} | \tilde{X}_P) = (\varepsilon_{P,\mathfrak{p}}) \cdot (G \cdot \tilde{P})$ and $G \cdot P = [\varepsilon_{P,\mathfrak{p}}] \cdot (G \cdot \tilde{P} | \tilde{X}_P)$. If $G = G_{E/S}$ then $R \subseteq R_G^{(X^0)}$ and, as $R_G^{(X^0)}$ is closed with respect to all fundamental operations, $R_f^{(X^0)} \subseteq R_G^{(X^0)}$. Since $G \cdot \tilde{P}$ and $R_G^{(X^0)}$ are equivalent (and even directly equivalent), to prove $R_f^{(X^0)} = R_G^{(X^0)}$ it is sufficient to show $G \cdot \tilde{P} \in R_f^{(X^0)}$.

Let $S = (E, R)$ be a structure under X^0 with card $X^0 \geqq \text{card}\, E$ and let $G = G_{E/S}$. Consider $S^* = (E, R \cup \neg R) = (E, R^*)$ where $\neg R = \{\neg r; r \in R\}$. Then S and S^* are equivalent below X^0 (really, $R \subseteq R^*$ and $R^* \subseteq R_f^{(X^0)}$) and $G_{E/S^*} = G_{E/S}$. Thus it suffices to prove $G\cdot\tilde{P}\in R_f^{*(X^0)}$, i.e., to prove $G\cdot\tilde{P}\in R_f^{(X^0)}$ under the hypothesis $R=\neg R$. In this case, by Remark 2 of Section 1, we have $G_{E/S}=D_{E/S}\cap S(E)$. As $\tilde{P}$ is bijective, $\delta\to\delta\cdot\tilde{P}$ is injective, implying $G\cdot\tilde{P}=D\cdot\tilde{P}\cap S(E)\cdot\tilde{P}$, where $D=D_{E/S}$. If δ is an injective, surjective or bijective self-mapping of E then $\delta\cdot\tilde{P}$ is injective, surjective or bijective as well, respectively. I. e., $S(E)\cdot\tilde{P}$ is the set of all bijective $\tilde{X}$-points while $G\cdot\tilde{P}$ is the set of all bijective points of $D\cdot\tilde{P}$. On the other hand, $r^*=D\cdot\tilde{P}\in R_{df}^{(X^0)}\subseteq\subseteq R_f^{(X^0)}$ has already been proved. Hence it is sufficient to show that the set of all bijective points of $r^*=D\cdot\tilde{P}$ can be obtained from this relation via a combination of fundamental operations.

First, an $\tilde{X}$-point $\tilde{Q}$ is not injective iff there exist $x, y\in\tilde{X}$, $x\neq y$, such that $\tilde{Q}\cdot x=\tilde{Q}\cdot y$, i.e., iff $\tilde{Q}\in\mathrm{ext}_{\tilde{X}}\cdot D_{x,y}$. Therefore the set of injective points of r is

$$r^{**}=r^*\cap(\neg\cdot\bigcup_{\substack{x,y\in\tilde{X}\\ x\neq y}}D_{x,y})=r^*\cap(\bigcap_{\substack{x,y\in\tilde{X}\\ x\neq y}}(\neg\cdot D_{x,y})).$$

As we do not want to use the axiom of choice, two cases have to be handled even if E is infinite.

(1) There exists no bijection from E (and also of $\tilde{X}$) onto any of its proper subsets. Then every injective $\tilde{X}$-point of E is surjective and r^{**} is the set of all bijective points of r^*. Now r^{**} is obtained from r^* and from simple diagonals via infinitary boolean operations, whence (cf. Remark 10 of Section 2 and the discussion of operations IV. 1-2 in the same section) r^{**} can be obtained from r^* via a combination of fundamental operations; which was to be proved.[1]

(2) There exist bijections from E onto some of its proper subsets. Then a set $\tilde{X}$ ($\tilde{X}\subseteq X^0$) and a mapping $\tilde{P}$ can be chosen so that $\tilde{X}\neq X^0$. Let y be an element of $X^0\setminus\tilde{X}$ and put $X'=\tilde{X}\cup\{y\}$. Let $\tilde{Q}\colon\tilde{X}\to E$ be an injective point and consider an arbitrary $\tilde{Q}'\in\{\tilde{Q}\}\times E^{(y)}$. Now, if $\tilde{Q}$ is bijective (i.e., surjective) then $\tilde{Q}'\cdot y$ belongs to $E=\tilde{Q}\cdot\tilde{X}=(\tilde{Q}'|\tilde{X})\cdot\tilde{X}$. Hence there exists an $x\in\tilde{X}$ such that $\tilde{Q}'\cdot x=\tilde{Q}'\cdot y$. I.e., $\tilde{Q}'$ cannot be injective. Conversely, if $\tilde{Q}$ is not surjective then there are an $e\in E$ and a $\tilde{Q}'$ such that $e\notin\tilde{Q}\cdot\tilde{X}$, $\tilde{Q}'\cdot y=e$ and $\tilde{Q}'$ is injective. So when $\tilde{Q}'$ ranges over the set of all injective points of $\mathrm{ext}_{X'}\cdot r^{**}$ then $(\tilde{Q}'|\tilde{X})$ ranges over the set of all non-bijective points of r^{**}. The set of injective points of $\mathrm{ext}_{X'}\cdot r^{**}$ is, visibly,

$$\mathrm{ext}_{X'}\cdot r^{**}\cap(\neg\cdot\bigcup_{x\in\tilde{X}}\mathrm{ext}_{X'}\cdot D_{x,y}).$$

[1]) Originally I proved the equivalence theorem of abstract Galois theory without using the axiom of choice under the assumption card $X^0\geqq$ card $E+1$. It was B. Poizat who found the present proof with card E instead of card $E+1$ for the case (1). P. Jurie proved that card E can be replaced even by card $E-1$ when E is finite.

Therefore the set of bijective points of r^{**} is

$$G\cdot\bar{P} = r^{***} = r^{**}\cap(\neg\cdot \mathrm{pr}_{\mathfrak{X}}\cdot(\mathrm{ext}_{X'}\cdot r^{**}\cap(\neg\cdot \bigcup_{x\in\mathfrak{X}} \mathrm{ext}_{X'}\cdot D_{x,y}))),$$

which is obtained from r^{**} and also from $r^{*}=D\cdot\bar{P}$ via a combination of fundamental operations. This completes the proof.

Remark 1. If $\operatorname{card} X^0 \geqq \operatorname{card} E$ then we have $\bar{R}^{(X^0)}=R_f^{(X^0)}$ and $\bar{\bar{R}}^{(X^0)}=R_{df}^{(X^0)}$.

Indeed, we have seen that $D_{E/\bar{\bar{R}}^{(X^0)}}=D_{E/R}$ and $G_{E/\bar{R}^{(X^0)}}=G_{E/R}$. Thus the equivalence theorems together with the fact that $\bar{\bar{R}}^{(X^0)}$ resp. $\bar{R}^{(X^0)}$ is closed with respect to direct resp. to all fundamental operations yield

$$R_{df}^{(X^0)} = (\bar{\bar{R}}^{(X^0)})_{df}^{(X^0)} = \bar{\bar{R}}^{(X^0)} \quad \text{and} \quad R_f^{(X^0)} = (\bar{R}^{(X^0)})_f^{(X^0)} = \bar{R}^{(X^0)}.$$

This means that a relation under X^0 belongs to $R_{df}^{(X^0)}$ iff it is stabilized by all selfmappings of E that stabilize every $r\in R$, and, analogously, it is in $R_f^{(X^0)}$ iff it is preserved by all permutations of E that preserve every $r\in R$.

Remark 2. Let S and S' be structures under X^0, where $\operatorname{card} X^0 \geqq \operatorname{card} E$. It follows from the equivalence theorems that it does not depend on the particular choice of X^0 whether S and S' are (directly) equivalent below X^0 or not. Therefore the notion of equivalence and that of direct equivalence can be defined without any reference to a particular X^0 in the following way: Two structures, S and S', are said to be equivalent resp. directly equivalent (in notation $S\sim S'$ resp. $S\underset{d}{\sim}S'$) if there exists a set X^0 such that $\operatorname{card} X^0 \geqq \operatorname{card} E$, S and S' are under X^0, and S and S' are equivalent resp. directly equivalent below X^0.

In particular, if R and R' are under X^0 and $X^{0\prime}\supseteq X^0$ then we have

$$R_{df}^{(X^{0\prime})}\cap R_E^{(X^0)} = R_{df}^{(X^0)},\quad R_f^{(X^{0\prime})}\cap R_E^{(X^0)} = R_f^{(X^0)},$$
$$R_{df}^{(X^{0\prime})} = (R_{df}^{(X^0)})_{df}^{(X^{0\prime})},\quad \text{and}\quad R_f^{(X^{0\prime})} = (R_f^{(X^0)})_{df}^{(X^{0\prime})}.$$

Thus a set of relations under $X^{0\prime}$, which is closed with respect to direct resp. all fundamental operations and its part below X^0, which is also closed with respect the same operations below X^0, mutually determine each other. More generally, if $\operatorname{card} X^0 \geqq \operatorname{card} E$ and $\operatorname{card} X^{0\prime} \geqq \operatorname{card} E$ and R is under $X^0\cap X^{0\prime}$ then $R_{df}^{(X^0)}$ and $R_{df}^{(X^{0\prime})}$ mutually determine each other (because any of them is characterized by $R_{df}^{(X^0\cup X^{0\prime})}$), and so do $R_f^{(X^0)}$ and $R_f^{(X^{0\prime})}$.

Remark 3. We are going to define two preorders, the thin (alias direct) Galois preorder $\underset{d}{\leqq}$ and the thick Galois preorder $\leqq$. For structures $S=(E, R)$ and $S'=(E, R')$ we write $S\underset{d}{\leqq}S'$ resp. $S\leqq S'$ if there is a set X^0 such that $\operatorname{card} X^0 \geqq \operatorname{card} E$, S and S' are under X^0 and $R_{df}^{(X^0)}\subseteq R_{df}^{\prime(X^0)}$ resp. $R_f^{(X^0)}\subseteq R_f^{\prime(X^0)}$. Note that $\underset{d}{\sim}$ and $\sim$ are the equivalence hulls of $\underset{d}{\leqq}$ and $\leqq$, respectively. For a structure S under

X^0 let $C_{de}^{(X^0)}(S)$ and $C_e^{(X^0)}(S)$ denote the $\underset{d}{\sim}$-class and $\sim$-class of S, respectively. Then, by the equivalence theorems, the mappings $C_{de}^{(X^0)}(S)\to D_{E/S}$ and $C_e^{(X^0)}(S)\to\to G_{E/S}$ are injective, are decreasing with respect to the orders $\underset{d}{\leqq}$ and $\leqq$, and do not depend on the choice of X^0. (Here the orders are induced by the similarly denoted preorders.) These mappings map $R_E^{(X^0)}/\underset{d}{\sim}$ and $R_E^{(X^0)}/\sim$, which do not depend on X^0, onto the set of semigroups of self-mappings of E that contain 1_E and onto the set of permutation groups on E, respectively. (This is an easy consequence of the existence theorems.)

Remark 4. Let $S=(E, R)$ be a structure under X^0, where card $X^0\geqq$card E, put $D=D_{E/S}$, let $\tilde{X}$ be a subset of X^0 with card $\tilde{X}$=card E, and let $\tilde{P}$: $\tilde{X}\to E$ be a bijective point. We have seen that any relation in $R_{df}^{(X^0)}=R_D^{(X^0)}=\bar{R}^{(X^0)}$ can be obtained from $D\cdot\tilde{P}$ via a combination of direct fundamental operations. (More precisely, first projections, then dilatations and finally an infinitary union have to be applied.)

Case of finite base set. In case E has only a finite number of elements, say m elements, it suffices to take a finite set $X^0=\{x_1, \dots, x_n\}$ with $n\geqq m$ in the equivalence and existence theorems. If $S=(E, R)$ is a structure under this X^0 then R is a finite set of relations. Further, the infinitary boolean operations are, in fact, the ordinary ones. So every structure can be considered as a model (with base set E) of some finite system $P_1(X_1), \dots, P_s(X_s)$ of predicates (with no axioms). (Here each $P_i(X_i)$ depends on a set $X_i\subseteqq X^0$ of object variables.) By a *model* (with base set E) of the previous system of predicates we mean a mapping r: $P_i(X_i)\to r(P_i)$ $(i=1, \dots, s)$ where $r(P_i)$ is an X_i-relation on E. This mapping will be extended, in the following way, to a mapping $F\to r(F)$ from the set of all formulas $F=F(P_1, \dots, P_s)$ obtained from $P_1, \dots, P_s$ via the operations of the predicate calculus with equality, used below X^0, into the set of relations under X^0. Firstly, these operations are generated by the following ones via superposition:

(1) *disjunction,* i.e., $(P, Q)\to P\vee Q$;

(2) *conjunction,* i.e., $(P, Q)\to P\& Q$;

(3) *negation,* i.e., $P\to\neg P$;

(4) *existential quantification,* i.e., $P(X)\to(\exists x)P(X)$ where $x\in X\subseteqq X^0$;

(5) *adjunction of a set of fictitious variables,* i.e., $P(X)\to P^{X'}(X')$ where $X\subseteqq\subseteqq X'\subseteqq X^0$ and $X'\setminus X$ is the set of fictitious variables;

(6) *floatages,* i.e., $P(X)\to P^{\lambda}(\lambda\cdot X)$ where λ is a bijection of $X=\{x_{i(1)}, x_{i(2)}, \dots, x_{i(t)}\}$ $(1\leqq i(1)<i(2)<\dots<i(t)\leqq n)$ onto a subset $\lambda\cdot X$ of X^0 and
$P^{\lambda}(\lambda\cdot x_{i(1)}, \dots, \lambda\cdot x_{i(t)})=P(x_{i(1)}, \dots, x_{i(t)})$;

(7) *adjunction of equality predicates* $x=y$ $(x, y \in X^0)$, which may be proper equality predicates if x and y are distinct or the *x-identity* predicates $x=x$.

It is not hard to see that any fundamental operation is a superposition of some of these seven kinds of operations. Really, $\cup$ and $\cap$ are iterations of disjunctions and conjunctions, pr_Y is a suitable iteration of (4), contractions can be composed from projections and (6), and any dilatation $[\psi]$ is a superposition of a floatage, of an extension, and of an intersection with some simple diagonals (see the discussion on the axiom of choice after the definition of direct fundamental operations). The above considerations allow us to extend r to all formulas $F(P_1, \ldots, P_s)$ below X^0 in the following obvious way, via induction: put

(1) $r(P \vee Q) = r(P) \cup r(Q)$,

(2) $r(P \,\&\, Q) = r(P) \cap r(Q)$,

(3) $r(\neg P) = \neg r(P)$,

(4) $r((\exists X) P(X)) = \mathrm{pr}_{X \setminus \{x\}} r(P(X))$,

(5) $r(P^{X'}) = \mathrm{ext}_{X'} r(P)$,

(6) $r(P^{\lambda}) = (\lambda) \cdot r(P)$,

(7) $r(x = y) = D_{x,y}$ if x and y are distinct while $r(x = x) = I(\{x\}; E) = E^{\{x\}}$.

Now it is clear that the set of all $r(F(P_1(X_1), \ldots, P_s(X_s)))$, where $F = F(T_1(X_1), \ldots, T_s(X_s))$ ranges over all formulas of the predicate calculus with equality sign that depend on the predicate variables $T_1(X_1), \ldots, T_s(X_s)$, coincides with the logical closure $R_f^{(X^0)}$ of $R = \{r(P_1), \ldots, r(P_s)\}$ below X^0. On the other hand, it is easy to see that the realizations (i.e., r-images) of the operations $T \vee T'$, $T \,\&\, T'$, $(\exists x) T(X)$ for $x \in X$, $T(X) \to T^{X'}(X')$ for $X \subseteq X'$, $T \to T^{\lambda}$, the adjunction of $x=y$ and also of $\neg(x=x)$ are direct fundamental operations. Conversely, every direct fundamental operation is the realization of an appropriate superposition of these operations. Let us call the part of predicate calculus (below X^0) generated by these operations *strictly positive predicate calculus* (below X^0). Then $R_{df}^{(X^0)}$ is the set of $r(F(P_1, P_2, \ldots, P_s))$ where $F(T_1, \ldots, T_s)$ ranges over the formulas of strictly positive predicate calculus (below X^0) that depend on the predicate variables $T_i = T_i(X_i)$ $(i = 1, 2, \ldots, s)$. So we have

Equivalence theorems for a finite base set E. *Let* $M = (E; P_i(X_i) \to r_i \subseteq E^{X_i}$ $(i = 1, \ldots, s))$ *be a model of a system of predicates* $\{P_i(X_i); i = 1, 2, \ldots, s\}$. *Then a relation* r *on* E *is of the form* $r(F(P_1, \ldots, P_s))$ *for some formula*

$F(T_1(X_1), \ldots, T_s(X_s))$ of the predicative resp. strictly predicative calculus below X^0 if and only if r is preserved resp. stabilized by all $\sigma \in S(E)$ resp. $\delta \in D(E)$ that preserve resp. stabilize all r_i $(i=1, \ldots, s)$. (The set X^0 is supposed to contain all X_i and to have at least as many elements as E.)

Examples of some classical structures

1. *The structure of the classical Galois theory.* Let E/k be a commutative field extension which is normal algebraic or algebraically closed. Consider two $\{x, y, z\}$-relations on E, $+(x, y, z)$ and $\times(x, y, z)$ such that $P \in +(x, y, z)$ iff $P \cdot x + P \cdot y = P \cdot z$ and $P \in \times(x, y, z)$ iff $(P \cdot x)(P \cdot y) = P \cdot z$. For $e \in E$ let $(x; e)$ denote the $\{x\}$-relation on E with the property $P \in (x; e)$ iff $P \cdot x = e$. Put $R_0 = \{+(x, y, z), \times(x, y, z)\} \cup \{(x; \alpha); \alpha \in k\}$, and let A be a subset of E. We can consider the structure $S = S_0(A) = (E, R_0 \cup \{(x; a);\ a \in A\})$. Then $G_{E/S}$ coincides with the ordinary Galois group of the field extension $E/k(A)$ while $D_{E/S}$ is the monoid of all isomorphisms of $E/k(A)$ into E. Note that $G_{E/S}$ and $D_{E/S}$ are the same when $E/k(A)$ is algebraic.

For $f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$ let $(f=0)$ denote the $\{x_1, \ldots, x_n\}$-relation on E such that an $\{x_1, \ldots, x_n\}$-point P belongs to $(f=0)$ iff $f(P \cdot x_1, \ldots, P \cdot x_n) = 0$. Then

$$+(x, y, z) = (x+y-z=0), \quad \times(x, y, z) = (xy-z=0) \quad \text{and} \quad (x; e) = (x-e=0).$$

Let

$$R_0' = \{(f=0);\ f \in k[x_1, x_2, \ldots] = k[\{x_n;\ n \in \mathbf{N}^*\}]\}$$

(here $\mathbf{N}^*$ stands for the set of positive natural numbers).

We claim that R_0 *and* R_0' *are deducible from one another* by means of direct fundamental operations. Really, a standard argument shows that the same self-mappings of E stabilize R_0 as R_0', whence the equivalence theorem of abstract Galois endotheory yields this assertion.

The aim of the classical Galois theory, as we have seen it in the introduction, is to determine the set $\bar{A}$ of all $\bar{a} \in E$ that are preserved by each $\sigma \in G_{E/k(A)}$. (Note that $\bar{A}$ coincides with the set $\bar{A}$ of all elements in E that are preserved by each $\delta \in D_{E/k(A)}$.) It is clear that $\sigma \cdot \bar{a} = \bar{a}$ is equivalent to $\sigma \cdot (x; \bar{a}) = (x; \bar{a})$. The abstract Galois theory answers this problem by describing $\bar{A}$ as follows:

$$\bar{a} \in \bar{A} \quad \text{iff} \quad (x; \bar{a}) \in (R_0 \cup \{(x; a);\ a \in A\})_f = (R_0' \cup \{(x; a);\ a \in A\})_f.$$

On the other hand, the classical Galois theory says that $\bar{A}$ is the closure of $k \cup A$ with respect to addition $x+y$, multiplication xy (both defined on $E \times E$), inversion x^{-1} (defined on $E^* = E \setminus \{0\}$) and, when the characteristic p of k is not 0, forming of

pth roots $\sqrt[p]{x}$ (defined on $E^p=\{x^p; x\in E\}$). The second result can be deduced from the first by means of "abstract Galois set theory", to be exposed in Section 6, and of the theory of "eliminative structures", to be exposed elsewhere. Although this deduction is quite complicated, it reveals deep reasons why the operations $x+y$, xy, x^{-1} and $\sqrt[p]{x}$, and only these operations, occur in the above-mentioned result of the classical Galois theory. Moreover, it can be shown that $\{G_{E/k(A)}; A\subseteq E\}$ is the set of all subgroups of $G_{E/k}$ that are closed with respect to the finite topology ("Krull topology") on $D(E)$, provided E/k is algebraic.

2. *A slightly different structure* is obtained if we replace $(x; e)=(x-e=0)$ by $(x, y; e)=(y-ex=0)$. I.e., R_0 is replaced by

$$R_0^* = \{+(x, y, z),\ \times(x, y, z)\}\cup\{(x, y; \alpha);\ \alpha\in k\}$$

and $R_0(A)$ by

$$R_0^*(A) = \{(x, y; a);\ a\in A\}\cup R_0^*.$$

Let $S_0^*(A)$ stand for $(E, R_0^*(A))$. Then we have $G_{E/S_0^*(A)}=G_{E/S_0(A)}$, i.e., $S_0^*(A)\sim$ $\sim S_0(A)$. Indeed,

$$(y-ex=0) = \mathrm{pr}_{\{x,y\}}\cdot((y-zx=0)\cap(z-e=0))$$

and

$$(x-e=0) = \mathrm{pr}_{\{x\}}\cdot((x-ey=0)\cap\mathrm{pr}_{\{x,y\}}\,\mathrm{ext}_{\{x,y,z,t\}}\cdot$$
$$\cdot((yz-t=0)\cap\neg(y+z-t=0)\cap D_{\{y,z,t\}}(E))).$$

Similarly, $D_{E/S_0^*(A)}=D_{E/S_0(A)}\cup\{0\}$, where 0 denotes $E\to\{0\}$, the zero homomorphism of E.

3. *Linear Galois theory.* Let k be a not necessarily commutative field, and let E/k be a field extension. We consider

$$R_0^{(L)} = \{+(x, y, z)\}\cup\{(x, y;\ \alpha);\ \alpha\in k\}$$

and

$$S_0^{(L)}(A) = (E, R_0^{(L)}\cup\{(x, y;\ a);\ a\in A\}).$$

The stability monoid $D_{E/S_0^{(L)}(A)}$ of $E/S_0^{(L)}(A)$ is the semigroup $\Lambda_{E/k(A)}$ of all linear transformations $\lambda: E\to E$ of the left vector space E over the field $k(A)$ (the field generated by $k\cup A$), while the Galois group $G_{E/S_0^{(L)}(A)}$ of $E/S_0^{(L)}(A)$ is the group of bijective linear transformations of the same vector space, i.e., it is the general linear group $GL_{k(A)}(E)$ of E over $k(A)$. The two main questions of this theory in classical algebra are the following: how to determine the set $\bar{A}$ of all $\bar{a}\in E$ such that every $\delta\in D_{E/S_0^{(L)}(A)}$ stabilizes $(x, y; \bar{a})$; and which submonoids of $D_{E/S_0^{(L)}}=\Lambda_{E/k}$ are of the form $D_{E/S_0^{(L)}(A)}$ for some $A\subseteq E$. It can be shown, in an elementary, way, that $\bar{A}=k(A)$. As regards the second question, Jacobson's density theorem yields the following answer. For a given $e \in E$ let (e) denote the linear transformation $x \to xe$ of the left vector space E/k, and put $(E)=\{(e);\ e \in E\}$. Then (E) is a field with respect to the addition and multiplication in $\Lambda_{E/k}$, and (E) is anti-isomorphic to E. Now, a submonoid Λ of $\Lambda_{E/k}$ is of the form $D_{E/S_0^{(L)}}$ for some $A \subseteq E$ if and only if Λ is a subring of $\Lambda_{E/k}$ containing (E) and closed with respect to the finite topology on $D(E)$, the set of self-mappings of E.

4. *Homogeneous Galois theory*. Let E/k be the same as in the first example; we put

$$R_0^{(h)} = \{+(x, y, z),\ \pi(x, y, z, t) = (xy - zt = 0)\} \cup \{(x; \alpha);\ \alpha \in k\}$$

and

$$R_0^{(h)}(A) = R_0^{(h)} \cup \{(x; a);\ a \in A\}.$$

It is easy to show that $G_{E/R_0^{(h)}(A)}$ and $D_{E/R_0^{(h)}(A)}$ are the group generated by the ordinary Galois group $G_{E/k(A)}$ together with the group

$$(E^*) = \{(e)\colon x \to xe;\ e \in E^* = E \setminus \{0\}\}$$

of multiplications by the non-zero elements of E, and the semigroup generated by the ordinary stability monoid $D_{E/k(A)}$ together with (E^*). In order to describe the set $\bar{A}$ of all elements $\bar{a} \in E$ that are preserved by every $\sigma \in G_{E/R_0^{(h)}(A)}$, which is the same as the set of elements preserved by all $\delta \in D_{E/R_0^{(h)}(A)}$, let $\hat{k}(B)$ denote the perfect closure (in E) of the field $k(B)$ generated by B (where $B \subseteq E$). Then $\bar{A} = a\hat{k}(a^{-1}A) = a\hat{k}(\{a^{-1}b;\ b \in A\})$ for some (moreover, for any) non-zero element a in A, and $\overline{\{0\}} = \{0\}$. Finally, note that $R_0^{(h)}$ can be replaced, up to direct equivalence, by the set

$$\{(f = 0);\ f \in k[x_1, x_2, \ldots] \text{ and } f \text{ is homogeneous}\}.$$

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