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A note on integral operators

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In 1974 A. V. BUHVALOV [1] proved that the set of all absolute integral operators from an ideal L into an ideal M of measurable functions is equal to the band generated by the integral operators of finite rank. A detailed discussion of this theorem can be found in [3] (Chapter 13). This result was proved under the additional hypothesis that the normal integrals on the domain L (i.e., linear functionals φ that can be written as $\varphi=\varphi_1-\varphi_2$, where φ_1 and φ_2 are positive order continuous linear functionals on L) separate the points, which guarantees the existence of sufficiently many non-zero integral operators. Some time ago it was conjectured by A. C. Zaanen that Buhvalov's result remains valid without the assumption concerning the normal integrals. In the present paper we prove that the conjecture is true. In particular it will be shown that if L is an ideal of measurable functions not possessing any non-zero normal integrals, then L cannot be the domain of any non-zero integral operator. This last situation occurs for example if (Y, Σ, ν) is a finite measure space not containing any atoms and if we take for L any of the ideals $L_p(Y, \nu)$ $(0<p<1)$, $L_{(1,\infty)}(Y, \nu)$ (the weak L_1 functions) or the space $L_0(Y, \nu)$ of all ν-measurable functions on X.

In this paper we restrict ourselves to considering real measurable functions only, since all the results can be extended easily to the complex case by means of complexification (see e.g. [3], sections 91 and 92).

1. We start with some notation and terminology. We refer to the book [2] for any unexplained terminology concerning Riesz spaces (vector lattices), such as band and the disjoint complement D^d of a subset D of a Riesz space. Let (Y, Σ, ν) be a σ-finite measure space. By $L_0(Y; \nu)$ we denote the space of all ν-measurable real functions on Y which are finite ν-a.e., with identification of functions which are equal ν-a.e. Let L be an ideal in $L_0(Y, \nu)$, i.e., L is a linear subspace with the additional property that $|g|\leqq|f|$, $g\in L_0(Y, \nu)$ and $f\in L$ implies that $g\in L$. A subset F of Y is called an L-null set if every $f\in L$ vanishes on F ν-a.e. There exists a maximal

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L-null set F_0, which is unique modulo null sets. The set $Y \setminus F_0$ is called the carrier of L. For the investigation of the ideal L we may assume that the carrier of L is equal to Y. We denote by $L_n^\sim$ the ideal in $L_0(Y, \nu)$ consisting of all functions g satisfying $\int_Y |fg|\, d\nu < \infty$ for all $f \in L$. The elements of $L_n^\sim$ can be identified in the obvious way with the linear functionals on L referred to in the introduction, which are called normal integrals. Let Y_0 be the carrier of $L_n^\sim$. In general Y_0 is a proper subset of Y. By way of example, if Y does not contain any atoms and $L = L_0(Y, \nu)$, then $Y_0 = \emptyset$ ([3], Example 85.1). Furthermore, $Y_0 = Y$ if and only if $L_n^\sim$ separates the points of L ([3], Theorem 95.2).

Now let (X, Λ, μ) be another σ-finite measure space and let M be an ideal in $L_0(X, \mu)$. The linear operator T from L into M is called an integral operator (or kernel operator) if there exists a $\mu \times \nu$-measurable function $T(x, y)$ on $X \times Y$, the kernel of T, such that for every $f \in L$,

$$(Tf)(x) = \int_Y T(x, y) f(y)\, d\nu(y) \quad \mu\text{-a.e. on } X.$$

Furthermore, T is called an absolute integral operator if the kernel $|T(x, y)|$ defines an integral operator from L into M as well. In fact, the integral operator T is absolute if and only if T is order bounded (i.e., T maps order intervals into order intervals), and in that case the absolute value $|T|$ of T in the Riesz space $\mathscr{L}_b(L, M)$ of all order bounded linear operators from L into M, is the integral operator with kernel $|T(x, y)|$ ([3], section 93). The set $\mathscr{I}(L, M)$ of all absolute integral operators from L into M is a band in $\mathscr{L}_b(L, M)$ ([3], Theorem 94.5). Observe that any integral operator T from L into M is order bounded as an operator from L into $L_0(X, \mu)$, since the kernel $|T(x, y)|$ defines an integral operator from L into $L_0(X, \mu)$.

As usual, for any $g \in L_n^\sim$ and $h \in M$ we denote by $g \otimes h$ the integral operator with kernel $h(x)g(y)$, and by $L_n^\sim \otimes M$ we denote the collection of all finite linear combinations of such operators. The elements of $L_n^\sim \otimes M$ are called integral operators of finite rank. It follows from $L_n^\sim \otimes M \subset \mathscr{I}(L, M)$ that the band $\{L_n^\sim \otimes M\}^{dd}$ generated by $L_n^\sim \otimes M$ satisfies $\{L_n^\sim \otimes M\}^{dd} \subset \mathscr{I}(L, M)$. In the next section we will show that $(L_n^\sim \otimes M)^{dd} = \mathscr{I}(L, M)$, without any extra assumption about the carrier of $L_n^\sim$.

2. Let (Y, Σ, ν) be a σ-finite measure space and L an ideal in $L_0(Y, \nu)$. It will be assumed that the carrier of L is equal to Y. We begin with a lemma which characterizes ideals L for which $L_n^\sim = \{0\}$.

Lemma 2.1. *The following statements are equivalent.*

(i) $L_n^\sim = \{0\}$.

(ii) *For any* $A \in \Sigma$ *with* $0 < \nu(A) < \infty$ *there exist disjoint sets* $\{A_n\}_{n=0}^\infty$ *in* Σ *such that* $A = \bigcup_{n=0}^\infty A_n$, $\sum_{n=1}^\infty n\nu(A_n) = \infty$ *and* $\sum_{n=1}^\infty n\chi_{A_n} \in L$.

Proof. First observe that $L_n^{\sim}=\{0\}$ if and only if for every $A\in\Sigma$ with $0<\nu(A)<\infty$ there exists $0\leqq f\in L$ such that $\int_A f\,d\nu=\infty$. Now it is clear that (ii) implies (i), by taking $f=\sum_{n=1}^{\infty} n\chi_{A_n}$. Now assume that $L_n^{\sim}=\{0\}$ and let $A\in\Sigma$ with $0<\nu(A)<\infty$ be given. By the remark above, there exists $0\leqq f\in L$ such that $\int_A f\,d\nu=\infty$. Define

$$A_n=\{y\in A\colon\ n\leqq f(y)<n+1\}\quad (n=0,1,2,\ldots).$$

Then $\{A_n\}_{n=0}^{\infty}$ is a disjoint sequence and $\bigcup_{n=0}^{\infty} A_n=A$ (modulo a nullset). Moreover, since $\nu(A)<\infty$, it follows easily that $\sum_{n=1}^{\infty} n\nu(A_n)=\infty$, and it follows from $0\leqq\sum_{n=1}^{\infty} n\chi_{A_n}\leqq f$ that $\sum_{n=1}^{\infty} n\chi_{A_n}\in L$.

In the next proposition it is shown that an ideal L with $L_n^{\sim}=\{0\}$ cannot be the domain of any non-zero integral operator.

Proposition 2.2. *Let L be an ideal in $L_0(Y,\nu)$ with $L_n^{\sim}=\{0\}$, and let (X,Λ,μ) be a σ-finite measure space. If T is an integral operator from L into $L_0(X,\mu)$, then $T=0$.*

Proof. Assume that T is a non-zero integral operator from L into $L_0(X,\mu)$ with kernel $T(x,y)$. Since T is order bounded (and hence T is the difference of two positive operators), we may assume that $T>0$. Furthermore, since X and Y are both σ-finite, there exist $X'\in\Lambda$ and $Y'\in\Sigma$ with $0<\mu(X')$, $\nu(Y')<\infty$ such that $T'(x,y)=T(x,y)\chi_{X'}(x)\chi_{Y'}(y)$ is not equal to zero $\mu\times\nu$-a.e. on $X\times Y$. Let L' be the ideal in $L_0(Y',\nu)$ consisting of all restrictions of elements in L to Y'. Clearly $(L')_n^{\sim}=\{0\}$ and $T'(x,y)$ defines a non-zero integral operator from L' into $L_0(X',\mu)$. Hence, we may assume that (X,Λ,μ) and (Y,Σ,ν) are both finite measure spaces with $\mu(X)=\nu(Y)=1$. Furthermore, there exists $\varepsilon>0$ such that the set $P=\{(x,y)\in X\times Y\colon T(x,y)>\varepsilon\}$ satisfies $(\mu\times\nu)(P)>0$. Now it follows from $T(x,y)\geqq\varepsilon\,\chi_P(x,y)$ that the kernel $\chi_P(x,y)$ defines a non-zero integral operator from L into $L_0(X,\mu)$, Therefore, we may assume without loss of generality that $\mu(X)=\nu(Y)=1$ and the integral operator T from L into $L_0(X,\mu)$ has kernel $\chi_P(x,y)$ with $P\subseteqq X\times Y$ and $(\mu\times\nu)(P)=\delta>0$. It follows from

$$\int_Y\Big\{\int_X \chi_P(x,y)\,d\mu(x)\Big\}\,d\nu(y)=\delta>0,$$

and from $\nu(Y)=1$ that the set $A=\Big\{y\in Y\colon \int_X\chi_P(x,y)\,d\mu(x)\geqq\delta/2\Big\}$ satisfies $\nu(A)>0$.

Observe that if $B \subseteqq A$, $B \in \Sigma$, then

$$\int_{X \times B} \chi_P \, d(\mu \times \nu) \geqq (\delta/2)\nu(B).$$

Since $L_n^{\sim} = \{0\}$, we can apply Lemma 2.1 to the set A, and so there exist disjoint sets $\{A_n\}_{n=0}^{\infty}$ in Σ with $\bigcup_{n=0}^{\infty} A_n = A$, $\sum_{n=1}^{\infty} n\nu(A_n) = \infty$ and $f = \sum_{n=1}^{\infty} n\chi_{A_n} \in L$. Put $g = Tf$. Since $\sum_{n=1}^{k} n\chi_{A_n} \uparrow_k f$, it follows from the theorem on integration of increasing sequences that $g = \sum_{n=1}^{\infty} nT\chi_{A_n}$ (μ-a.e. convergent series in $L_0(X, \mu)$). For $k = 1, 2, \ldots$ let $E_k = \{x \in X: g(x) \leqq k\}$. Then $E_k \uparrow X$. Since $\mu(X) < \infty$, there exists k such that $\mu(X \setminus E_k) < \delta/4$. Let this k be fixed.

Since $A_n \subseteqq A$ $(n = 1, 2, \ldots)$, it follows from the observation above that

$$(\delta/2)\nu(A_n) \leqq \int_{X \times A_n} \chi_P \, d(\mu \times \nu) = \int_{(X \setminus E_k) \times A_n} \chi_P \, d(\mu \times \nu) + \int_{E_k \times A_n} \chi_P \, d(\mu \times \nu) \leqq$$

$$\leqq \mu(X \setminus E_k)\nu(A_n) + \int_{E_k} \Big\{ \int_Y \chi_P(x, y)\chi_{A_n}(y) \, d\nu(y) \Big\} \, d\mu(x) \leqq$$

$$\leqq (\delta/4)\nu(A_n) + \int_{E_k} (T\chi_{A_n})(x) \, d\mu(x) \quad (n = 1, 2, \ldots),$$

and hence

$$(\delta/4)\nu(A_n) \leqq \int_{E_k} (T\chi_{A_n})(x) \, d\mu(x) \quad (n = 1, 2, \ldots).$$

This implies that

$$(\delta/4) \sum_{n=1}^{\infty} n\nu(A_n) \leqq \int_{E_k} \Big\{ \sum_{n=1}^{\infty} n(T\chi_{A_n})(x) \Big\} \, d\mu(x) = \int_{E_k} g(x) \, d\mu(x) \leqq k\mu(E_k) < \infty,$$

which is a contradiction. This completes the proof of the proposition.

Corollary 2.3. *Let (X, Λ, μ) and (Y, Σ, ν) be σ-finite measure spaces and let L and M be ideals in $L_0(Y, \nu)$ and $L_0(X, \mu)$ respectively. Denote by Y_0 the carrier of $L_n^{\sim}$ and by X_0 the carrier of M. Let T be an integral operator from L into M with kernel $T(x, y)$. Then $T(x, y) = 0$ $\mu \times \nu$-a.e. outside $X_0 \times Y_0$.*

Proof. Let K denote the ideal in $L_0(Y \setminus Y_0, \nu)$ consisting of all restrictions of elements in L to $Y \setminus Y_0$. It is clear that $K_n^{\sim} = \{0\}$. Let $S(x, y)$ be the restriction of $T(x, y)$ to $X \times (Y \setminus Y_0)$. Then the kernel $S(x, y)$ defines an integral operator S from K into M. It follows now from the above proposition that $S = 0$, and hence $S(x, y) = 0$ $\mu \times \nu$-a.e. on $X \times (Y \setminus Y_0)$ (see [3], Theorem 93.1). Therefore $T(x, y) = 0$ $\mu \times \nu$-a.e. on $X \times (Y \setminus Y_0)$. Furthermore, the integral operator from L into M with kernel $T(x, y)\chi_{X\setminus X_0}(x)$ is the zero operator, hence $T(x, y)\chi_{X\setminus X_0}(x)=0$ $\mu\times\nu$-a.e. on $X\times Y$, i.e., $T(x, y)=0$ $\mu\times\nu$-a.e. on $(X\setminus X_0)\times Y$. We may conclude, therefore, that $T(x, y)=0$ $\mu\times\nu$-a.e. outside $X_0\times Y_0$.

In the next theorem we will show that the band of absolute integral operators from the ideal L into the ideal M is equal to the band generated by the integral operators of finite rank. Under the additional assumption that the carrier of the ideal $L_n^\sim$ is equal to the whole space Y this result was proved by A. V. Buhvalov [1] (see also [3], Theorem 95.1).

Theorem 2.4. *Let (Y, Σ, ν) and (X, Λ, μ) be σ-finite measure spaces and L and M ideals in $L_0(Y, \nu)$ and $L_0(X, \mu)$ respectively. Then the band $\mathscr{I}(L, M)$ of all absolute integral operators from L into M is equal to the band $\{L_n^\sim \otimes M\}^{dd}$.*

Proof. It is sufficient to show that any positive integral operator T from L into M is contained in $\{L_n^\sim \otimes M\}^{dd}$. Let $T(x, y)$ be the kernel of T. Denote by Y_0 the carrier of $L_n^\sim$ and by X_0 the carrier of M. By the above corollary we have $T(x, y)=0$ $\mu\times\nu$-a.e. on $(X\times Y)\setminus(X_0\times Y_0)$. Since Y_0 is the carrier of $L_n^\sim$, there exists a sequence $\{Y_n\}_{n=1}^\infty$ in Σ such that $Y_n\uparrow Y_0$ and $\chi_{Y_n}\in L_n^\sim$ for all n. Similarly, there exists a sequence $\{X_n\}_{n=1}^\infty$ in Λ with $X_n\uparrow X_0$ and $\chi_{X_n}\in M$ $(n=1, 2, \ldots)$. Now define

$$T_n = \inf\{T, n\chi_{Y_n}\otimes\chi_{X_n}\} \quad (n = 1, 2, \ldots),$$

which is an integral operator with kernel

$$T_n(x, y) = \inf\{T(x, y),\ n\chi_{X_n}(x)\chi_{Y_n}(y)\}$$

(see [3], Theorem 94.3). Note that $0\leqq T_n\in\{L_n^\sim \otimes M\}^{dd}$. Since $T(x, y)=0$ $\mu\times\nu$-a.e. outside $X_0\times Y_0$, it follows that $0\leqq T_n(x, y)\uparrow T(x, y)$ $\mu\times\nu$-a.e. on $X\times Y$, and hence $0\leqq T_n\uparrow T$ in $\mathscr{L}_b(L, M)$ ([3], Theorem 94.5). This shows that $T\in\{L_n^\sim \otimes M\}^{dd}$.

Remark. 2.5. In [1] Buhvalov presented an important characterization of integral operators. Recall that the sequence $\{f_n\}_{n=1}^\infty$ in $L_0(Y, \nu)$ is called star convergent to zero (denoted by $f_n \xrightarrow{*} 0$) if every subsequence of $\{f_n\}_{n=1}^\infty$ contains a subsequence which is converging to zero ν-a.e. on Y. Let L and M be ideals in $L_0(Y, \nu)$ and $L_0(X, \mu)$ respectively. We say that the operator T from L into M satisfies Buhvalov's condition if it follows from $0\leqq u_n\leqq u\in L$ $(n=1, 2, \ldots)$ with $u_n \xrightarrow{*} 0$ that $Tu_n\to 0$ pointwise ν-a.e. on X. It was proved by Buhvalov that the operator T from L into M is an integral operator if and only if T satisfies Buhvalov's condition. In the proof of this theorem it is assumed first that the carrier of $L_n^\sim$ is equal to Y (see also [3], Theorems 96.5 and 96.8), and then it is observed that the general case can be reduced easily to this special situation. Indeed, consider the ideal $K = L \cap L_\infty(Y; \nu)$ and let T_0 be the restriction of T to K. If T satisfies Buhvalov's condition, then T_0 satisfies Buhvalov's condition as well and the carrier of $K_n^\sim$ is equal to Y. Hence T_0 is an integral operator with kernel $T(x, y)$. Now it is easy to see that T is an integral operator with kernel $T(x, y)$.

References

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