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A generalization of a theorem of Dieudonné for k-triangular set functions

ENDRE PAP

1. Introduction

Although non-additive set functions occur frequently in mathematics (semi-variations of measures with values in abstract spaces, outer measures, capacities, etc.), just recently are they studied in detail. In recent years several authors considered non-additive set functions.

As it is well-known, the Nikodym boundedness theorem for measures in general fails for algebras of sets (see Example 5., DIESTEL, UHL [2], p. 18). But there are uniform boundedness theorems in which the initial boundedness conditions are on some subfamilies of a given σ-algebra; those subfamilies may not be σ-algebras. A famous theorem of DIEUDONNÉ [3] states that for compact metric spaces the pointwise boundedness of a family of Borel regular measures on open sets implies its uniform boundedness on all Borel sets. We shall generalize Dieudonné's theorem on a wider class of set functions. The class of finitely additive regular Borel set functions gives nothing new, because each finitely additive regular Borel set functions (also in the case of vector measures) is necessarily countably additive (KUPKA [7]).

We take in this paper a wider class of real valued set functions, the so-called k-triangular set functions ([5], [6]). We prove a generalized Dieudonné type theorem for this class of set functions. Using some modifications we obtain also a generalization of Dieudonné's theorem for semigroup valued set functions.

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2. k-triangular set functions

Let T be a locally compact space and $\mathcal{S}$ a class of subsets of T such that $\emptyset \in \mathcal{S}$. First some definitions.

Definition 1 (DINCULEANU [4], p. 303). A set function $\mu: \mathcal{S} \to R$ is said to be *regular* if for every $A \in \mathcal{S}$ and every $\varepsilon > 0$ there exist a compact set $K \subset A$ and an open set $G \supset A$ such that for every set $A' \in \mathcal{S}$, $K \subset A' \subset G$, we have

$$|\mu(A) - \mu(A')| < \varepsilon.$$

Definition 2. A set function $\mu: \mathcal{S} \to R_+$ is said to be *k-triangular* with $k \in (0, +\infty)$ if for every $A, B \in \mathcal{S}$, such that $A \cap B = \emptyset$ and $A \cup B \in \mathcal{S}$, we have

$$\mu(A) - k\mu(B) \leqq \mu(A \cup B) \leqq \mu(A) + k\mu(B)$$

and $\mu(\emptyset) = 0$.

The following theorem is important for further characterization of set functions which are both regular and triangular.

Theorem 1. *Let $\mathcal{S}$ be a ring of subsets of T. If a set function $\mu: \mathcal{S} \to R$ is regular and superadditive, i.e.*

$$\mu(A \cup B) \geqq \mu(A) + \mu(B) \quad \textit{for every} \quad A, B \in \mathcal{S}, \quad A \cap B = \emptyset,$$

then it satisfies the following condition

(R) *For every $A \in \mathcal{S}$ and every number $\varepsilon > 0$ there exist a compact set $K \subset A$ and an open set $G \supset A$ such that for every set $B \in \mathcal{S}$ with $B \subset G \setminus K$ we have $|\mu(B)| < \varepsilon$.*

Proof. It is enough to adapt the proof of Proposition 1 on the page 304 in [4].

Corollary 1. *If a set function $\mu: \mathcal{S} \to R$ ($\mathcal{S}$ is a ring), $\mu(\emptyset) = 0$, has regular variation, where the variation $|\mu|$ is defined in the usual way, i.e.*

$$|\mu|(E) := \sup_{\pi} \sum_{A \in \pi} |\mu(A)| \quad (E \in \mathcal{S})$$

and the supremum is taken over all partitions π of E into a finite number of pairwise disjoint members of $\mathcal{S}$, then μ satisfies condition (R).

Proof. Since $|\mu|$ is superadditive ([4], p. 34), we can apply Theorem 1 for $|\mu|$. Then the inequality $\mu \leqq |\mu|$ implies our statement.

Definition 3. A set function $\mu: \mathcal{S} \to R$ is said to be *exhaustive* whenever given a sequence (E_n) of pairwise disjoint members of $\mathcal{S}$, $\lim_{n \to \infty} \mu(E_n) = 0$.

It is obvious that a k-triangular set function μ with regular variation is itself regular.

3. Uniform boundedness theorem

We take from now on for the class $\mathscr{S}$ the collection $\mathscr{B}$ of all Borel sets of a Hausdorff locally compact topological space T. Now we formulate the main theorem.

Theorem 2. *Let $\mathscr{M}$ be a family of k-triangular set functions defined on $\mathscr{B}$ with regular variations. If the set $\{\mu(O); \mu\in\mathscr{M}\}$ is bounded for every open set O, then*

$$\{\mu(B);\ \mu\in\mathscr{M},\ B\in\mathscr{B}\}$$

is a bounded set.

Remark 1. We shall assume in the following proofs that T is a compact Hausdorff space. Namely, we can replace T with an Alexandrov one point ω compactification $T\cup\{\omega\}$, taking $\mu(\omega)=0$ $(\mu\in\mathscr{M})$.

We easily obtain the following

Corollary 2. *Let $\mathscr{M}$ be a family of regular scalar measures defined on $\mathscr{B}$. If the set $\{|\mu(O)|; \mu\in\mathscr{M}\}$ is bounded for every open set O, then*

$$\{|\mu(B)|;\ \mu\in\mathscr{M},\ B\in\mathscr{B}\}$$

is a bounded set.

Proof. Let $v(B):=|\mu(B)|$ $(B\in\mathscr{B},\ \mu\in\mathscr{M})$. It is obvious that the family $\mathscr{F}$ of all such set functions v satisfies the conditions of Theorem 2 (by Proposition 24 from [4], p. 319, $|v|=|\mu|$ is also regular). So we apply Theorem 2.

In the proof of Theorem 2 we need two lemmas.

Lemma 1. *Let μ be a k-triangular set function defined on $\mathscr{B}$ with regular variation. Then μ is k-σ-subadditive on each sequence of disjoint open sets (O_n), i.e.*

$$\mu\Big(\bigcup_{j=1}^{\infty} O_j\Big) \leqq k \sum_{j=1}^{\infty} \mu(O_j).$$

Proof of Lemma 1. First we shall prove that μ is order continuous on open sets, i.e. for each sequence (U_n) of open sets such that $U_j\supset U_{j+1}$ $(j\in N)$ and $\bigcap_{j=1}^{\infty} U_j=\emptyset$, we have

$$\lim_{j\to\infty} \mu(U_j) = 0.$$

For each $\varepsilon>0$ there exists a sequence of compact sets (K_n) such that $K_j\subset U_j$ and

$$(1)\qquad |\mu|(U_j\setminus K_j) < \begin{cases} \varepsilon/2^j & \text{for} \quad 0<k\leqq 1 \\ \varepsilon/2^j k & \text{for} \quad k>1 \end{cases} \qquad (j\in N).$$

Then there exists $n_0\in N$ such that $\bigcap_{j=1}^{n} K_j=\emptyset$ for all $n\geqq n_0$. Let $n\geqq n_0$. Then we have

$$\mu(U_n) = \mu(U_n \setminus \bigcap_{j=1}^{n} K_j) = \mu(\bigcup_{j=1}^{n} (U_n \setminus K_j)) \leqq |\mu|(\bigcup_{j=1}^{n} (U_n \setminus K_j)).$$

Hence, since $|\mu|$ is k-subadditive (i.e. $|\mu|(A \cup B) \leqq |\mu|(A) + k|\mu|(B)$ for every pair A, B of not necessarily disjoint sets from $\mathscr{B}$, see [4], pp. 35—36 and p. 16) and non-decreasing, we obtain by (1) for $k \geqq 1$ (for $0 < k < 1$ we take $k = 1$)

$$\mu(U_n) \leqq k \sum_{j=1}^{n} |\mu|(U_j \setminus K_j) < \varepsilon$$

for all $n \geqq n_0$. Now, let (O_n) be a sequence of disjoint open sets. Then we have

$$\mu(\bigcup_{j=1}^{\infty} O_j) \leqq k \sum_{j=1}^{n} \mu(O_j) + \mu(\bigcup_{j=n+1}^{\infty} O_j).$$

Taking $n \to \infty$ we obtain

$$\mu(\bigcup_{j=1}^{\infty} O_j) \leqq k \sum_{j=1}^{\infty} \mu(O_j).$$

The following lemma is given by C. SWARTZ [12] as an extract from the elementary proof of the Antosik—Miskusiński diagonal theorem [1].

Lemma 2. *Let X be a Banach space. If $x_{ij} \in X$ $(i, j \in N)$ such that*

$$\lim_{j \to \infty} x_{ij} = 0 \quad (i \in N), \quad \lim_{i \to \infty} x_{ij} = 0 \quad (j \in N)$$

and $\|x_{ii}\| \geqq \delta > 0$ $(i \in N)$, then there exist a sequence (i_n) of natural numbers and a sequence (ε_n) of positive real numbers such that

$$\Big\|\sum_{k=1}^{n-1} x_{i_n i_k}\Big\| = (1/2 - \varepsilon_n)\|x_{i_n i_n}\|, \quad \|x_{i_n i_{n+q}}\| < 2^{-q}\varepsilon_n \|x_{i_n i_n}\|$$

(in [12] δ is instead of $\|x_{i_n i_n}\|$).

Proof of Theorem 2. Firstly, let us suppose that $k = 1$. It suffices to prove that every point in T belongs to an open set O on which holds

$$\sup\{\mu(A) \colon A \subset O \quad (A \in \mathscr{B}),\ \mu \in \mathscr{M}\} < \infty. \tag{2}$$

Suppose that this is not true. Then there exists a point $x \in T$ such that (2) does not hold for every open set O such that $x \in O$. We shall prove that there exists a sequence of pairwise disjoint open sets (E_n) and a sequence (μ_n) from $\mathscr{M}$ such that $\mu_i(E_i) > i$ $(i \in N)$. For any open set O such that $x \in O$ there exists a Borel set $B \subset O$ and $\mu_1 \in \mathscr{M}$ such that

$$\mu_1(B) > 4 + 2 \sup_{\mu \in \mathscr{M}} \mu(\{x\}). \tag{3}$$

It is easy to prove that the preceding supremum is finite. Since μ_1 has regular variation, by Corollary 1 there exists a compact set $K\subset B$ and an open set $O'\subset O$, $B\subset O'$ such that $\mu_1(B')<1$ for each $B'\subset O'\setminus K$. We have by the subadditivity of μ_1

$$\mu_1(K)+\mu_1(B\setminus K) \geqq \mu_1(B).$$

Using the preceding inequality, the inequality $\mu_1(B\setminus K)<1$ and (3), we obtain

$$\mu_1(K) > 3+2 \sup_{\mu\in\mathcal{M}} \mu(\{x\}).$$

Let $K_1=K\cup\{x\}$. Then the last inequality implies (directly for $x\in K$) by the triangularity of μ_1 (for $x\notin K$) that

$$\mu(K_1) > 3+ \sup_{\mu\in\mathcal{M}} \mu(\{x\}).$$

By the regularity of μ_1 there exists an open set U such that $O\supset U\supset K_1$ and $\mu_1(B'')<1$ for every $B''\subset U\setminus K_1$. The preceding inequality together with the inequality

$$\mu_1(U) \geqq \mu_1(K_1)-\mu_1(U\setminus K_1)$$

implies

(4) $$\mu_1(U) > 2+ \sup_{\mu\in\mathcal{M}} \mu(\{x\}).$$

Again by the regularity of μ_1 there exists an open set W such that $\{x\}\subset W\subset U$ and

(5) $$\mu_1(B''') < 1$$

for every $B'''\subset W\setminus\{x\}$.

Let H be an open set such that $x\in H\subset\bar{H}\subset W$ (where $\bar{H}$ is the closure of the set H). Then we have

$$\mu_1(\bar{H}) \leqq \sup_{A\subset\bar{H}\setminus\{x\}} \mu_1(A)+\mu_1(\{x\}) \leqq \sup_{B\subset W\setminus\{x\}} \mu_1(B)+\mu_1(\{x\}).$$

Hence by (5) we obtain

(6) $$\mu_1(\bar{H}) < 1+ \sup_{\mu\in\mathcal{M}} \mu(\{x\}).$$

Let $E_1=U\setminus\bar{H}$. Then we have $E_1\subset O$ and $E_1\cap\bar{H}=\emptyset$. By the inequality

$$\mu_1(E_1)+\mu_1(\bar{H}) \geqq \mu_1(U),$$

(4) and (6) we obtain $\mu_1(E_1)>1$. Using the preceding procedure, taking in the inequality (3) the constant 5 instead of 4 and taking into account the facts that $x\in H$ and the family $\mathcal{M}$ is not bounded on H, we obtain open sets E_2, H_1 $(H_1\subset H)$ and $\mu_2\in\mathcal{M}$ such that $E_2\cap H_1=\emptyset$, $x\in H_1$ and $\mu_2(E_2)>2$. We have $E_1\cap E_2=\emptyset$. Continuing this procedure we obtain a sequence (μ_i) from $\mathcal{M}$ and a sequence (E_i)

of pairwise disjoint open sets such that

$$\mu_i(E_i) > i \quad (i \in N). \tag{7}$$

We shall prove that μ_i $(i \in N)$ are exhaustive on the sequence (E_n) of disjoint open sets, i.e.

$$\lim_{j\to\infty} \mu_i(E_j) = 0 \quad (i \in N). \tag{8}$$

Since $\bigcup_{j=1}^{\infty} E_j$ is an open set and $|\mu_i|$ are regular, for $\varepsilon>0$ by Corollary 1 there exists a compact set $K' \subset \bigcup_{j=1}^{\infty} E_j$ such that $\mu_i(C)<\varepsilon$ for each $i \in N$ and each $C \subset \bigcup_{j=1}^{\infty} E_j \setminus K'$. Since (E_i) is an open cover of K', there exists $n_0 \in N$ such that $K' \subset \bigcup_{j=1}^{n_0} E_j$. Then we have for $m \geqq n_0+1$

$$\mu_i(E_m) \leqq \sup_{C'} \mu_i(C') \leqq \sup_{C} \mu_i(C) < \varepsilon \quad (i \in N)$$

where $C' \subset E_m \cup (\bigcup_{j=1}^{n_0} E_j \setminus K')$ and $C \subset \bigcup_{j=1}^{\infty} E_j \setminus K'$. So we obtain (8).

Let $x_{ij} = \mu_i(E_j)/i$. We have by (8) $\lim_{j\to\infty} x_{ij} = 0$ $(i \in N)$. We obtain by the boundedness assumption of the theorem that $\lim_{i\to\infty} x_{ij} = 0$ $(j \in N)$. Applying Lemma 2 for the infinite matrix $[x_{ij}]$ $(i, j \in N)$ we obtain a sequence (i_n) from N and a sequence (ε_n) of positive real numbers such that

$$\sum_{k=1}^{n-1} x_{i_n i_k} = (1/2 - \varepsilon_n) x_{i_n i_n}, \tag{9}$$

$$x_{i_n i_{n+q}} < 2^{-q} \varepsilon_n x_{i_n i_n} \quad (n \in N). \tag{10}$$

Using the triangularity of μ_{i_n} $(n \in N)$ and Lemma 1 we obtain

$$\mu_{i_n}\Big(\bigcup_{k=1}^{\infty} E_{i_k}\Big) \geqq \mu_{i_n}(E_{i_n}) - \sum_{k=1}^{n-1} \mu_{i_n}(E_{i_k}) - \sum_{k=n+1}^{\infty} \mu_{i_n}(E_{i_k}) \quad (n \in N).$$

Hence by (9) and (10)

$$i_n^{-1} \mu_{i_n}\Big(\bigcup_{k=1}^{\infty} E_{i_k}\Big) \geqq x_{i_n i_n} - \sum_{k=1}^{n-1} x_{i_n i_k} - \sum_{k=n+1}^{\infty} x_{i_n i_k} \geqq x_{i_n i_n}/2 \quad (n \in N),$$

i.e.,

$$\mu_{i_n}\Big(\bigcup_{k=1}^{\infty} E_{i_k}\Big) \geqq \mu_{i_n}(E_{i_n})/2 \quad (n \in N).$$

Then by (7) we obtain

$$\mu_{i_n}\Big(\bigcup_{k=1}^{\infty} E_{i_k}\Big) \geqq i_n/2 \quad \text{for each } n \in N.$$

Since $\bigcup_{k=1}^{\infty} E_{i_k}$ is an open set we obtain a contradiction with the boundedness of (μ_{i_n}) on open sets.

Finally, we reduce the general case $k\in(0, +\infty)$ to the preceding one. Namely, for $r\in(0, 1]$ this follows from the fact that each r-triangular set function is 1-triangular. Let us suppose now that $\mu_n{}'$ $(n\in\mathrm{N})$ are k-triangular for some k such that $k>1$. Since for any number k such that $k>1$ and any $r\in(0, 1]$ there exists $m\in\mathrm{N}$ such that $k\leqq mr$, it follows that the set functions $\nu_n, \nu_n=m\cdot\mu_n$ $(n\in\mathrm{N})$, are r-triangular. Now an application of the first part to the set functions $\{\nu_n\}$ completes the proof.

4. Semigroup valued k-triangular set functions

Let X be a commutative semigroup with a neutral element O. Let d: $X\to[0, +\infty)$ be a pseudometric such that satisfies the following condition

$$(\mathrm{d}_+) \qquad d(x+x_1, y+y_1) \leqq d(x, y)+d(x_1, y_1)$$

for all $x, x_1, y, y_1\in X$.

Example. WEBER [13] has proved that for every commutative complete uniform semigroup there exists a family of pseudometrics which satisfy (d_+) and which generate its uniformity.

Let X be endowed with a pseudometric d which satisfies (d_+). Now we can extend the definition of the regularity of a set function ν: $\mathscr{S}\to X$ taking only in the Definition 1 ν and "$d(\nu(A), \nu(A'))<\varepsilon$" instead of μ and "$|\nu(A)-\nu(A')|<\varepsilon$", respectively.

The pseudometric d induces a *triangular functional* [8], [10] in the following way:

$$f(x):=d(x, 0) \quad (x\in X).$$

The functional f satisfies

(F_1) $f(x+y)\leqq f(x)+f(y)$, and

(F_2) $f(x+y)\geqq |f(x)-f(y)|$ for all $x, y\in X$.

Now we define the variation $|\nu|$ of a set function ν: $\mathscr{S}\to X$ with $\nu(\emptyset)=0$ in the following way:

$$|\nu|(E):=\sup_{\pi} \sum_{A\in\pi} f(\nu(A)) \quad (E\in\mathscr{S})$$

where the supremum is taken over all partitions π of E into a finite number of pairwise disjoint members of $\mathcal{S}$. It is easy to see that $|\nu|$ is superadditive.

A set function $\nu: \mathcal{S} \to X$ is said to be a semigroup valued k-triangular set function if satisfies $\nu(\emptyset)=0$, and

$$f(\nu(A))-kf(\nu(B)) \leqq f(\nu(A\cup B)) \leqq f(\nu(A))+kf(\nu(B))$$

for $A, B\in\mathcal{S}$ with $A\cap B=\emptyset$.

Now we have the following generalization of Theorem 2.

Theorem 3. *Let $\mathcal{F}$ be a family of semigroup valued k-triangular set functions with regular variations defined on $\mathcal{B}$. If the set $\{f(\nu(O));\ \nu\in\mathcal{F}\}$ is bounded for every open set O, then*

$$\{f(\nu(B));\ \nu\in\mathcal{F},\ B\in\mathcal{B}\}$$

is a bounded set.

Proof. We take $\mu(B):=f(\nu(B))$ $(B\in\mathcal{B}, \nu\in\mathcal{F})$ and we apply Theorem 2.

Remark 2. Diagonal theorems ([1], [8], [9], [12]) are very useful in the elementary proofs of many important theorems in functional analysis and measure theory.

References

[1] P. Antosik, On the Mikusiński diagonal theorem, *Bull. Acad. Polon. Sci., Ser. Math. Astronom. Phys.*, 19 (1971), 305—310.

[2] J. Diestel, J. J. Uhl, *Vector measures*, Math. Surveys 15, Amer. Math. Soc. (Providence, R. I., 1977).

[3] J. Dieudonné, Sur la convergence des suites de mesures de Radon, *Anais Acad. Brasil. Ci.*, 23 (1951), 21—38, 277—281.

[4] N. Dinculeanu, *Vector Measures*, Pergamon Press (New York, 1967).

[5] L. Drewnowski, On the continuity of certain non-additive set functions, *Coll. Math.*, 38 (1978), 243—253.

[6] Н. С. Гусельников, О продолжении квазилипшицевых функций множества, *Математические заметки*, 17 (1975), 21—31.

[7] J. Kupka, Uniform boundedness principles for regular Borel vector measures, *J. Austral. Math. Soc. Series A*, 29 (1980), 206—218.

[8] E. Pap, A generalization of the diagonal theorem on a block-matrix, *Mat. ves.*, 11 (26) (1974), 66—71.

[9] E. Pap, On the continuous functions with the values in a semigroup, in: *Topology* (Proc. Conf. Budapest, 1978), Coll. Math. Soc. János Bolyai, Vol. 23, North-Holland (Amsterdam, 1980); pp. 931—938.

[10] E. Pap, Triangular functionals on commutative semigroups, in: *Proc. of the Conf. on Convex Structures in Schwerin*, Akad. der Wiss. der DDR (to appear).

[11] J. D. STEIN, A uniform-boundedness theorem for measures, *Mich. Math. J.*, **19** (1972), 161—165.

[12] C. SWARTZ, Applications of the Mikusiński's diagonal theorem, *Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys.*, **26** (1978), 421—424.

[13] W. WEBER, Fortsetzung von Massen mit Werten in uniformen Halbgruppen, *Arch. Math.*, **27** (1976), 412—423.

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