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Completely bounded maps and hypo-Dirichlet algebras

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1. An important result of VON NEUMANN [16] is the theorem that the closed unit disk $\mathbf{D}^-$ is a spectral set for every contraction T defined on a complex Hilbert space $\mathfrak{H}$. SZ.-NAGY showed in [25] that every contraction T possesses a unitary dilation U on some larger Hilbert space $\mathfrak{K}$, and thus obtained an alternate proof of this. The investigation of the relation of T and U forms the basis for [26]. In [24] SZ.-NAGY also proved that an invertible operator T satisfying $\sup\{\|T^k\|: k\in Z\}<\infty$ is similar to a unitary operator. His question of whether the assumption that $\sup\{\|T^k\|: k=1,2,\ldots\}<\infty$, implies that T is similar to a contraction was shown to have a negative answer by FOGUEL [6]. HALMOS then reformulated the question [11] to ask if every polynomially bounded operator is similar to a contraction.

All of these results and questions can be reformulated to involve mappings from a function algebra A to the algebra $\mathfrak{L}(\mathfrak{H})$ of bounded operators on a Hilbert space $\mathfrak{H}$. Von Neumann's result concerns the extendibility of the mapping $p(z)\to p(T)$ to a contractive unital homomorphism from the disk algebra $A(\mathbf{D})$ to $\mathfrak{L}(\mathfrak{H})$. Sz.-Nagy's theorem shows that every such map dilates to a *-homomorphism from $C(\partial\mathbf{D})$ to $\mathfrak{L}(\mathfrak{K})$ for some Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$. Halmos' question asks whether every bounded unital homomorphism φ from $A(\mathbf{D})$ to $\mathfrak{L}(\mathfrak{H})$ is similar to a contractive unital homomorphism. ANDO [2] has shown that Sz.-Nagy's dilation theorem generalizes to the bidisk algebra $A(\mathbf{D}^2)$, and on the basis of PARROTT's example [17], one can show that this is false for $A(\mathbf{D}^n)$ for $n>2$ (cf. [26]). VAROPOULOS showed in [28] that the analogous result is also false for the ball algebra $A(\mathbf{B}^n)$ for $n>2$.

The above results form the core of dilation theory and the theory of spectral sets. Dilation theory is concerned with which linear maps from a function algebra to $\mathfrak{L}(\mathfrak{H})$ dilate to a representation of a self-adjoint algebra containing the function

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algebra, on some possibly larger Hilbert space. The theory of spectral sets is concerned with determining when a particular set Ω is spectral for an operator T and if it is, deciding whether or not T possesses a normal dilation whose spectrum is contained in $\partial\Omega$, i.e., a $\partial\Omega$-normal dilation. That is, deciding whether or not the induced contractive homomorphism, $r(z)\to r(T)$, on the uniform closure $R(\Omega)$ of the rational functions with poles off Ω, dilates to $C(\partial\Omega)$.

Foiaş in [7] and then independently Berger [4] and Lebow [13] (cf. Sarason [21]) showed that if $R(\Omega)$ is a Dirichlet algebra on $\partial\Omega$, then all unital contractive homomorphisms on $R(\Omega)$ dilate to $C(\partial\Omega)$.

In this paper we investigate problems for mappings from function algebras defined for finitely connected domains in $\mathbf{C}$. Such domains yield algebras $R(\Omega)$ which are hypo-Dirichlet. We show, in Section 2, that operators having such domains as spectral sets are similar to operators which have a normal dilation. We prove, more generally, that all unital contractive homomorphisms of hypo-Dirichlet algebras are similar to homomorphisms that dilate.

While these results still leave open the question of whether or not operators having such domains as spectral sets have normal dilations, other results suggest that perhaps this is the wrong question. The search for criteria which insure that a set is spectral has been in some ways unsuccessful. Shields [33] remarks that for an invertible operator T, the annulus with radii determined by the norms of T and T^{-1} need not be spectral for T. In [14], Misra shows that this occurs even when T is a 2 by 2 matrix. However, in Section 3, we show that such an operator is always similar to an operator for which the original annulus is spectral, and which moreover possesses a normal dilation. Thus, criteria for a set to be a spectral set, up to similarity, can be more readily obtainable. An analogous phenomenon occurs for all multiply connected domains—adding a similarity allows one to consider simpler domains.

In [3], Arveson reformulated dilation theory and generalized some of the previous results. If φ is a linear mapping from a function algebra A to $\mathfrak{L}(\mathfrak{H})$, then there is a natural linear mapping φ_n from $M_n(A)$ to $M_n(\mathfrak{L}(\mathfrak{H}))\cong\mathfrak{L}(\mathfrak{H}\otimes\mathbf{C}^n)$, where $M_n(A)$ denotes the algebra of $n\times n$ matrices with entries from A, defined by applying φ entrywise. If φ is bounded, then so is φ_n and $\|\varphi\|\leqq\|\varphi_2\|\leqq\|\varphi_3\|\leqq\ldots$, but it is not necessarily true that $\|\varphi_n\|=\|\varphi\|$ or even that the sequence of norms is uniformly bounded. When it is, the map is said to be *completely bounded* and we set $\|\varphi\|_{cb}=\sup\{\|\varphi_n\|: n=1, 2, \ldots\}$. The map is said to be *n-contractive* if $\|\varphi_n\|\leqq 1$ and *completely contractive* if it is n-contractive for all n.

Generalizing earlier work of Stinespring [23], Arveson showed in [3] that a unital contractive map is dilatable if and only if it is completely contractive. Moreover, he showed that all unital contractive maps on Dirichlet algebras are completely contractive, thus generalizing the results of Sz.-Nagy and Foiaş—Berger—

Lebow. Ando's result shows that all unital contractive homomorphisms on the bidisk algebra are completely contractive, but the question for unital contractive linear maps is open.

In Section 4, we show that Arveson's result on Dirichlet algebras does not extend to hypo-Dirichlet algebras. We give an example for the annulus algebra of a unital contraction which is not even a 2-contraction, although we leave unanswered the question for homomorphisms. (See Note added in proof.)

A necessary condition that a homomorphism be similar to a dilatable homomorphism is that it be completely bounded. Extending the work of ARVESON [3], and HADWIN [10], the second author has shown that this is also sufficient [18] (see also, [9], [31], [32]). In fact, if φ is a unital completely bounded homomorphism then there is always a similarity S, with $\|S\|\cdot\|S^{-1}\|=\|\varphi\|_{cb}$ such that $S\varphi S^{-1}$ is completely contractive [19]. Thus, in particular an operator T is similar to a contraction if and only if the mapping $p(z)\to p(T)$ extends to be completely bounded on $A(\mathbf{D})$. The problem of whether T being polynomially bounded implies that the resulting map is completely bounded is still unsolved.

2. A uniformly closed unital subalgebra A of $C(X)$ is said to be hypo-Dirichlet of codimension n, if the closure $\mathfrak{S}$ of $A+\bar{A}$ is of finite codimension n in $C(X)$. A Dirichlet algebra is obviously hypo-Dirichlet as is the algebra $R(\Omega)$ generated by the rational function on a finitely connected domain in $\mathbf{C}$. We consider in this section the problem of dilating unital contractive mappings from a hypo-Dirichlet algebra A to $\mathfrak{L}(\mathfrak{H})$. In Section 4 we present an example showing that Arveson's result doesn't extend even to the annulus algebra although our example is not a homomorphism. We show instead that for $\varphi\colon A\to\mathfrak{L}(\mathfrak{H})$ a unital contraction, one has $\|\varphi\|_{cb}\leqq 2n+1$, where n is the codimension of $\mathfrak{S}$ in $C(X)$. In particular, this implies that if $\varphi\colon A\to\mathfrak{L}(\mathfrak{H})$ is a unital contractive homomorphism, then there exists an invertible operator S such that $\|S^{-1}\|\,\|S\|\leqq 2n+1$ and the mapping $\varphi_1\colon A\to\mathfrak{L}(\mathfrak{H})$ defined by

$$\varphi_1(f)=S^{-1}\varphi(f)S$$

is dilatable to $C(X)$.

We begin with the following lemma about bases in finite dimensional Banach spaces.

Lemma 2.1. *Let A be a unital C^*-algebra and let $1\in\mathfrak{S}\subseteqq A$ be a self-adjoint subspace of codimension n. Then for every $\varepsilon>0$ there exists a positive map $\varrho\colon A\to\mathfrak{S}$ and a positive linear functional s on $\mathfrak{S}$ such that $\|\varrho\|\leqq n+1+\varepsilon$, $\|s\|\leqq n+\varepsilon$, and*

$$\varrho(f)=f+s(f)\cdot 1 \quad \textit{for } f \textit{ in } \mathfrak{S}.$$

Proof. Let $\pi\colon A\to A/\mathfrak{S}$ be the quotient map. It is not difficult to show that there exist self-adjoint linear functionals $l_1', l_2', \dots, l_n'$ on $A/\mathfrak{S}$ and self-adjoint elements $h'_1, h'_2, \ldots, h'_n$ in $A/\mathfrak{S}$ which form a basis, such that $\|l'_i\|=\|h'_i\|=1$ and $l'_i(h'_j)=\delta_{ij}$, the Kronecker delta. Let $h_1, h_2, \ldots, h_n$ be self-adjoint elements in A such that $\pi(h_i)=h'_i$ and $\|h_i\|\leqq 1+\varepsilon/n$. Also, let $l_i=l'_i\circ\pi$, so that $\mathfrak{S}=\{f\in A\colon l_i(f)=0, i=1, 2, \ldots, n\}$ with $\|l_i\|=1$ and $l_i(h_j)=\delta_{ij}$.

By [20, 1.14.3] we can write $l_i=p_i-q_i$, where $\|p_i\|\leqq 1$ $\|q_i\|\leqq 1$ and $\|p_i+q_i\|\leqq 1$ with p_i and q_i positive linear functionals in A. For every g in A

$$f=g-\sum_{i=1}^{n} l_i(g)h_i,$$

is in $\mathfrak{S}$. We define a positive map $\varrho\colon A\to\mathfrak{S}$ by

$$\varrho(g)=g+\sum_{i=1}^{n} q_i(g)(\|h_i\|+h_i)+\sum_{i=1}^{n} p_i(g)(\|h_i\|-h_i).$$

Since each of the three expressions defines a positive map, we need only check that the range of ϱ is contained in $\mathfrak{S}$. An easy calculation shows for $g=f+\sum_{i=1}^{n} l_i(g)h_i$ that $\varrho(g)=f+\sum_{i=1}^{n}(p_i(g)+q_i(g))\|h_i\|$. If we set $s(g)=\sum_{i=1}^{n}(p_i(g)+q_i(g))\|h_i\|$, then

$$\|s\|\leqq\sum_{i=1}^{n}\|p_i+q_i\|\,\|h_i\|\leqq n+\varepsilon$$

and $\varrho(f)=f+s(f)\cdot 1$ for f in $\mathfrak{S}$. Finally, since ϱ is positive we have $\|\varrho\|=\|\varrho(1)\|=\|1+s(1)\|\leqq n+1+\varepsilon$.

Theorem 2.2. *If $A\subseteqq C(X)$ is a hypo-Dirichlet algebra of codimension n, and $\varphi\colon A\to\mathfrak{L}(\mathfrak{H})$ is a unital contraction, then $\|\varphi\|_{cb}\leqq 2n+1$.*

Proof. Fix $\varepsilon>0$, let $\mathfrak{S}$ be the closure of $A+\bar{A}$ in $C(X)$ and let $\varrho\colon C(X)\to\mathfrak{S}$ and s be as in the previous Lemma. If we extend φ to $\tilde{\varphi}\colon\mathfrak{S}\to\mathfrak{L}(\mathfrak{H})$ by $\tilde{\varphi}(f+\bar{g})=\varphi(f)+\varphi(g)^*$, then $\tilde{\varphi}$ will be positive by [3, pp. 152—153]. Thus $\tilde{\varphi}\circ\varrho\colon C(X)\to\mathfrak{L}(\mathfrak{H})$ is positive and hence completely positive [23]. Finally for f in A, we have

$$\varphi(f)=\tilde{\varphi}\circ\varrho(f)-s(f)\cdot 1_{\mathfrak{H}}$$

so that,

$$\|\varphi\|_{cb}\leqq\|\tilde{\varphi}\circ\varrho\|_{cb}+\|s\|_{cb}=\|\tilde{\varphi}\circ\varrho(1)\|+\|s(1)\|\leqq 2n+1+2\varepsilon,$$

since $\tilde{\varphi}\circ\varrho$ and s are completely positive (see [3, Proposition 1.2.10]).

Corollary 2.3. *If $A\subseteqq C(X)$ is a hypo-Dirichlet algebra of codimension n, and $\varphi\colon A\to\mathfrak{L}(\mathfrak{H})$ is a unital contractive homomorphism, then φ is similar to a homomorphism that dilates to $C(X)$. Furthermore, the similarity S may be chosen such that $\|S\|\cdot\|S^{-1}\|\leqq 2n+1$.*

Corollary 2.4 *Let Ω be a spectral set for T in $\mathfrak{L}(\mathfrak{H})$ with $R(\Omega)$ a hypo-Dirichlet subalgebra of $C(\partial\Omega)$ of codimension n. Then there exists an invertible operator S on $\mathfrak{H}$ with $\|S\|\cdot\|S^{-1}\|\leqq 2n+1$ such that $S^{-1}TS$ has a $\partial\Omega$-normal dilation.*

We are unable to determine whether these results are the best one can do even in the case of the annulus.

3. As we mentioned in the first section the prototypical example of a hypo-Dirichlet algebra is $R(\Omega)$ for Ω the closure of a finitely connected domain in $\mathbf{C}$. As might be expected a more natural proof of Corollary 2.4 is possible in this case which has other consequences that we explore. We make no attempt at working with the most general domains possible.

A compact subset Ω of $\mathbf{C}$ is said to be a K-spectral set for T in $\mathfrak{L}(\mathfrak{H})$ if $\sigma(T)\subseteq\Omega$ and $\|f(T)\|\leqq K\|f\|$ for f in $R(\Omega)$, where $\|f\|$ denotes the supremum norm on $\partial\Omega$. We call Ω a *complete K-spectral set* for T if, in addition,

$$\|(f_{i,j}(T))\|\leqq K\|(f_{i,j})\|$$

for all matrices $(f_{i,j})$ in $M_n(R(\Omega))$, where $\|(f_{i,j})\|$ denotes the supremum of the matrix norm on $\partial\Omega$ and $(f_{i,j}(T))$ denotes the operator in $M_n(\mathfrak{L}(\mathfrak{H}))\cong\mathfrak{L}(\mathfrak{H}\otimes\mathbf{C}^n)$. For $K=1$ we obtain the usual notion of *spectral* or *complete spectral set.*

We will need also to consider certain unbounded subsets of $\mathbf{C}$. If Ω is a closed unbounded subset of $\mathbf{C}$, then we let $R(\Omega)$ denote the function algebra obtained from the uniform closure of the rational functions on Ω with poles off Ω and which vanish at infinity. We shall call such an Ω a *(complete) K-spectral set* for T provided that the above inequalities hold for all f in $R(\Omega)$ ($(f_{i,j})$ in $M_n(R(\Omega))$).

We shall call a compact subset Ω of $\mathbf{C}$ a *Dirichlet algebra domain* if $\{\mathrm{Re}\,(f): f\in R(\Omega)\}$ is uniformly dense in the algebra of real-valued continuous functions on $\partial\Omega$. More generally, we shall call an arbitrary closed subset Ω a *Dirichlet algebra domain* if there is a λ in $C\setminus\Omega$ such that the compact set

$$\Omega_\lambda^{-1}=\{(z-\lambda)^{-1}\colon z\in\Omega\}\cup\{0\}$$

is a Dirichlet algebra domain.

We record the following facts about these "unbounded" Dirichlet domains for further use.

Proposition 3.1. *Let Ω be a closed subset of $\mathbf{C}$ containing a neighborhood of infinity and let λ be in $\mathbf{C}\setminus\Omega$.*

(i) *If Ω is a (complete) K-spectral set for T, then Ω_λ^{-1} is a (complete) $(2K+1)$-spectral set for $(T-\lambda)^{-1}$.*

(ii) *If Ω_λ^{-1} is a (complete) K-spectral set for $(T-\lambda)^{-1}$, then Ω is a (complete) K-spectral set for T.*

(iii) *The set* Ω *is a spectral set for* T *if and only if* Ω_λ^{-1} *is a spectral set for* $(T-\lambda)^{-1}$.

(iv) *If* Ω *is a Dirichlet algebra domain, then* Ω *is a spectral set for* T *if and only if* Ω *is a complete spectral set for* T.

Proof. If Ω is a K-spectral set for T, then for every f in $R(\Omega_\lambda^{-1})$, with $f(0)=0$, $g(z)=f((z-\lambda)^{-1})$ is in $R(\Omega)$ and hence,

$$\|f((T-\lambda)^{-1})\| = \|g(T)\| \leqq K\|g\| = K\|f\|.$$

Now, if f is an arbitrary element of $R(\Omega_\lambda^{-1})$, then,

$$\|f((T-\lambda)^{-1})\| \leqq \|f((T-\lambda)^{-1})-f(0)I\|+\|f(0)I\| \leqq (2K+1)\|f\|.$$

The proofs of the "complete" case and of (ii) are similar. If Ω is a spectral set for T, and f is in $R(\Omega_\lambda^{-1})$, with $\|f\|\leqq 1$, let $x=f(0)$ and let $\varphi_x(z)=(z-x)/(1-\bar{x}z)$ be the conformal mapping of the disc into itself that carries x to 0. We have that $\|\varphi_x\circ f\|<1$, and since $\varphi_x\circ f(0)=0$, by the calculations above,

$$\|\varphi_x\circ f((T-\lambda)^{-1})\| < 1.$$

But now by von Neumann's inequality,

$$\|f((T-\lambda)^{-1})\| = \|\varphi_{-x}\circ\varphi_x\circ f((T-\lambda)^{-1})\| < 1.$$

This conformal mapping technique was introduced by WILLIAMS [30]. (iv) follows easily from the case of bounded domain using (ii).

We shall call a compact subset Ω of $\mathbf{C}$ *decomposable* if $\Omega=\bigcap\Omega_j$ (possibly infinite), Ω_j is a Dirichlet algebra domain and there is a constant K such that for any n and f in $M_n(R(\Omega))$, we can write $f=\sum f_j$ (norm convergent) with f_j in $M_n(R(\Omega_j))$ and

$$\sum\|f_j\| \leqq K\|f\|.$$

We shall let K_Ω denote the minimum of such constants K. We show the usefulness of this notion after establishing that nice finitely connected subsets of $\mathbf{C}$ are decomposable.

Proposition 3.2. *If* Ω *is a compact subset of* $\mathbf{C}$ *such that* $\mathbf{C}\setminus\Omega$ *consists of finitely many open connected subsets whose boundaries are disjoint and each is a rectifiable simple closed curve, then* Ω *is decomposable.*

Proof. Let $\mathbf{C}\setminus\Omega=\Delta_0\cup\Delta_1\cup\ldots\cup\Delta_n$ with Δ_0 the connected component of infinity and let $\Gamma_i=\partial\Delta_i$ be given a parametrization such that $\Gamma=\bigcup\Gamma_i$ has winding number 1 for every point in the interior of Ω and 0 for every point in $\mathbf{C}\setminus\Omega$. Let $\Omega_i=\mathbf{C}\setminus\Delta_i$ so that Ω_i is a Dirichlet algebra domain and Ω_0 is bounded.

For λ in the interior of Ω and f in $M_k(R(\Omega))$, let

$$f_j(\lambda) = (1/2\pi i) \int_{\Gamma_j} (f(z)/(z-\lambda))\, dz$$

so that $f(\lambda) = \sum_{i=0}^{n} f_i(\lambda)$ for λ in Ω. Note that f_j clearly extends to a continuous function on Γ_i for $i \neq j$ and hence to a continuous function in Γ_j by letting $f_j(\lambda) = f(\lambda) - \sum_{i\neq j} f_i(\lambda)$. Moreover, f_0 is analytic on the interior of Ω_0, continuous on Ω_0 and hence in $M_k(R(\Omega_0))$ [8, Chapter VIII, 8.4]. Similarly, for $i \neq 0$, f_i is in $M_k(R(\Omega_i))$.

Now for fixed λ in Γ_i, $i \neq j$,

$$\|f_j(\lambda)\| \leqq d(\Gamma_i, \Gamma_j)^{-1} \cdot |\Gamma_j| \cdot \|f\|$$

where $|\Gamma_j|$ is the length of Γ_j and $d(\Gamma_i, \Gamma_j)$ denotes the usual distance between Γ_i and Γ_j. Finally, for λ in Γ_j, we have

$$\|f_j(\lambda)\| \leqq \|f\| + \sum_{i\neq j} \|f_i(\lambda)\| \leqq \big(1 + \sum_{i\neq j} d(\Gamma_i, \Gamma_j)^{-1} \cdot |\Gamma_i|\big) \|f\|$$

and thus $\sum \|f_i\| \leqq K \|f\|$ which completes the proof.

In [14] an example is given of an operator T for which $\{z\colon |z| \leqq R_1\}$ and $\{z\colon |z| \geqq R_0\}$ are both spectral sets for T, while the annulus $\{z\colon R_0 \leqq |z| \leqq R_1\}$ is not a spectral set for T. The use of decomposability shows that no such example can exist if we consider K-spectral sets instead.

Theorem 3.3. *Let Ω be decomposable with $\Omega = \bigcap_{j\geqq 1} \Omega_j$. If each Ω_j is a (complete) K_j-spectral set for T and $\sup_j K_j < \infty$, then there exists a K such that Ω is a (complete) K-spectral set for T.*

Moreover, if each Ω_j is a spectral set for T, then Ω is a complete K_Ω-spectral set for T.

Proof. Let f be in $M_k(R(\Omega))$ and write $f = \sum_j f_j$ with f_j in $M_k(R(\Omega_j))$. If we set $M = \sup_j K_j$, then

$$\|f(T)\| \leqq \sum_j \|f_j(T)\| \leqq MK_\Omega \|f\|$$

from which the first result follows.

If each Ω_j is spectral for T, then since each Ω_j is a Dirichlet algebra domain, it is completely spectral by Proposition 3.1. Hence for f in $M_k(R(\Omega))$ we have

$$\|f(T)\| \leqq \sum_j \|f_j(T)\| \leqq K_\Omega \|f\|.$$

Corollary 3.4. *If $\Omega=\bigcap\Omega_j$ is decomposable and each Ω_j is spectral for T, then there exists an invertible operator S with $\|S^{-1}\|\cdot\|S\|\leqq K_\Omega$ such that $S^{-1}TS$ has a $\partial\Omega$-normal dilation.*

Proof. Apply the Theorem together with [18].

We note that if Ω is decomposable and spectral for T then the hypotheses of Corollary 3.4 are certainly met. Thus we obtain that for Ω a "nice" n-holed domain whose holes are separated, then up to similarity every spectral operator is dilatable. The hypothesis that the holes of Ω are separated is necessary only for the proof we gave and many domains not satisfying this are decomposable. For example, the domain $\Omega=\Omega_0\cap\Omega_1$, where $\Omega_0=\{z\colon |z|\leqq 2\}\cap\{z\colon |z+1|\geqq 1\}$ and $\Omega_1=\{z\colon |z-3/4|\geqq 1/4\}\cap\{z\colon |z-5/4|\geqq 1/4\}$ is decomposable with Dirichlet algebra domains Ω_0 and Ω_1. By combining these techniques with the Mlak decomposition theorem [15], one can extend considerably the class of sets for which spectral implies similar to a dilatable operator. However, it is not clear at this time whether or not these techniques yield any sets Ω for which $R(\Omega)$ is not hypo-Dirichlet. It would also be interesting to have more particular information on the decomposability constant K_Ω versus the value $2n+1$ in the hypo-Dirichlet case.

We conclude this section with a different application of the notion of decomposability.

Corollary 3.5. *If T is a bounded invertible operator on $\mathfrak{H}$ for which there exist invertible operators S_1 and S_2 satisfying*

$$\|S_1^{-1}TS_1\| \leqq \alpha \quad \text{and} \quad \|S_2^{-1}T^{-1}S_2\| \leqq \beta,$$

then there exists an invertible operator S such that both

$$\|S^{-1}TS\| \leqq \alpha \quad \text{and} \quad \|S^{-1}T^{-1}S\| \leqq \beta.$$

Proof. If $\alpha\beta\leqq 1$, then $\sigma(T)\subseteqq\{z\colon |z|\leqq\alpha\}$ and $\sigma(T^{-1})\subseteqq\{z\colon |z|\leqq\beta\}$ which implies that

$$\sigma(T)\subseteqq\{z\colon \beta^{-1}\leqq|z|\leqq\alpha\} \text{ and hence } \alpha\beta=1.$$

Setting $R=\alpha^{-1}T$ we have that $\|R^n\|\leqq\|S_1^{-1}\|\,\|S_1\|$ and $\|R^{-n}\|\leqq\|S_2^{-1}\|\,\|S_2\|$ for $n>0$. Therefore, by the result of Sz.-Nagy [24] there exists an invertible operator S such that $S^{-1}RS$ is unitary which implies that $\|S^{-1}TS\|\leqq\alpha$ and $\|S^{-1}T^{-1}S\|\leqq\beta$.

If $\alpha\beta>1$, then $\Omega_0=\{z\colon |z|\leqq\alpha\}$ is a spectral set for $S_1^{-1}TS_1$, and hence a complete spectral set since Ω_0 is a Dirichlet algebra domain. This implies for f in $M_k(R(\Omega_0))$ that

$$\|f(T)\| = \|(S_1\otimes 1_k)f(S_1^{-1}TS_1)(S_1\otimes 1_k)^{-1}\| \leqq \|S_1^{-1}\|\,\|S_1\|\,\|f\|$$

and so Ω_0 is a complete $(\|S_1^{-1}\|\,\|S_1\|)$-spectral set for T. Similarly, $\Omega_1=\{z\colon |z|\geqq\beta^{-1}\}$ is a complete $(\|S_2^{-1}\|\,\|S_2\|)$-spectral set for T. Therefore, by Theorem 3.3, $\Omega=\Omega_0\cap\Omega_1$, is a complete K-spectral set for T for some K. Again applying [18], we have that there exists an invertible operator S with $\|S^{-1}\|\,\|S\|\leqq K$ and such that Ω is a complete spectral set for $S^{-1}TS$. In particular, this implies that $\|S^{-1}TS\|\leqq\alpha$ and $\|S^{-1}T^{-1}S\|\leqq\beta$.

One difficulty mitigating the usefulness of results on dilating completely bounded maps is the lack of uniqueness. However, even in the class of operators for which the annulus is a complete spectral set and hence dilatable, there is no uniqueness to the normal dilation as was observed in [1]. In the next section we pursue a more restrictive notion of dilation on the annulus for which uniqueness persists.

4. In this section we show that even for the annulus algebra $R(A)$, the analogue of Arveson's result for Dirichlet algebras is false. Namely, we show that there exists a unital contractive linear map $\varphi\colon R(A)\to\mathfrak{L}(\mathfrak{H})$ which is not completely contractive. We remind the reader that in this case the closure of the real parts of the rational functions on A with poles off A is a subspace of codimension one in the algebra of continuous real-valued functions on ∂A.

We begin by tying together some results on completely positive maps. The equivalence of statements (1) to (5) is certainly known [3], while (6) is based on an idea of Choi [5]. Recall that a linear map φ defined from a self-adjoint subspace $\mathfrak{S}$ of a C^*-algebra A to $\mathfrak{L}(\mathfrak{H})$ is said to be *completely positive* if $\varphi_n\colon M_n(\mathfrak{S})\to\mathfrak{L}(\mathfrak{H}\otimes\mathbf{C}_n)$ is positive for $n=1, 2, 3, \dots$.

Proposition 4.1. *Let A be a unital C^*-algebra, $1\in\mathfrak{R}\subset A$ a subalgebra and let $\mathfrak{S}=\mathfrak{R}+\mathfrak{R}^*$ (or its closure). Then the following are equivalent:*

(1) *Every unital contraction $\varphi\colon\mathfrak{R}\to\mathfrak{L}(\mathfrak{H})$ is a complete contraction,*

(2) *Every unital, positive $\varphi\colon\mathfrak{S}\to\mathfrak{L}(\mathfrak{H})$ is completely positive,*

(3) *Every positive $\varphi\colon\mathfrak{S}\to\mathfrak{L}(\mathfrak{H})$ is completely positive,*

(4) *For all n, every unital positive $\varphi\colon\mathfrak{S}\to M_n$ is completely positive,*

(5) *For all n, every positive $\varphi\colon\mathfrak{S}\to M_n$ is completely positive, and*

(6) *For all n, the convex hull $h(\mathfrak{S}^+\otimes M_n^+)$ of $\{f\otimes p\colon f\in\mathfrak{S}^+, p\in M_n^+\}$ is norm dense in $(\mathfrak{S}\otimes M_n)^+$.*

Proof. The fact that (2) implies (1) follows from [3, Proposition 1.2.8 and Theorem 1.2.9]. The equivalence of (2) and (4) follows from restricting $\mathfrak{H}$ and φ to finite dimensional subspaces. Similarly (3) and (5) are equivalent. The equivalence of (4) and (5) follows by considering $\varphi(1)=R$, restricting to the subspace where it is invertible and replacing φ by $R^{-1/2}\varphi(\cdot)R^{-1/2}$. Since $\mathfrak{R}$ is a subalgebra, by von Neumann's inequality, unital positive maps on $\mathfrak{S}$ are contractive on $\mathfrak{R}$ and so (1) implies (2).

We now establish the equivalence of (5) and (6). To do this we need to recall that there is a one to one correspondence between linear maps from $\mathfrak{S}$ into M_n

and linear functionals on $\mathfrak{S}\otimes M_n$, such that under this correspondence a linear map is completely positive if and only if the associated linear functional is positive [12] (see also [22]). For $\varphi: \mathfrak{S}\to M_n$ a linear map, the corresponding linear functional s_φ is given by $s_\varphi((f_{i,j}))=\sum \varphi(f_{i,j})_{i,j}$, where the outer subscript indicates taking the (i,j)-th entry of the matrix.

Now if φ is positive, f is in $\mathfrak{S}^+$, p is in M_n^+, and we set $Q=\varphi(f)$, then

$$s_\varphi(f\otimes p) = \sum p_{i,j}q_{i,j}$$

which is easily recognizable as the sum of the entries of the Schur product $(p_{i,j}q_{i,j})$ of p and Q. Since the Schur product of positive matrices is positive and since the sum of the entries of a positive matrix is positive, we have that $s_\varphi(f\otimes p)\geqq 0$. Hence, if φ is positive, then s_φ is positive on every thing in the convex hull $h(\mathfrak{S}^+\otimes M_n^+)$ of $\{f\otimes p: f\in\mathfrak{S}^+, p\in M_n^+\}$. Thus, when $h(\mathfrak{S}^+\otimes M_n^+)$ is dense in $(\mathfrak{S}\otimes M_n)^+$, s_φ will be positive and consequently, φ will be completely positive.

Conversely, if $h(\mathfrak{S}^+\otimes M_n^+)$ is not dense, then choose a linear functional s which is positive on $h(\mathfrak{S}^+\otimes M_n^+)$ but negative somewhere on $(\mathfrak{S}\otimes M_n)^+$. If $\varphi_s: \mathfrak{S}\to M_n$ is the associated linear map from $\mathfrak{S}$ to M_n, then φ_s is not completely positive. However, if f is in $\mathfrak{S}^+$ and $x=(x_1, x_2, ..., x_n)$ is a vector in $\mathbf{C}^n$, then

$$\langle\varphi_s(f)x, x\rangle = \sum \varphi_s(f)_{i,j}\cdot x_j\bar{x}_i = s(f\otimes p) \geqq 0,$$

where $p=(\bar{x}_i x_j)$ is a matrix in M_n^+. Thus φ is positive but not completely positive.

One advantage of this result is that it allows one to replace problems involving the selection of measures, by problems involving the approximation of matrix-valued functions. As an example of this we present a new proof of the well-known result that every positive map on $C(X)$ is completely positive [23].

Corollary 4.2. *Every positive map* $\varphi: C(X)\to\mathfrak{L}(\mathfrak{H})$ *is completely positive.*

Proof. It is enough by the Proposition to show that p in $\big(C(X)\otimes M_n\big)^+$ can be approximated by a sum $\sum f_i p_i$, when f_i is in $C(X)^+$ and p_i is in M_n^+. To this end fix $\varepsilon>0$ and choose a finite open covering $\{U_i\}$ of X such that for x, y in U_j we have $\|p(x)-p(y)\|<\varepsilon$. Let $f_1, ..., f_n$ be a positive partition of unity of X subordinate to the U_j's. Fix points x_j in U_j, set $p_j=p(x_j)$ and note that

$$\|p-\sum f_i p_j\| < \varepsilon.$$

We remark that if one were interested in the question of when a unital k-contractive (k-positive) map is completely contractive (positive), there is an appropriate analogue of Proposition 4.1. One just replaces contractive by k-contractive and positive by k-positive in (1) to (4) and $h(\mathfrak{S}^+\otimes M_n^+)$ by a bigger set (a set which contains, in particular, all elements of the form, $f\oplus 0$ with f in $(\mathfrak{S}\otimes M_k^+)$). The standard results, such as, the fact that every k-positive map of $C(X)\otimes M_k$ is completely positive can then be proved along the same lines as that for Corollary 4.2. Further analogous results for C^*-algebras having only finite dimensional irreducible representations of bounded dimension can also be obtained.

We turn our attention now to the annulus $A=\{z: R_0 \leqq |z| \leqq R_1\}$, where $0<R_0<R_1$ and let $\mathfrak{S}$ denote the closure of $R(A)+\overline{R(A)}$ in $C(\partial A)$. It was proved in [29] that $\mathfrak{S}$ is a subspace of $C(\partial A)$ of codimension one and, in fact, that $C(\partial A)$ is the span of $\mathfrak{S}$ and $\log |z|$. From these facts, it is clear that $C(\partial A)$ is also the span of $\mathfrak{S}$ and $h(z)$, where we set

$$h(z) = \begin{cases} 1, & |z| = R_1, \\ 0, & |z| = R_0. \end{cases}$$

We also need another characterization of $\mathfrak{S}$. For this we define positive linear functions s_j, $j=0, 1$, on $C(\partial A)$ by

$$s_j(f) = (1/2\pi) \int_0^{2\pi} f(R_j \cdot e^{i\theta})\, d\theta, \quad j = 0, 1.$$

Note that if $f(z)= \sum_{j=-N}^{N} a_j z^j$ is a Laurent polynomial, then $s_0(f)=s_1(f)=a_0$. Thus $s_0(f)=s_1(f)$ for all f in $\mathfrak{S}$. Since $\mathfrak{S}$ has codimension one in $C(\partial A)$, it follows that

$$\mathfrak{S} = \{f \in C(\partial A):\ s_0(f) = s_1(f)\}.$$

Note also that $s_j(h)=j$, $j=0, 1$. We fix these notations for the remainder of this section.

We begin with the negative result.

Theorem 4.3. *There exists a unital contractive map* φ: $R(A) \to M_2$ *which is not completely contractive.*

Proof. By virtue of Proposition 4.1, it is sufficient to construct an F in $(\mathfrak{S} \otimes M_2)^+$ which is not approximable by elements of $h(\mathfrak{S}^+ \otimes M_2^+)$.

We define $F=(f_{i,j})$ as follows:

$$f_{12}(R_1 e^{i\theta}) = f_{21}(R_1 e^{i\theta}) = 0,$$

$$f_{12}(R_0 e^{i\theta}) = f_{21}(R_0 e^{i\theta}) = \begin{cases} 2\theta/\pi, & 0 \leqq \theta \leqq \pi/2, \\ 2(\pi-\theta)/\pi, & \pi/2 \leqq \theta \leqq 3\pi/2, \\ 2(\theta-2\pi)/\pi, & 3\pi/2 \leqq \theta \leqq 2\pi, \end{cases}$$

$$f_{11}(R_0 e^{i\theta}) = f_{22}(R_0 e^{i\theta}) = 1,$$

$$f_{11}(R_1 e^{i\theta}) = f_{22}(R_1 e^{-i\theta}) = \begin{cases} 8\theta/\pi, & 0 \leqq \theta \leqq \pi/2, \\ 8(\pi-\theta)/\pi, & \pi/2 \leqq \theta \leqq \pi, \\ 0, & \pi \leqq \theta \leqq 2\pi. \end{cases}$$

Note that $s_0(f_{i,j})=s_1(f_{i,j})$ and that F is positive at each point so that F is in $(\mathfrak{S}\otimes M_2)^+$.

Fix $\varepsilon>0$ and suppose there exist g_l in $\mathfrak{S}^+$ and P_l in M_2^+, $l=1, 2, \ldots, n$ such that $\|F-\sum_{l=1}^{n} g_l P_l\|<\varepsilon$. Let

$$a_{l,j} = (1/\pi)\int_0^{\pi} g_l(R_j e^{i\theta})\, d\theta, \quad j=0, 1; \quad l = 1, 2, \ldots, n,$$

and let

$$b_{l,j} = (1/\pi)\int_{\pi}^{2\pi} g_l(R_j e^{i\theta})\, d\theta, \quad j=0, 1; \quad l = 1, 2, \ldots, n.$$

Note that $a_{l,0}+b_{l,0}=2s_0(g_l)=2s_1(g_l)=a_{l,1}+b_{l,1}$.

Also let

$$A_j = (1/\pi)\int_0^{\pi} F(R_j e^{i\theta})\, d\theta, \quad B_j = (1/\pi)\int_{\pi}^{2\pi} F(R_j e^{i\theta})\, d\theta, \quad j = 0, 1,$$

so that

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } B_0 = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}.$$

Note that

$$\|A_j-\sum_l a_{l,j}P_l\| < \varepsilon \quad\text{and}\quad \|B_j-\sum_l b_{l,j}P_l\| < \varepsilon, \quad j = 0, 1,$$

since all are integrated over the same intervals.

Let $\sum_p$ denote summation over those indices for which $\operatorname{Re}((P_l)_{12})\geqq 0$. We then have that

$$(\textstyle\sum_p a_{l,1}P_l)_{11} \leqq (\sum a_{l,1}P_l)_{11} < (A_1)_{11}+\varepsilon = 2+\varepsilon$$

and

$$(\textstyle\sum_p a_{l,1}P_l)_{22} \leqq (\sum a_{l,1}P_l)_{22} < (A_1)_{22}+\varepsilon = \varepsilon.$$

Since $\det(\sum_p a_{l,1}P_l)\geqq 0$, we have that

$$\operatorname{Re}(\textstyle\sum_p (a_{l,1}P_l)_{12}) \leqq |(\sum_p a_{l,1}P_l)_{12}| < \sqrt{(2+\varepsilon)\varepsilon}.$$

Similarly, we find that

$$\operatorname{Re}(\textstyle\sum_p (b_{l,1}P_l)_{12}) \leqq |(\sum_p b_{l,1}P_l)_{12}| \leqq \sqrt{(2+\varepsilon)\varepsilon}.$$

However, since $0\leqq a_{l,0}\leqq a_{l,1}+b_{l,1}$, we see that

$$\operatorname{Re}((\textstyle\sum a_{l,0}P_l)_{12}) \leqq \operatorname{Re}((\sum_p a_{l,0}P_l)_{12}) \leqq \operatorname{Re}((\sum_p (a_{l,1}+b_{l,1})P_l)_{12}) < 2\sqrt{(2+\varepsilon)\varepsilon}.$$

Now, it is clear that for ε sufficiently small, this contradicts the fact that

$$\varepsilon > \|A_0-\textstyle\sum a_{l,0}P_l\| \geqq |(A_0-\sum a_{l,0}P_l)_{12}| = |1/2-(\sum a_{l,0}P_l)_{12}|.$$

This contradiction shows that $h(\mathfrak{S}^+ \otimes M_2^+)$ is not dense in $(\mathfrak{S} \otimes M_2)^+$ which completes the proof of the Theorem.

Before continuing let us make a remark. If φ is a unital contractive linear map from $\mathfrak{S}$ to M_2 which is not completely contractive it is clear by Proposition 4.1 that it is not two-contractive so that $\|\varphi_2\| > 1$. It may be true for the annulus algebra $R(A)$ that a unital two contractive linear φ must be completely contractive. Although the above techniques can be used in principle to resolve this problem, we have been unable to do it.

We now turn our attention to a completely positive map from $C(\partial A)$ to $\mathfrak{S}$ related to that given in Lemma 2.1 which in the case of the annulus we can describe explicitly.

Theorem 4.4. *The map* $\varrho\colon C(\partial A) \to \mathfrak{S}$ *defined by*

$$\varrho(g) = [g + s_0(g)h + s_1(g)(1-h)]/2$$

is a unital, completely positive map with range $\mathfrak{S}$.

Proof. The proof is straightforward.

The explicit nature of ϱ allows us to construct dilations, something like the ϱ-dilations of [23]. Note in particular, that $\varrho(z^n) = z^n/2$ for $n \neq 0$.

Theorem 4.5. *Let* $A = \{z\colon R_0 \leqq |z| \leqq R_1\}$ *be a spectral set for* T *in* $\mathfrak{L}(\mathfrak{H})$. *Then there is a Hilbert space* $\mathfrak{K} \supseteqq \mathfrak{H}$ *and a normal operator* N *on* $\mathfrak{K}$ *with* $\sigma(N) \subseteqq \partial A$ *such that*

$$T^n = 2P_{\mathfrak{H}} N^n | \mathfrak{H}, \quad n \neq 0, \quad \text{and} \quad (R_0^2 + R_1^2)/2 = P_{\mathfrak{H}} N^* N | \mathfrak{H}.$$

Moreover, if $\mathfrak{K}$ *is the smallest reducing subspace for* N *containing* $\mathfrak{H}$, *then* N *is unique up to unitary equivalence.*

Proof. Let ψ be the unital contractive homomorphism from $R(A)$ to $\mathfrak{L}(\mathfrak{H})$ defined by $\psi(f) = f(T)$ and $\tilde{\psi}$ the positive extension of ψ from $\mathfrak{S}$ to $\mathfrak{L}(\mathfrak{H})$. If we let $\varphi = \tilde{\psi} \circ \varrho$, then φ is a unital positive map from $C(\partial A)$ to $\mathfrak{L}(\mathfrak{H})$ and hence is a unital completely positive map. Applying Stinespring's Theorem to φ we obtain a Hilbert space $\mathfrak{K} \supseteqq \mathfrak{H}$ and a *-homomorphism $\pi\colon C(\partial A) \to \mathfrak{L}(\mathfrak{H})$ such that $\varphi(\cdot) = P_{\mathfrak{H}} \pi(\cdot) | \mathfrak{H}$. Let $N = \pi(z)$ so that N is normal, $\sigma(N) \subseteqq \partial A$, and

$$2P_{\mathfrak{H}} N^n | \mathfrak{H} = 2\tilde{\psi} \circ \varrho(z^n) = \psi(z^n) = T^n.$$

Also since $|z|^2 = (R_1^2 - R_0^2)h(z) + R_0^2$ we have that

$$P_{\mathfrak{H}} N^* N | \mathfrak{H} = \tilde{\psi} \circ \varrho((R_1^2 - R_0^2)h(z) + R_0^2) = (R_0^2 + R_1^2)/2.$$

Finally, the uniqueness statement comes from the uniqueness of a minimal Stinespring representation.

Note added in proof. Recently Jim Agler in "Rational Dilation on an Annulus" (Ann. Math., 121 (1985), 537—564) has proven that every operator for which the annulus A is a spectral set, possesses a ∂A-normal dilation.

References

[1] M. B. Abrahamse and R. G. Douglas, Operators on multiply-connected domains, *Proc. Royal Irish Acad,* 74A (1974), 135—141.

[2] T. Ando, On a pair of commutative contractions, *Acta Sci. Math.*, 24 (1963), 88—90.

[3] W. B. Arveson, Subalgebras of C^*-algebras, *Acta Math.*, 123 (1969), 141—224.

[4] C. A. Berger, *Normal dilations*, Doctoral dissertation, Cornell University, 1963.

[5] M.-D. Choi, Positive linear maps, in: *Operator algebras and applications*, Proc. Symp. Pure Math., Vol. 38, Part II, Amer. Math. Soc. (Providence, R. I., 1982); pp. 583—590.

[6] S. R. Foguel, A counterexample to a problem of Sz.-Nagy, *Proc. Amer. Math. Soc.*, 15 (1964), 788—790.

[7] C. Foiaş, Certaines applications des ensembles spectraux. I. Mesure harmonique-spectrale, *Stud. Cerc. Mat.*, 10 (1959), 365—401.

[8] T. W. Gamelin, *Uniform Algebras*, Prentice Hall (Englewood Cliffs, N. J., 1969).

[9] U. Haagerup, Solution of the similarity problem for cyclic representations of C^*-algebras, *Ann. Math.*, 118 (1983), 215—240.

[10] D. Hadwin, Dilations and Hahn decompositions for linear maps, *Canad. J. Math.*, 33 (1981), 826—839.

[11] P. R. Halmos, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.*, 70 (1970), 887—933.

[12] C. Lance, On nuclear C^*-algebras, *J. Functional Analysis*, 12 (1973), 157—176.

[13] A. Lebow, On von Neumann's theory of spectral sets, *J. Math. Anal. Appl.* 7 (1963), 64—90.

[14] G. Misra, Curvature inequalities and extremal properties of bundle shifts, *J. Operator Theory*, 11 (1984), 305—318.

[15] W. Mlak, Decompositions and extensions of operator valued representations of function algebras, *Acta Sci. Math.*, 30 (1969), 181—193.

[16] J. von Neumann, Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes, *Math. Nachr.*, 4 (1931), 258—281.

[17]. S. Parrott, Unitary dilations for commuting contractions, *Pacific J. Math.*, 34 (1973), 481—490.

[18] V. I. Paulsen, Every completely polynomially bounded operator is similar to a contraction, *J. Functional Analysis*, 55 (1984), 1—17.

[19] V. I. Paulsen, Completely bounded homomorphisms of operator algebras, *Proc. Amer. Math. Soc.*, 92 (1984), 225—228.

[20] S. Sakai, *C^*-algebras and W^*-algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60, Springer-Verlag (Berlin, 1971).

[21] D. E. Sarason, On spectral sets having connected complement, *Acta Sci. Math.*, 26 (1965), 289—299.

[22] R. Smith and J. Ward, Matrix ranges for Hilbert space operators, *Amer. J. Math.*, 102 (1980), 621—631.

[23] W. F. Stinespring, Positive functions on C^*-algebras *Proc. Amer. Math. Soc.* 6 (1955), 211—216.

[24] B. Sz.-Nagy, On uniformly bounded linear transformations in Hilbert space, *Acta Sci. Math.*, **11** (1947), 152—157.
[25] B. Sz.-Nagy, Sur les contractions de l'espace de Hilbert, *Acta Sci. Math.*, **15** (1953), 87—92.
[26] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, American Elsevier (New York, 1970).
[27] M. Takesaki, *Theory of Operator Algebras. I*, Springer-Verlag (New York, 1979).
[28] N. Th. Varopoulos, On an inequality of von Neumann and application of the metric theory of tensor products to operators theory, *J. Functional Analysis*, **16** (1974), 83—100.
[29] J. L. Walsh, The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions, *Bull. Amer. Math. Soc.*, **35** (1929), 499—544.
[30] J. P. Williams, Minimal spectral sets of compact operators, *Acta Sci. Math.*, **28** (1967), 93—106.
[31] G. Wittstock, Ein operatorwertiger Hahn—Banach Satz, *J. Functional Analysis*, **40** (1981), 127—150.
[32] G. Wittstock, *Extensions of completely bounded C*-module homomorphisms*, Monographs and Studies in Math., Vol. 18, Pitman (Boston, 1984); pp. 238—250.
[33] A. Shields, Weighted shift operators and analytic function theory, in: *Topics in operator theory*, Mathematical Surveys, Vol. 13, Amer. Math. Soc. (Providence, R. I., 1974); pp. 49—128.

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