Acta Sci. Math., 50 (1986), 133—142

Error bounds for certain classes of quintic splines

S. SALLAM

1. Introduction. Interpolation of functions by quintic splines has been extensively studied by many authors (see e.g. [5], [6] and [7]). Recently, SALLAM [4] has presented some new types of quintic splines where an analysis of their corresponding error bounds in L^2-norm was presented. Our object is to continue this study and obtain error bounds for these new types of interpolatory splines in L^∞-norm.

In Section 2, we developed some preliminary formulas of interpolatory quintic splines under different continuity requirements and different given data together with a lacunary interpolation by quintic splines of certain functions without using function values. Section 3 is devoted to studying various convergence results of the presented interpolatory splines, namely the error bounds in L^∞-norm, and it is shown that the order of convergence remains the same as if function values are considered.

2. Construction of some quintic splines. Let

$$\Delta\colon\ 0 = x_0 < x_1 < \ldots < x_{N+1} = 1$$

denote a partition of $I=[0, 1]$. Denote by $\mathcal{S}^{(l)}_{N,5}$, $l=2, 3$, the class of quintic splines $q_s(x)$ such that:

(i) $q_s(x)\in C^l(I)$;

(ii) $q_s(x)$ is a quintic in each $[x_i, x_{i+1}]$, $i=0(1)N$. We set $f^{(k)}(x_j)=f_l^k$ and $q_s^{(k)}(x_j)=q_j^k$ stand for $D^k f(x_j)$ and $D^k q_s(x_j)$, respectively. Further define $h_i= =x_{i+1}-x_i$ and $h=\max_i h_i$.

We now discuss the possibility of interpolation of some given function by elements in $\mathcal{S}^{(l)}_{N,5}$. It is well known that the following procedures are well defined according to Theorems 1 and 2 (cf. [4]).

Received February 27, 1984.

Definition 1. Given numbers f_i, f_i^2, $i=0(1)N+1$ and f_i^1, $i=0, N+1$; there exists a unique quintic spline $\hat{q}_s(x)\in\mathcal{S}_{N,5}^{(3)}$ such that

$$\begin{aligned} &\hat{q}_s(x_i) = f_i, \\ (2.1)\qquad &\hat{q}_s^{(2)}(x_i) = f_i^2, \quad i = 0(1)N+1, \\ &\hat{q}_s^{(1)}(x_i) = f_i^1, \quad i = 0, N+1, \end{aligned}$$

provided that $h_i > h_{i-1}$, for all i.

Definition 2. Given numbers f_i^1, f_i^2, $i=0(1)N+1$ and f_i, $i=0, N+1$, there exists a unique quintic spline $\overset{*}{q}_s(x)\in\mathcal{S}_{N,5}^{(3)}$ such that

$$\begin{aligned} &\overset{*}{q}_s^{(1)}(x) = f_i^1, \\ (2.2)\qquad &\overset{*}{q}_s^{(2)}(x) = f_i^2, \quad i = 0(1)N+1, \\ &\overset{*}{q}_s(x_i) = f_i, \quad i = 0, N+1. \end{aligned}$$

We now turn to prove the following.

Theorem 1. *Given numbers* f_i, f_i^3, $i=0(1)N+1$ *and* f_i^2, $i=0, N+1$, *there exists a unique quintic spline* $\bar{q}_s(x)\in\mathcal{S}_{N,5}^{(3)}$ *such that*

$$\begin{aligned} &\bar{q}_s(x_i) = f_i, \\ (2.3)\qquad &\bar{q}_s^{(3)}(x_i) = f_i^3, \quad i = 0(1)N+1, \\ &\bar{q}_s^{(2)}(x_i) = f_i^2, \quad i = 0, N+1. \end{aligned}$$

Theorem 2. *Given numbers* f_i^2, f_i^3, $i=0(1)N+1$ *and* f_i, $i=0, N+1$, *there exists a unique quintic spline* $\tilde{q}_s(x)\in\mathcal{S}_{N,5}^{(3)}$ *such that*

$$\begin{aligned} &\tilde{q}_s^{(2)}(x_i) = f_i^2, \\ &\tilde{q}_s^{(3)}(x_i) = f_i^3, \quad i = 0(1)N+1, \\ &\tilde{q}_s(x_i) = f_i, \quad i = 0, N+1. \end{aligned}$$

To prove the above theorems, we need the following well-known lemma (cf. [2]).

Lemma 1. *If* $p(x)$ *is a quintic on* $[0, 1]$, *then*

$$(2.5)\qquad \begin{aligned} p(x) = {} & p(0)B_0(1-x)+p(1)B_0(x)+p^{(2)}(0)B_1(1-x)+p^{(2)}(1)B_1(x)- \\ & -p^{(3)}(0)B_2(1-x)+p^{(3)}(1)B_2(x) \end{aligned}$$

where

$$(2.6)\qquad \begin{aligned} &B_0(x) = x, \quad B_1(x) = (x^4-x^5)/10+(3/20)(x^4-x), \\ &B_2(x) = (x^5-x^4)/20+(x-x^4)/30. \end{aligned}$$

Proof of Theorem 1. In order to establish Theorem 1, one can express $\bar{q}_s(x)$ in $[x_i, x_{i+1}]$ using (2.5) and $x=x_i+th_i$, $0\leqq t\leqq 1$, in the following form

$$\bar{q}_s(x) = f_i B_0(1-t)+f_{i+1}B_0(t)+h_i^2\bar{q}_i^2 B_1(1-t)+h_i^2\bar{q}_{i+1}^2 B_1(t)- -h_i^3 f_i^3 B_2(1-t)+h_i^3 f_{i+1}^3 B_2(t). \tag{2.7}$$

Similarly, in $[x_{i-1}, x_i]$ $\bar{q}_s(x)$ may be represented as

$$\bar{q}_s(x) = f_{i-1}B_0(1-t)+f_i B_0(t)+h_{i-1}^2\bar{q}_{i-1}^2 B_1(1-t)+h_{i-1}^2\bar{q}_i^2 B_1(t)- -h_{i-1}^3 f_{i-1}^3 B_2(1-t)+h_{i-1}^3 f_i^3 B_2(t). \tag{2.8}$$

Straightforward calculation shows that $\bar{q}_s(x)\in C^3(I)$ is equivalent to

(2.9)

$$(3/10)h_{i-1}\bar{q}_{i-1}^2+(7/10)(h_{i-1}+h_i)\bar{q}_i^2+(3/10)h_i\bar{q}_{i+1}^2 = 2h_{i-1}^{-1}f_{i-1}-2(h_{i-1}^{-1}+h_i^{-1})f_i+ +2h_i^{-1}f_{i+1}-(1/15)h_{i-1}^2 f_{i-1}^3-(1/10)(h_i^2-h_{i-1}^2)f_i^3+(1/15)h_i^2 f_{i+1}^3, \quad i = 1(1)N.$$

In matrix form (2.9) can be written as $A\bar{\mathbf{q}}=\mathbf{b}$, where $A=(a_{ij})$ with

$$a_{ij} = \begin{cases} v_i = (3/7)h_i/(h_i+h_{i-1}), & i<j, \\ 1, & i=j, \\ \mu_i = (3/7)h_{i-1}/(h_i+h_{i-1}), & i>j. \end{cases} \tag{2.10}$$

Clearly A is a diagonally dominant matrix and hence the quintic spline is determined uniquely. In the case of uniform distribution, (2.9) becomes

$$(3/10)\bar{q}_{i-1}^2+(14/10)\bar{q}_i^2+(3/10)\bar{q}_{i+1}^2 = 2h^{-2}(f_{i-1}-2f_i+f_{i+1})-(h/15)(f_{i-1}^3-f_{i+1}^3).$$

Proof of Theorem 2. Similar to that of the previous theorem, the system (2.5) is now replaced by

(2.11)

$$-h_{i-1}^{-1}\tilde{q}_{i-1}+(h_{i-1}^{-1}+h_i^{-1})\tilde{q}_i-h_i^{-1}\tilde{q}_{i+1} = -(3/20)h_{i-1}f_{i-1}^2-(7/20)(h_{i-1}+h_i)f_i^2- -(3/20)h_i f_{i+1}^2-(1/30)h_{i-1}^2 f_{i-1}^3+(1/20)(h_{i-1}^2-h_i^2)f_i^3+(1/30)h_i^2 f_{i+1}^3, \quad i = 1(1)N.$$

Again the matrix of coefficients is diagonally dominant. In the case of a uniform partition, (2.10) will take the form

$$\tilde{q}_{i-1}-2\tilde{q}_i+\tilde{q}_{i+1} = (3/20)h^2(f_{i-1}^2+(14/3)f_i^2+f_{i+1}^2)+(h^3/30)(f_{i-1}^3-f_{i+1}^3).$$

We now proceed to develop a new lacunary quintic spline, which interpolates the lacunary data (function values and third derivatives) midway between the knots and first derivatives at the knots. The problem is stated in the following theorem.

Theorem 3. *Given arbitrary numbers* f_i^1, $i=0(1)N+1$; $f^{(p)}(z_i)$, $i=0(1)N$, $p=0, 3$, $z_i=(1/2)(x_i+x_{i+1})$ *and* f_i, $i=0, N+1$, *then there exists a unique quintic* *spline* $Q(x) \in \mathcal{S}^{(2)}_{N,5}$, *such that*

$$Q^{(1)}(x_i) = f_i^1, \quad i = 0(1)N+1,$$

(2.12)
$$Q^{(p)}(z_i) = f^{(p)}(z_i), \quad i = 0(1)N; \quad p = 0,3,$$

$$Q(x_i) = f_i, \quad i = 0, N+1,$$

provided that $h_i > h_{i-1}$ *for all* i.

In order to establish Theorem 3, we need the following lemma (see [3]).

Lemma 2. *If* $p(x)$ *is quintic on* $[0, 1]$, *then*

(2.13)
$$p(x) = p(0)A_0(x) + p(1/2)A_1(x) + p(1)A_2(x) + p^{(1)}(0)A_3(x) + \\ + p^{(1)}_{(1)}A_4(x) + p^{(3)}(1/2)A_5(x)$$

where

$$A_0(x) = 4x^5 - 18x^4 + 26x^3 - 13x^2 + 1, \quad A_1(x) = 16(x^4 - 2x^3 + x^2),$$
$$A_2(x) = A_0(1-x), \quad A_3(x) = 2x^5 - 7x^4 + 9x^3 - 5x^2 + x,$$
$$A_4(x) = -A_3(1-x), \quad A_5(x) = (1/6)(-2x^5 + 5x^4 - 4x^3 + x^2).$$

Proof of Theorem 3. In $[x_i, x_{i+1}]$ $Q(x)$ can be expressed as

(2.14)
$$Q(x) = Q_iA_0(t) + f(z_i)A_1(t) + Q_{i+1}A_2(t) + h_if_i^1A_3(t) + h_if_{i+1}^1A_4(t) + h_i^3f^{(3)}(z_i)A_5(t).$$

Similarly in $[x_{i-1}, x_i]$ $Q(x)$ will be

(2.15)
$$Q(x) = Q_{i-1}A_0(t) + f(z_{i-1})A_1(t) + Q_iA_2(t) + h_{i-1}f_{i-1}^1A_3(t) + \\ + h_{i-1}f_i^1A_4(t) + h_{i-1}^3f^{(3)}(z_{i-1})A_5(t).$$

In order that $Q(x) \in C^2(I)$, we obtain

(2.16)
$$6h_{i-1}^{-2}Q_{i-1} + 26(h_{i-1}^{-2} + h_i^{-2})Q_i - 6h_i^{-2}Q_{i+1} = 32(h_{i-1}^{-2}f(z_{i-1}) - \\ - h_i^{-2}f(z_i)) + 10(h_i^{-1} + h_{i-1}^{-1})f_i^1 + (1/3)(h_if^{(3)}(z_i) - h_{i-1}f^{(3)}(z_{i-1})).$$

In matrix form (2.16) can be written as $A\mathbf{Q} = \mathbf{b}$, where $A = (a_{ij})$ and

(2.17)
$$a_{ij} = \begin{cases} \nu_i = -(3/13)h_{i-1}^2/(h_i^2 - h_{i-1}^2), & i < j, \\ 1, & i = j, \\ \mu_i = (3/13)h_i^2/(h_i^2 - h_{i-1}^2), & i > j. \end{cases}$$

It is clear that $\nu_i < 0$ if $h_i > h_{i-1}$ for all i and hence A is nonsingular [1]. Thus $Q(x)$ is uniquely determined.

If the nodes are uniformly distributed on I, then (2.16) reduces to

$$Q_{i-1} - Q_{i+1} = (16/3)(f(z_{i-1}) - f(z_i)) + (10/3)hf_i^1 - (h^3/18)(f^{(3)}(z_{i-1}) - f^{(3)}(z_i)).$$

The coefficient matrix is skew-symmetric and hence nonsingular if N is even.

3. Error bounds. The purpose of this section is to obtain error estimates for hese new types of interpolatory quintic and interpolation by lacunary quintic splines which introduced in [4] and in Section 2, in L^∞-norm. We follow the idea of Prasad and Varma [3].

Throughout $K_{r,l}$, $K'_{r,l}$, etc. denote generic constants independent of the functions considered and maximum mesh spacing h. However, these constants in general do depend in particular upon the order of various derivatives.

In the sequel we treat, for the sake of brevity, only estimates for error bounds of the quintic splines $\bar{q}_s(x)$ and $Q(x)$, respectively. Error bounds for other types can be handled in a similar manner. We begin with the following theorem.

Theorem 4. *Let Δ be an arbitrary partition of I. If $f\in C^l(I)$, $l=3(1)6$, then, for the unique quintic spline $\bar{q}_s(x)$ associated with f and satisfying* (2.3), *we have*

(3.1)
$$|\bar{q}_s^{(r)}(x)-f^{(r)}(x)| \leqq K_{r,l}h^{l-r}\|f^{(l)}\|_\infty+K'_{r,l}h^{l-r}\omega(f^{(l)}, h), \quad r = 0(1)3; \quad l = 3(1)5,$$

(3.2)
$$|\bar{q}_s^{(r)}(x)-f^{(r)}(x)| \leqq K_{r,6}h^{6-r}\|f^{(6)}\|_\infty, \quad r = 0(1)3; \quad l = 6,$$

where $\omega(f^{(l)}, h)$ denotes the modulus of continuity of $f^{(l)}$.

We shall prove Theorem 4 for $l=6$, the proof needs the following lemma.

Lemma 3. *Let $f\in C^6(I)$, then*

(3.3)
$$|\bar{q}_s^{(2)}(x_i)-f^{(2)}(x_i)| \leqq K_{2,6}h^4\|f^{(6)}\|_\infty.$$

Proof. The condition that $\bar{q}_s(x)\in C^3(I)$ is equivalent to the system (2.9). Using Taylor's formula, it can be shown that

(3.4)
$$(3/10)h_{i-1}f_{i-1}^2+(7/10)(h_{i-1}+h_i)f_i^2+(3/10)h_i f_{i+1}^2 = 2h_{i-1}^{-1}f_{i-1}-2(h_{i-1}^{-1}+h_i^{-1})f_i+$$
$$+2h_i^{-1}f_{i+1}-(1/15)h_{i-1}^2f_{i-1}^3-(1/10)(h_i^2-h_{i-1}^2)f_i^3+(1/15)h_i^2 f_{i+1}-$$
$$-(1/720)(h_i^5+h_{i-1}^5)f^{(6)}(\xi_i), \quad \xi_i\in(x_{i-1}, x_{i+1}), \quad i = 1(1)N.$$

From (2.9) and (3.4), we deduce that

$$(3/10)h_{i-1}(\bar{q}_{i-1}^2-f_{i-1}^2)+(7/10)(h_i+h_{i-1})(\bar{q}_i^2-f_i^2)+(3/10)h_i(\bar{q}_{i+1}^2-f_{i+1}^2) =$$
$$= (1/720)(h_{i-1}^5+h_i^5)f^{(6)}(\xi_i)$$

which can be written as $Ae=\mathbf{z}$ and the entries of A are given by (2.10), $e_i^2=\bar{q}_i^2-f_i^2$, $i=1(1)N$ and

$$z_i = (1/504)\big((h_{i-1}^5+h_i^5)/(h_{i-1}+h_i)\big)f^{(6)}(\xi_i).$$

Let $A=I+B$ where $\|B\|_\infty=3/7$, we have $e^2=(I+B)^{-1}z$. Hence $\|e^2\|_\infty \leqq \leqq (7/4)\|z\|_\infty$ i.e.,

$$\|e^2\|_\infty \leqq K_{2,6}h^4\|f^{(6)}\|_\infty,$$

where $K_{2,6}=1/288$. This completes the proof of Lemma 3.

It can be easily seen that the following identities are valid (cf. (2.6)):

$$\begin{aligned} &B_0(1-t)+B_0(t) = 1,\\ &B_0(t)+2B_1(1-t)+2B_1(t) = t^2,\\ &B_0(t)+6B_1(t)-6B_2(1-t)+6B_2(t) = t^3,\\ &B_0(t)+12B_1(t)+24B_2(t) = t^4,\\ &B_0(t)+20B_1(t)+60B_2(t) = t^5. \end{aligned} \tag{3.5}$$

Proof of Theorem 4. From (2.7), it follows that

$$h_i^3\bar{q}_s^{(3)}(x) = -h_i^2\bar{q}_i^2B_1'''(1-t)+h_i^2\bar{q}_{i+1}^2B_1'''(t)+h_i^3f_i^3B_2'''(1-t)+h_i^3f_{i+1}^3B_2'''(t), \tag{3.6}$$

where $B'''(z)=d^3B/dz^3$, which can be written as

$$h_i^3\bar{q}^{(3)}(x) = \lambda_i(t)+\mu_i(t) \tag{3.7}$$

where

$$\lambda_i(t) = -h_i^2(\bar{q}_i^2-f_i^2)B_1'''(1-t)+h_i^2(\bar{q}_{i+1}^2-f_{i+1}^2)B_1'''(t)$$

and

$$\mu_i(t) = -h_i^2f_i^2B_1'''(1-t)+h_i^2f_{i+1}^2B_1'''(t)+h_i^3f_i^2B_2'''(1-t)+h_i^3f_{i+1}^3B_2'''(t).$$

On using Lemma 3, we obtain

$$|\lambda_i(t)| \leqq K_{2,6}ah^4\|f^{(6)}\|_\infty, \tag{3.8}$$

where $a=\max_{0\leqq z\leqq 1}|B_1'''(z)|$.

Using Taylor's expansion and the identities (3.5) together with

$$f^{(3)}(x) = \sum_{k=3}^{5}\left(f^{(k)}(x_i)/(k-3)!\right)(th_i)^{k-3}+\left(f^{(6)}(\eta_1)/5!\right)(th_i)^3, \tag{3.9}$$

it is not difficult to show that

(3.10)

$$\mu_i(t) = h_i^3f_i^3+h_i^4tf_i^4+(h_i^5/2)t^2f_i^5+(h_i^6/6!)[30f^{(6)}(\eta_2)B_1'''(t)+120f^{(6)}(\eta_3)B_2'''(t)].$$

Thus, (3.9) and (3.10) yield

$$\mu_i(t) = h_i^3f^3(x)+(h_i^6/6!)\nu_i(t), \tag{3.11}$$

where

$$\nu_i(t) = 30f^{(6)}(\eta_2)B_1'''(t)+120f^{(6)}(\eta_3)B_2'''(t)-120t^3f^{(6)}(\eta_1). \tag{3.12}$$

Now combining (3.7) with (3.12), gives

$$h_i^3(\bar{q}_s^{(3)}(x)-f^{(3)}(x)) = \lambda_i(t)+(h_i^6/6!)\,v_i(t). \tag{3.13}$$

It is clear from (3.12) that

$$|v_i(t)| \leqq \{30|B_1'''(t)|+120|B_2'''(t)|+120|t^5|\} \max_{x\in I} |f^{(6)}(x)| \leqq C\|f^{(6)}\|_\infty. \tag{3.14}$$

Hence (3.13), (3.14) and (3.8) give

$$|\bar{q}_s^{(3)}(x)-f^{(3)}(x)| \leqq K_{3,6}h^3\|f^{(6)}\|_\infty.$$

This proves (3.2) for $r=3$. To prove (3.2) for $r=2$, observe that for $x_i \leqq x < x_{i+1}$ and using Lemma 3, it follows that

$$|\bar{q}_s^{(2)}(x)-f^{(2)}(x)| = \Big|\int_{x_i}^{x} (q_s^{(3)}(t)-f^{(3)}(t))\,dt+q_i^2-f_i^2\Big| \leqq K_{2,6}h^4\|f^{(6)}\|_\infty.$$

Since $\bar{q}_s(x)-f(x)$ vanishes at x_i and x_{i+1}, there exists a point ξ_i, $\xi_i\in(x_i, x_{i+1})$, such that $\bar{q}_s^{(1)}(\xi_i)=f^{(1)}(\xi_i)$. Hence

$$|\bar{q}_s^{(1)}(x)-f^{(1)}(x)| \leqq \int_{\xi_i}^{x} |\bar{q}_s^{(2)}(t)-f^{(2)}(t)|\,dt \leqq K_{1,6}h^5\|f^{(6)}\|_\infty.$$

This proves (3.2) for $r=1$. Formula (3.2), for $r=0$, follows immediately by using a similar argument. For $l=3(1)5$ the proof is analogous to that of Theorem 5.

We now turn to the derivation of error bounds for the interpolation error, $Q(x)-f(x)$, and its derivatives. We begin with the following main result.

Theorem 5. *Let $f\in C^l(I)$ and $h_i>\sqrt{8/5}\,h_{i-1}$ for all i or the partition be uniform. Then for the unique quintic spline $Q(x)$ associated with f and satisfying* (2.12), *we have*

$$|Q^{(r)}(x)-f^{(r)}(x)| \leqq K_{r,l}h^{l-r}\omega(f^{(l)}, h), \quad r=0(1)2; \quad l=3(1)5, \tag{3.15}$$

$$|Q^{(r)}(x)-f^{(r)}(x)| \leqq K_{r,6}h^{6-r}\|f^{(6)}\|_\infty+h^{6-r}\omega(f^{(6)}, h), \quad r=0(1)2; \quad l=6. \tag{3.16}$$

To prove Theorem 5, we need the following lemma

Lemma 4. *Let $f\in C^l(I)$ and $h_i>\sqrt{8/5}\,h_{i-1}$ for all i, then*

$$|Q(x_i)-f(x_i)| \leqq K_l(h_i^{l-1}h_{i-1}^{l-1}/(h_i^{l-2}-h_{i-1}^{l-2}))\,\omega(f^{(l)}, h), \quad l=3(1)5, \tag{3.17}$$

$$|Q(x_i)-f(x_i)| \leqq K_6h_i^2h_{i-1}^2(h_i^2+h_{i-1}^2)\|f^{(6)}\|_\infty, \quad l=6. \tag{3.18}$$

Proof. We will prove the lemma for $l=3$. The condition that $Q(x)\in C^2(I)$ is equivalent to the system (2.16). Using Taylor's expansion, it is easy to show that

$$(3.19)\qquad 6h_{i-1}^{-2}(Q_{i-1}-f_{i-1})+26(h_{i-1}^{-2}-h_i^{-2})(Q_i-f_i)-6h_i^{-2}(Q_{i+1}-f_{i+1}) =$$
$$= (2/3)h_i\big(f^{(3)}(\eta_{1,i})-f^{(3)}(\eta_{3,i})\big)+(2/3)h_{i-1}\big(f^{(3)}(\eta_{2,i})-f^{(3)}(\eta_{4,i})\big)+$$
$$+(h_i/3)\big(f^{(3)}(\eta_{1,i})-f^{(3)}(z_i)\big)+(h_{i-1}/3)\big(f^{(3)}(\eta_{2,i})-f^{(3)}(z_{i-1})\big),$$

where $x_{i-1}<\eta_{2,i}<x_i$, $x_i<\eta_{1,i}<x_{i+1}$, $x_i<\eta_{3,i}<z_i$, $z_{i-1}<\eta_{4,i}<x_i$.

In matrix form (3.19) can be written as $A\mathbf{e}=\mathbf{z}$ with $e_i=Q(x_i)-f(x_i)$, $i=1(1)N$. Multiplying $A\mathbf{e}=\mathbf{z}$ by the diagonal matrix $D=(d_{ii})$, $d_{ii}=(1/26)h_i^2h_{i-1}^2/(h_i^2-h_{i-1}^2)$, the matrix DA will be $I+B$ with $\|B\|_\infty<1$ if $h_i>\sqrt{8/5}\,h_{i-1}$ for all i. Since

$$\|\mathbf{e}\| \leqq \|(I+B)^{-1}\|_\infty\|D\mathbf{z}\|_\infty \leqq \big(1/(1-\|B\|_\infty)\big)\|D\mathbf{z}\|_\infty.$$

It follows that

$$|Q(x_i)-f(x_i)| \leqq K_3\big(h_i^2h_{i-1}^2/(h_i-h_{i-1})\big)\omega(f^{(3)}, h)$$

with $K_3=5/13$. This proves (3.17) for $l=3$. The proof is similar in the other cases.

For equally spaced knots and N is even Lemma 4 will be modified as follows.

Lemma 5. *Let $f\in C^l(I)$, then*

$$(3.20)\qquad |Q(x_i)-f(x_i)| \leqq K_lh^l\omega(f^{(l)}, h),\quad l = 3(1)5.$$

Proof. We prove the lemma for $l=3$. The condition that $Q(x)\in C^2(I)$ and the partition be uniform is equivalent to (cf. (3.19))

$$(Q_{i-1}-f_{i-1})-(Q_{i+1}-f_{i+1}) = (h^3/9)\big(f^{(3)}(\eta_{1,i})-f^{(3)}(\eta_{3,i})\big)+$$
$$+(h^3/9)\big(f^{(3)}(\eta_{2,i})-f^{(3)}(\eta_{4,i})\big)+(h^3/18)\big(f^{(3)}(\eta_{1,i})-f^{(3)}(z_i)\big)+$$
$$+(h^3/9)\big(f^{(3)}(\eta_{2,i})-f^{(3)}(z_{i-1})\big).$$

Or, in the matrix form $M\mathbf{e}=\mathbf{z}$, where $M=(m_{ij})$ with

$$m_{ij} = \begin{cases} -1, & i<j \\ 0, & i=j \\ 1, & i>j. \end{cases}$$

Since $\|Mx\|_\infty\geqq 1$ for $\|x\|_\infty=1$, it follows that $\|M^{-1}\|_\infty\leqq 1$. Hence

$$\|\mathbf{e}\|_\infty \leqq \|M^{-1}\|_\infty\|\mathbf{z}\|_\infty \leqq K_3h^3\omega(f^{(3)}, h).$$

This completes the proof of Lemma 5.

It is well known that the following identities are valid (see [3]).

$$A_0(t)+A_1(t)+A_2(t) = 1,$$
$$A_1(t)+2A_2(t)+2A_3(t)+2A_4(t) = 2t,$$
$$(1/4)A_1(t)+A_2(t)+2A_4(t) = t^2,$$
$$(1/8)A_1(t)+A_2(t)+3A_4(t)+6A_5(t) = t^3,$$
$$(1/16)A_1(t)+A_2(t)+4A_4(t)+12A_5(t) = t^4.$$

Proof of Theorem 5. The proof will be carried out only for $l=3$. From (2.14), it follows that

$$(3.21)\qquad h_i^2Q^{(2)}(x) = Q_iA_0''(t)+f(z_i)A_1''(t)+Q_{i+1}A_2''(t)+h_if_i^1A_3''(t)+ +h_if_{i+1}^1A_4''(t)+h_i^3f^{(3)}(z_i)A_5''(t)$$

which can be written as

$$(3.22)\qquad h_i^2Q^{(2)}(x) = \lambda_i(t)+\mu_i(t),$$

where

$$\lambda_i(t) = \big(Q(x_i)-f(x_i)\big)A_0''(t)+\big(Q(x_{i+1})-f(x_{i+1})\big)A_2''(t),$$

and

$$\mu_i(t) = f(x_i)A_0''(t)+f(x_{i+1})A_2''(t)+f(z_i)A_1''(t)+h_if^{(1)}(x_i)A_3''(t)+ +h_if^{(1)}(x_{i+1})A_4''(t)+h_i^3f^{(3)}(z_i)A_5''(t).$$

Using Lemma 4, for $l=3$, we obtain

$$(3.23)\qquad |\lambda_i(t)| \leqq K_3ah_i^2\left\{\frac{h_{i-1}^2}{h_i-h_{i-1}}+\frac{h_{i+1}^2}{h_{i+1}-h_i}\right\}\omega(f^{(3)}, h)$$

where $a=\max\{l_0, l_2\}$, $l_r=\max\limits_{0\leqq t\leqq 1}|A_r''(t)|$.

Also it can be easily seen that

$$(3.24)\qquad \mu_i(t) = h_i^2f^{(2)}(x)+(h_i^3/3!)\nu_i(t),$$

where

$$\nu_i(t) = (1/8)\big(f^{(3)}(\eta_{2,i})-f^{(3)}(z_i)\big)A_1''(t)+\big(f^{(3)}(\eta_{1,i})-f^{(3)}(\eta_{6,i})\big)A_2''(t)+ +3\big(f^{(3)}(\eta_{3,i})-f^{(3)}(\eta_{6,i})\big)A_4''(t)+6\big(f^{(3)}(z_i)-f^{(3)}(\eta_{6,i})\big)A_5''(t).$$

Consequently

$$(3.25)\qquad |\nu_i(t)| \leqq a^*\omega(f^{(3)}, h),$$

where $a^*=\max\{l_1, l_2, l_4, l_5\}$.

Combining (3.22) and (3.24), we have

$$h_i^2\big(Q^{(2)}(x)-f^{(2)}(x)\big) = \lambda_i(t)+(h_i^3/3!)\nu_i(t).$$

Thus

$$(3.26)\quad |Q^{(2)}(x)-f^{(2)}(x)| \leqq \left\{K_3 a\left(\frac{h_{i-1}^2}{h_i-h_{i-1}}+\frac{h_{i+1}^2}{h_{i+1}-h_i}\right)+\frac{a^*}{3!}h_i\right\}\omega(f^{(3)}, h) \leqq$$
$$\leqq K_{2,3} h\omega(f^{(3)}, h).$$

This proves (3.15) for $r=2$. The proof for $r=1$ and $r=0$ follows immediately using the fact that $Q^{(1)}(x)-f^{(1)}(x)$ vanishes at x_i and x_{i+1} and that $Q(x)-f(x)$ vanishes at $x=z_{i-1}$ and z_i.

Remark. It is worth noting that even in the absence of the function values at the mesh points, it is possible to construct quintic and lacunary quintic splines, which, for some cases, requires certain partition restrictions. Also the order of convergence achieved is almost the same when using function values as given data. We believe that this observed phenomenon may have some applications in physics and engineering problems as well.

References

[1] A. Bleyer and S. Sallam, Interpolation by cubic splines, *Periodica Poly.*, **22** (1978), 91—105.

[2] A. Meir and A. Sharma, Lacunary interpolation by splines, *SIAM J. Numer. Anal.*, **10** (1973), 423—442.

[3] J. Prasad and A. K. Varma, Lacunary interpolation by quintic splines, *SIAM J. Numer. Anal.*, **16** (1978), 1075—1079.

[4] S. Sallam, On interpolation by quintic splines, *Bull. Fac. Sci., Assiut University*, **11** (1982), 97—106.

[5] A. Sharma, Some poised and nonpoised problems of interpolation, *SIAM Rev.*, **14** (1972), 129—151.

[6] B. K. Swartz and R. Varga, A note on lacunary interpolation by splines, *SIAM J. Numer. Anal.*, **10** (1973), 443—447.

[7] B. K. Swartz and R. Varga, Error bounds for spline and *L*-spline interpolation, *J. Approx. Theory*, **6** (1972), 6—49.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KUWAIT
P. O. BOX 5969
KUWAIT