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On norms of projections

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Let $(X, \|\cdot\|)$ be a normed space. A continuous linear mapping $P: X\to X$ is said to be a projection if $P^2=P$. As usual, the range and the null space of P is denoted by $\mathscr{R}(P)$ and $\mathscr{N}(P)$, respectively. Further, the norm of P is defined as $\|P\|=\sup\{\|Px\| \,|\, \|x\|\le 1\}$. Clearly $\|P\|\ge 1$ excepting $P=0$ and $\|I\|=1$. (Here 0 and I denotes the zero and the identity operator on X, respectively.)

Let Y be a one-dimensional subspace of X. It follows immediately from Hahn—Banach theorem [3] that there exists a projection $P: X\to X$ for which $\mathscr{R}(P)=Y$ and $\|P\|=1$.

The aim of this paper is to investigate the question of the existence of normed spaces for which $P\ne I$ and $\dim \mathscr{R}(P)>1$ imply $\|P\|>1$.

By a density theorem, we solve the problem in finite dimensions. The infinite dimensional case seems to be entirely open.

From now on, let X be an n-dimensional real vector space. Assume that $n\ge 3$. Let $\langle\cdot,\cdot\rangle: X\times X\to \mathbf{R}$ be a fixed scalar product. Let $e_1, \ldots, e_n$ be a fixed orthonormal system with respect to the scalar product $\langle\cdot,\cdot\rangle$. Every $x\in X$ has a unique representation of the form $x=\sum_{i=1}^{n}\alpha_i e_i$; $\alpha_i\in\mathbf{R}$. Thus the basis $e_1, \ldots, e_n$ determines a one to one correspondence between vectors of X and n-tuples (column vectors) in $\mathbf{R}^n$. Given $x_1, \ldots, x_n\in X$, now it is possible to define the determinant function $\det(x_1, \ldots, x_n)$, as a function of column vectors in $\mathbf{R}^n$.

The set of norms defined on X will be denoted by $N(X)$. $N(X)$ can be made into a metric space in a very natural way. The distance between two norms $\|\cdot\|_1, \|\cdot\|_2: X\to\mathbf{R}^+$ can be defined as $d(\|\cdot\|_1, \|\cdot\|_2)=\sup\{|\|x\|_1-\|x\|_2| \,|\, \langle x, x\rangle=1\}$. As any two scalar products (moreover, any two norms) induce the same topology on X, the topology on $N(X)$ induced by d does not depend on the particular choice of the scalar product $\langle\cdot,\cdot\rangle$. Therefore, we are justified in speaking about openness and density of subsets of X as well as of $N(X)$ without referring to any particular scalar product.

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Now we are in a position to formulate our main result.

Theorem. *Let X be an n-dimensional real vector space. Assume that* $n \geqq 3$. *Define*

$$N_1(X) =$$
$$= \{\|\cdot\| \in N(X) \mid \text{for any projection } P\colon X \to X;\ P \neq I \text{ and } \dim \mathcal{R}(P) > 1 \text{ imply } \|P\| > 1\}.$$

Then $N_1(X)$ is open and dense in $N(X)$.

Proof of the openness of $N_1(X)$. Pick a norm $\|\cdot\|_1$ in $N_1(X)$. There exists a constant $K>0$ such that $(\langle x, x\rangle)^{1/2} \leqq K\|x\|_1$. If $d(\|\cdot\|_1, \|\cdot\|_2) \leqq \eta$, then $|\|x\|_1 - \|x\|_2| \leqq \eta(\langle x, x\rangle)^{1/2}$, and consequently, $(1-\eta K)\|x\|_1 \leqq \|x\|_2 \leqq (1+\eta K)\|x\|_1$.

It follows easily from a compactness argument that $\inf\{\|P\|_1 \mid P\colon X \to X$ is a projection, $P \neq I$, $\dim \mathcal{R}(P) > 1\} > 1$, i.e. for any projection $P\colon X \to X$ there holds $\|P\|_1 \geqq 1+\alpha$ for some fixed $\alpha > 0$ provided that $P \neq I$, $\dim \mathcal{R}(P) > 1$. Therefore, $\|Px\|_2 \geqq (1-\eta K)\|Px\|_1 \geqq (1-\eta K)(1+\alpha)\|x\|_1 \geqq (1-\eta K)(1+\alpha)(1+\eta K)^{-1}\|x\|_2$. Consequently, η being sufficiently small implies $\|P\|_2 > 1$, $\|\cdot\|_2 \in N_1(X)$.

The proof of the density of $N_1(X)$ requires more difficult considerations. If $S \subset X$, the set of all linear combinations of elements of S, i.e. the subspace spanned by S is denoted by $\operatorname{Span}(S)$. The orthogonal complement of $\operatorname{Span}(S)$ is denoted by $\operatorname{Span}^\perp(S)$. Let us recall that $\dim \operatorname{Span}(S) + \dim \operatorname{Span}^\perp(S) = n$.

Definition. Let N be a fixed positive integer. For sake of brevity, we call a set $\{x_1, \ldots, x_N\} \subset X$ to be independent if for any $Y \subset \{x_1, \ldots, x_N\}$ and for any partition $Y_1 = \{{}^1x_1, \ldots, {}^1x_{n_1}\}, \ldots, Y_k = \{{}^kx_1, \ldots, {}^kx_{n_k}\}$ of Y ($k \geqq 1$, $Y_i \cap Y_j = \emptyset$ if $i \neq j$; $i, j = 1, \ldots, k$; $n_j \geqq 0$; $\bigcup_{j=1}^k Y_j = Y$) satisfying $n_j \leqq n$, $j = 1, \ldots, k$ there holds

$$\dim\Big(\bigcap_{j=1}^k \operatorname{Span}(Y_j)\Big) = \max\Big\{0,\ n - \sum_{j=1}^k (n-n_j)\Big\}. \tag{1}$$

Further, we say that a vector $\bar{x} = (x_1, \ldots, x_N) \in X^N$ is of type $\mathcal{I}$ if the set of its coordinate vectors $\{x_1, \ldots, x_N\}$ is independent.

Remark 1. In case of $N=n$, $k=1$ one arrives at the usual definition of linear independence.

Remark 2. On the account of

$$\Big(\bigcap_{j=1}^k \operatorname{Span}(Y_j)\Big)^\perp = \operatorname{Span}(\{\operatorname{Span}^\perp(Y_1), \ldots, \operatorname{Span}^\perp(Y_k)\}),$$

(1) can be reformulated as

$$\dim(\operatorname{Span}(\{\operatorname{Span}^\perp(Y_1), \ldots, \operatorname{Span}^\perp(Y_k)\})) = \min\Big\{n,\ \sum_{j=1}^k (n-n_j)\Big\}. \tag{1'}$$

Lemma 1. *The set of vectors of type $\mathcal{I}$ is dense in X^N.*

Proof. Pick a vector $\bar{x}=(x_1, \dots, x_N)\in X^N$. Consider a partition $Y_1=\{{}^1x_1, \dots, {}^1x_{n_1}\}, \dots, Y_k=\{{}^kx_1, \dots, {}^kx_{n_k}\}$ of a subset $Y\subset\{x_1, \dots, x_N\}$ satisfying $n_j\leqq n$, $j=1, \dots, k$.

For $j=1, \dots, k$; $l=1, \dots, n-n_j$ define

$${}^jw_l = \det({}^jx_1, \dots, {}^jx_{n_j}, e_1, \dots, e_{l-1}, e_{l+1}, \dots, e_{n-n_j}, \operatorname{col}(e_1, \dots, e_n)).$$

The vectors ${}^jw_1, \dots, {}^jw_{n-n_j}$ form a basis for $\operatorname{Span}^\perp(Y_j)$ provided that

$$\det({}^jx_1, \dots, {}^jx_{n_j}, {}^jw_1, \dots, {}^jw_{n-n_j}) \neq 0. \tag{2}$$

Suppose that any $p\leqq n$ vectors in $\bigcup_{j=1}^{k}\{{}^jw_1, \dots, {}^jw_{n-n_j}\}$, say $z_1, \dots, z_p$ satisfy

$$\det(z_1, \dots, z_p, e_1, \dots, e_{n-p}) \neq 0. \tag{3}$$

It is obvious that (2) and (3) imply (1'). Thus the set of vectors of type $\mathcal{I}$ contains the complement of a real algebraic variety, consequently [4], it is dense in X^N.

Suppose now $x_1, \dots, x_n$ is a basis of X. The $(n-1)$-simplex $\sigma[x_1, \dots, x_n]$ with vertices $x_1, \dots, x_n$ and its interior $\operatorname{int}(\sigma[x_1, \dots, x_n])$ is defined as the set of all $x\in X$ of the form

$$x = \sum_{i=1}^{n} \alpha_i x_i \quad \text{where} \quad \alpha_i \geqq 0; \quad i = 1, \dots, n; \quad \sum_{i=1}^{n}\alpha_i = 1$$

and

$$x = \sum_{i=1}^{n} \alpha_i x_i \quad \text{where} \quad \alpha_i > 0; \quad i = 1, \dots, n; \quad \sum_{i=1}^{n}\alpha_i = 1$$

respectively. The vector $v(\sigma[x_1, \dots, x_n])$ defined by

$$\det(x_2-x_1, \dots, x_n-x_1, \operatorname{col}(e_1, \dots, e_n))$$

is a nonzero element in $\operatorname{Span}^\perp(\{x_2-x_1, \dots, x_n-x_1\})$, i.e. it is a normal vector to $\sigma[x_1, \dots, x_n]$.

Lemma 2. *Let $\bar{x}=(x_1, \dots, x_N)\in X^N$ be a vector of type $\mathcal{I}$. Given $\varepsilon>0$, then there exists an N-tuple of real numbers $(\varepsilon_1, \dots, \varepsilon_N)\in\mathbf{R}^N$ satisfying $0<\varepsilon_l<\varepsilon$; $l=1, \dots, N$ such that for any $Z\subset\{x_1, \dots, x_N, -x_1, \dots, -x_N\}$ and for any partition $Z_1=\{{}^1x_1, \dots, {}^1x_n\}, \dots, Z_n=\{{}^nx_1, \dots, {}^nx_n\}$ of Z $(Z_i\cap Z_j=\emptyset$ if $i\neq j$; $i,j=1, \dots, n$; $\bigcup_{j=1}^{n} Z_j=Z)$ satisfying*

$${}^ix_j \neq \pm {}^px_r \quad \textit{for} \quad |i-p|+|j-r| \neq 0$$

there holds

$$\text{Span}(\{v(\sigma[(1+{}^1\varepsilon_1)^1x_1, \ldots, (1+{}^1\varepsilon_n)^1x_n]), \ldots \ldots, v(\sigma[(1+{}^n\varepsilon_1)^n x_1, \ldots, (1+{}^n\varepsilon_n)^n x_n])\}) = X. \tag{4}$$

Proof. (4) is equivalent to

$$\det(v(\sigma[(1+{}^1\varepsilon_1)^1x_1, \ldots, (1+{}^1\varepsilon_n)^1x_n]), \ldots \ldots, v(\sigma[(1+{}^n\varepsilon_1)^n x_1, \ldots, (1+{}^n\varepsilon_n)^n x_n])) \neq 0. \tag{4'}$$

As in the proof of Lemma 1, the desired result follows from [4].

A bounded set $K\subset X$ is said to be a centrally symmetric convex polyhedron if there exist nonzero linear functionals $f_s\colon X\to\mathbf{R}$, $s=1, \ldots, t$ such that $K=\bigcap_{s=1}^{t}\{x\in X \mid |f_s(x)|\leq 1\}$. The bounding hyperplanes of K are defined as $\{x\in X \mid f_s(x)=1\}$, $\{x\in X \mid f_s(x)=-1\}$, $s=1, \ldots, t$. Assume that

(5) for any bounding hyperplane H, the intersection $H\cap K$ is an $(n-1)$-simplex. (Such simplices are called the facets of K.)

For sake of brevity, we say that the facets $\sigma[\tilde{x}_1, \ldots, \tilde{x}_n]$ and $\sigma[\tilde{\tilde{x}}_1, \ldots, \tilde{\tilde{x}}_n]$ are non-neighbouring if there holds

$$\{\tilde{x}_1, \ldots, \tilde{x}_n, -\tilde{x}_1, \ldots, -\tilde{x}_n\}\cap\{\tilde{\tilde{x}}_1, \ldots, \tilde{\tilde{x}}_n, -\tilde{\tilde{x}}_1, \ldots, -\tilde{\tilde{x}}_n\} = \emptyset.$$

If $M\subset X$ is a symmetric (i.e. symmetric with respect to the origin) convex set with the origin in its interior, then its Minkowsky functional $\Phi_M\colon X\to\mathbf{R}^+$ defined by $\Phi_M(x)=\inf\{\alpha>0 \mid x\in\alpha M\}$ is a norm. Conversely, if we are given a norm in X, then the unit ball it defines is a symmetric convex set with the origin in its interior, and it is the corresponding Minkowsky functional.

Proof of the density of $N_1(X)$.

Step 1. Pick a norm $\|\cdot\|$ in $N(X)$. Given $\varepsilon>0$, then there exists a norm $\|\cdot\|_K$ in $N(X)$ with the following properties:

(6) $d(\|\cdot\|, \|\cdot\|_K)<\varepsilon$;

(7) $\|\cdot\|_K=\Phi_K$, the Minkowsky functional of a centrally symmetric convex polyhedron K satisfying (5);

(8) denoting the vertices of K by $x_1, \ldots, x_N$, $-x_1, \ldots, -x_N$, the vector $\bar{x}=(x_1, \ldots, x_N)\in X^N$ is of type $\mathscr{S}$;

(9) if $\sigma_1, \ldots, \sigma_n$ are non-neighbouring facets of K, there holds

$$\text{Span}(\{v(\sigma_1), \ldots, v(\sigma_n)\}) = X;$$

and

(10) for any two-dimensional (linear) subspace $W\subset X$, the number of pairwise non-neighbouring facets of K intersecting W at a segment, is at least $n(n-1)$.

The existence of $\|\cdot\|_K$ satisfying (6)—(9) follows from the lemmas. (10) is automatically satisfied if

$$\max\{\|\tilde{x}-\tilde{\tilde{x}}\| \mid \text{there exists a facet } \sigma \text{ of } K \text{ such that } \tilde{x}, \tilde{\tilde{x}}\in\sigma\}$$

is sufficiently small.

Step 2. We show that $\|\cdot\|_K\in N_1(X)$. Let us observe first that (8) implies the following property of K:

(11) if the facets $\sigma_1, \dots, \sigma_{n-1}$ are pairwise non-neighbouring and a two-dimensional (linear) subspace $W\subset X$ intersects each of them at a segment, then, for some $k^*\in\{1, \dots, n-1\}$, W intersects $\operatorname{int}(\sigma_{k^*})$.

To the contrary, let us suppose that there exists an $l(k)\in\{1, \dots, n\}$ such that

$$W \subset Y_k = \operatorname{Span}(\{\overset{k}{\smile}x_1, \dots, \overset{k}{\smile}x_{l(k)-1}, \overset{k}{\smile}x_{l(k)+1}, \dots, \overset{k}{\smile}x_n\}),$$

for each $k=1, \dots, n-1$. Since $\dim(\operatorname{Span}(Y_k))=n-1$, (1) yields

$$2 = \dim W \leqq \dim\big(\bigcap_{k=1}^{n-1} \operatorname{Span}(Y_k)\big) = \max\big\{0, n-\sum_{k=1}^{n-1} 1\big\} = 1,$$

a contradiction.

Step 3. Let us suppose now that $P\colon X\to X$ is a projection for which $\|P\|_K=1$, $\dim\mathscr{R}(P)>1$. We have to show that $P=I$.

Consider a two-dimensional (linear) subspace $W\subset\mathscr{R}(P)$. Assume that for a facet σ of K there holds $W\cap\operatorname{int}(\sigma)\neq\emptyset$. We claim that $v(\sigma)\in\mathscr{N}^\perp(P)$. Pick a $z\in W\cap\operatorname{int}(\sigma)$. It is sufficient to show that $x\in\mathscr{N}(P)$ implies $x\in\operatorname{Span}^\perp(v(\sigma))$. In fact, we have $\|z\|_K=\|P(z+\lambda x)\|_K\leqq\|z+\lambda x\|_K$ for arbitrary $\lambda\in\mathbf{R}$. On the other hand, $z\in\operatorname{int}(\sigma)$ implies $(z+\lambda x)\in\sigma$ for $|\lambda|$ sufficiently small. Consequently, $x=((z+\lambda x)-z)/\lambda\in\operatorname{Span}^\perp(v(\sigma))$.

By the same reasoning, (10) and (11) imply the existence of pairwise non-neighbouring facets $\sigma_1, \dots, \sigma_n$ of K such that $v(\sigma_1), \dots, v(\sigma_n)\in\mathscr{N}^\perp(P)$. Applying (9) we obtain $X\subset\mathscr{N}^\perp(P)$, which, in turn, implies that $P=I$.

For applications of the Theorem, see [1], [2].

Remark 3. The Theorem remains valid if X is allowed to be a complex finite dimensional vector space.

The following problems arise naturally:

Problem 1. What is the minimum number of vertices of centrally symmetric convex polyhedra satisfying $\|\cdot\|_K\in N_1(X)$? (In the three-dimensional real case it is not hard to construct a centrally symmetric convex polyhedron K with twelve vertices for which $\|\cdot\|_K\in N_1(X)$. On the other hand, it seems plausible that there are no such polyhedra with ten vertices. Nevertheless, we are not able to prove it.)

Problem 2. Give upper and lower bounds for

$$\sup\left\{\inf\{\|P\| \mid P\colon X \to X \text{ is a projection satisfying } \dim \mathscr{R}(P) > 1,\ P \neq I\} \,\middle|\, \|\cdot\| \in N_1(X)\right\}.$$

Problem 3. The infinite dimensional case.

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