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Compact weighted composition operators on $L^2(\lambda)$

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1. Introduction. Let $(X, \mathcal{S}, \lambda)$ be a sigma-finite measure space and let $T\colon X\to X$ be a non-singular measurable transformation such that the composition transformation C_T defined as $C_Tf=f\circ T$ is bounded linear operator on $L^2(\lambda)$. If $\theta\in L^\infty(\lambda)$, then the multiplication operator M_θ defined as $M_\theta f=\theta\cdot f$ is a bounded linear operator on $L^2(\lambda)$. The product C_TM_θ is an operator on $L^2(\lambda)$ and we call it a weighted composition operator on $L^2(\lambda)$. This class of operators includes some of the well known operators such as multiplication operators, weighted shifts and composition operators [1].

In this note we are interested in studying compact weighted composition operators on $L^2(\lambda)$.

By $B(H)$ we denote the C^*-algebra of all bounded linear operators on a Hilbert space H. If C_T is a composition operator on $L^2(\lambda)$, then f_0 denotes the Radon—Nikodym derivative of the measure λT^{-1} with respect to the measure λ. For any complex valued function on X, $Z_\theta=\{x\colon \theta(x)=0\}$ and Z'_θ is the complement of Z_θ.

2. Some general results. It has been proved in [4] that if $C_T\in B(L^2(\lambda))$, then $C_T^*C_T=M_{f_0}$, where f_0 is the Radon—Nikodym derivative of the measure λT^{-1} with respect to the measure λ. Also it has been proved in [6] that $f_0\circ T\neq 0$ (a.e.). By using these results we prove the following theorem.

Theorem 2.1. *Let $C_T\in B(L^2(\lambda))$. Then C_T has dense range if and only if $C_TC_T^*=M_{f_0\circ T}$.*

Proof. Suppose C_T has dense range and let $f\in L^2(\lambda)$. Then there exists a sequence $\{f_n\}$ in $L^2(\lambda)$ such that $\{C_Tf_n\}$ converges to f. Now

$$C_TC_T^*f=\lim_n (C_TC_T^*C_Tf_n)=\lim_n (C_TM_{f_0}f_n)=M_{f_0\circ T}(\lim_n C_Tf_n)=M_{f_0\circ T}f.$$

Hence $C_TC_T^*=M_{f_0\circ T}$.

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Conversely, suppose $C_T C_T^* = M_{f_0 \circ T}$. Since $f_0 \circ T \neq 0$ (a.e.), $M_{f_0 \circ T}$ and hence $C_T C_T^*$ is an injection. This implies that $\ker C_T^* = \ker C_T C_T^* = \{0\}$ and hence C_T has dense range.

Corollary 2.2. *Let $C_T \in B(l^2)$. Then $C_T C_T^* = M_{f_0 \circ T}$ if and only if C_T is onto.*

Proof. Since the range of C_T is always closed in l^2, the result follows immediately.

Theorem 2.3. *Let $C_T M_\theta \in B(L^2(\lambda))$. Then $(C_T M_\theta)^* C_T M_\theta = M_{|\theta|^2 f_0}$.*

Proof. If $C_T M_\theta \in B(L^2(\lambda))$, then

$$(C_T M_\theta)^* C_T M_\theta = M_{\bar{\theta}} C_T^* C_T M_\theta = M_{|\theta|^2 f_0}.$$

Corollary 2.4. *Let $\theta \in L^\infty(\lambda)$ be such that $Z_\theta \subset (\operatorname{ran} T)'$, the complement of the range of T. Then $(C_T M_\theta)(C_T M_\theta)^* = M_{(|\theta|^2 f_0) \circ T}$ if and only if $C_T M_\theta$ has dense range.*

Proof. Suppose $(C_T M_\theta)(C_T M_\theta)^* = M_{(|\theta|^2 f_0) \circ T}$. Since $Z_\theta \subset (\operatorname{ran} T)'$, $\theta \circ T \neq 0$ (a.e.) and hence $(|\theta|^2 f_0) \circ T \neq 0$ (a.e.). This implies that $\ker (C_T M_\theta)^* = \{0\}$ and hence $C_T M_\theta$ has dense range.

The converse of this theorem follows from Theorem 2.1.

Theorem 2.5. *Let $C_T, M_\theta \in B(L^2(\lambda))$. Then*

(i) $C_T M_\theta = M_{\theta \circ T} C_T$,

(ii) $M_\theta C_T = 0$ *if and only if* $\theta = 0$ *(a.e.),*

(iii) $C_T M_\theta = 0$ *if and only if* $\theta \circ T = 0$ *(a.e.), and*

(iv) $C_T M_\theta = M_\theta C_T$ *if and only if* $\theta = \theta \circ T$ *(a.e.).*

Proof. (i) Let $f \in L^2(\lambda)$. Then $C_T M_\theta f = (\theta \circ T)(f \circ T) = M_{\theta \circ T} C_T f$. Hence $C_T M_\theta = M_{\theta \circ T} C_T$.

(ii) Suppose $M_\theta C_T = 0$. Then $M_\theta C_T f = 0$ for every f in $L^2(\lambda)$. Since $(X, \mathcal{S}, \lambda)$ is a sigma-finite measure space, there exists an $f \in L^2(\lambda)$ such that $f \neq 0$ (a.e.). Hence $\theta \cdot (f \circ T) = 0$ (a.e.) implies that $\theta = 0$ (a.e.).

The converse is obvious.

(iii) Since $C_T M_\theta = M_{\theta \circ T} C_T$, the proof follows from (ii).

(iv) The sufficiency of this result is obvious. To prove the necessary part, suppose $C_T M_\theta = M_\theta C_T$. Then $M_{\theta \circ T} C_T = M_\theta C_T$ and hence $M_{\theta \circ T - \theta} C_T = 0$. Thus the result follows from (ii).

The following examples illustrate that there are C_T and M_θ in $B(l^2)$, such that C_T commutes with M_θ. Here l^2 denotes the Hilbert space of square summable sequences of complex numbers.

Example 2.6. Let $X=N$, the set of natural numbers and λ be the counting measure on it. Define $T: X\to X$ by $T(n)=1$, if $n=1, 2$ and $T(3n+m)=n+2$, if $m=0, 1, 2$ and $n\in N$. Then $C_T\in B(l^2)$. Define $\theta: X\to \mathbf{C}$ by $\theta(n)=2$, if $n=1, 2$ and $\theta(n)=3$, if $n>2$. Then $M_\theta\in B(l^2)$ and C_T commutes with M_θ.

Example 2.7. If $\theta\in L^\infty(\lambda)$ is a constant, then M_θ commutes with every $C_T\in B(L^2(\lambda))$.

Example 2.8. Let $C_T\in B(L^2(\lambda))$ be such that $T(E)=E$ for some $E\in\mathscr{S}$ and $0<\lambda(E)<\infty$. Define $\theta=X_E$, the characteristic function of E. Then C_T commutes with M_θ.

3. Compact weighted composition operators. Let $(X, \mathscr{S}, \lambda)$ be a sigma-finite measure space. An element $E\in\mathscr{S}$ is said to be an atom if for every non-null measurable subset F of E, either $\lambda(F)=0$ or $\lambda(F)=\lambda(E)$. A measure space $(X, \mathscr{S}, \lambda)$ is said to be atomic if every element of $\mathscr{S}$ contains an atom. A measure space $(X, \mathscr{S}, \lambda)$ is said to be non-atomic if it does not contain any atom. It has been proved in [4], that no composition operator on L^2 of a non-atomic measure space is compact. It is interesting to note that the weighted composition operator on $L^2(\lambda)$ is compact if and only if it is the zero operator. This is evident from the following theorem.

Theorem 3.1. *The weighted composition operator $C_T M_\theta$ on L^2 of a non-atomic measure space is compact if and only if $\theta=0$ (a.e.) on Z'_{f_0}.*

Proof. Suppose $C_T M_\theta$ is compact. Then $C_T^* C_T M_\theta$ and hence $M_{\theta f_0}$ is compact. By a theorem of [5], $\theta f_0=0$ (a.e.). If $\theta\neq 0$ (a.e.) on Z'_{f_0}, then $f_0=0$ (a.e.). This implies that $C_T=0$. But no non-singular measurable transformation induces the zero operator. Hence $\theta=0$ (a.e.) on Z'_{f_0}. The converse is obvious.

Corollary 3.2. *The weighted composition operator $C_T M_\theta$ on $L^2(\lambda)$ is compact if and only if it is the zero operator.*

Proof. Suppose $C_T M_\theta$ is compact. Then $(C_T M_\theta)^* C_T M_\theta$ and hence $M_{|\theta|^2 f_0}$ is the zero operator. Hence $C_T M_\theta$ is the zero operator.

Corollary 3.3. *No composition operator on $L^2(\lambda)$ is compact.*

Let $\theta\in L^\infty(\lambda)$. We denote

$$X_\theta^\delta = \{x\in X: \theta(x)>\delta\} \quad \text{and} \quad M_\delta^\theta = \{f\in L^2(\lambda): f(x) = 0 \text{ on } X-X_\delta^\theta\}.$$

It has been proved in [5] that the multiplication operator M_θ on $L^2(\lambda)$ is compact if and only if M_δ^θ is finite dimensional. We shall characterize compact weighted composition operators on L^2 of an atomic measure space. Since $(X, \mathscr{S}, \lambda)$ is a sigma finite measure space, without loss of generality we write X as a countable union of atoms and we denote the ith atom by i.

Theorem 3.4. *Let $C_T M_\theta \in B(L^2(\lambda))$. Then $C_T M_\theta$ is compact if and only if either $\{f_0(i)\}$ or $\{\theta(i)\}$ converges to zero.*

Proof. Suppose $C_T M_\theta$ is compact. Then $M_{\theta f_0}$ is compact and hence $M_\delta^{\theta f_0}$ is finite dimensional. This shows that $X_\delta^{\theta f_0}$ contains finite number of atoms. It follows from this that the sequence $\{\theta f_0(i)\}$ converges to zero. Since θ and f_0 are essentially bounded functions, either $\{f_0(i)\}$ or $\{\theta(i)\}$ converges to zero. This completes the necessary part of the theorem.

Conversely, suppose either $\{f_0(i)\}$ or $\{\theta(i)\}$ converges to zero. Then either C_T or M_θ is compact. Hence $C_T M_\theta$ is compact.

It follows from this theorem that there are plenty of compact weighted composition operators on L^2 of an atomic measure space as is shown in the following example.

Example 3.5. Let $X=N$ and $\lambda(n)=a^n, 0<a<1$. Then l_a^2 denotes the weighted sequence space. Define $T: X \to X$ by $T(n)=n+1$, if n is odd and $T(n)=n-1$, if n is even. Then $C_T \in B(l_a^2)$ and $f_0(n)=a^{n-1}(1+a)$. Hence C_T is compact. If M_θ is any multiplication operator on l_a^2, then $C_T M_\theta$ is always compact.

The following theorem characterizes compact weighted composition operators on l^2, the Hilbert space of square summable sequences of complex numbers on N, the set of natural numbers.

Theorem 3.6. *Let $C_T M_\theta \in B(l^2)$. Then $C_T M_\theta$ is compact if and only if $\{\theta(n)\}$ converges to zero.*

Proof. Suppose $C_T M_\theta$ is compact. Then $M_\delta^{\theta f_0}$ is finite dimensional and hence $N_\delta^{\theta f_0}$ contains finite number of elements of N. If N_δ^θ contains infinite number of elements of N, then $N_\delta^{f_0}$ must contain only finite number of elements of N. This shows that $f_0=0$ for all but finitely many elements of N and hence the range of T contains finitely many elements of N. By taking $T(N)=E$, we have $\lambda T^{-1}(E) \not\leqq M\lambda(E)$ for any finite $M>0$. Hence by Theorem 1 of [3], C_T is not bounded. This proves that N_δ^θ contains finitely many elements of N. Hence $\{\theta(n)\}$ converges to zero.

Conversely, if $\{\theta(n)\}$ converges to zero, then M_θ is compact and hence $C_T M_\theta$ is compact.

Corollary 3.7. *No composition operator on l^2 is compact.*

Proof. The proof follows from Theorem 3.6, when $\theta(x)=1$.

Theorem 3.6 implies that the necessary condition for a weighted composition operator $C_T M_\theta$ on l^2 to be compact is that θ is not bounded away from zero. But this condition is not sufficient as is shown in the following example.

Example 3.8. Let $X=N$ and let $C_T \in B(l^2)$. Define $\theta: X \to \mathbf{C}$ by $\theta(1)=0$ and $\theta(n)=1$, if $n>2$. Hence θ is not bounded away from zero, but $C_T M_\theta$ is not compact.

Definition. A subalgebra $\mathcal{A}$ of $B(H)$ is said to be transitive if $\mathcal{A}$ is weakly closed, contains the identity operator and $\mathrm{Lat}\,\mathcal{A}=\{0, H\}$, where $\mathrm{Lat}\,\mathcal{A}= = \bigcap_{A\in\mathcal{A}} \mathrm{Lat}\,A$. It has been proved in [2] that if $\mathcal{A}$ is a transitive algebra of $B(H)$ containing a compact operator, then $\mathcal{A}=B(H)$.

Let $\{w_n\}$ be a bounded sequence of non-zero complex numbers and let $\{e_n\}$ be an orthonormal basis of H. The operator W on H defined by the requirements $We_0=0$ and $We_n=w_n e_{n-1}$ $(n=1, 2, \ldots)$ is called a weighted unilateral (backward) shift with the weight sequence $\{w_n\}$.

Corollary 3.9. *The weighted shift W on l^2 is compact if and only if the sequence of weights $\{w_n\}$ converges to zero.*

Corollary 3.10. *If $\mathcal{A}$ is a transitive algebra of $B(l^2)$ containing a weighted composition operator $C_T M_\theta$ such that $\theta(n)\to 0$ as $n\to\infty$, then $\mathcal{A}=B(l^2)$.*

The following result of YADAV and CHATTARJEE [7] follows immediately from Theorem 3.6.

Corollary 3.11. *If $\mathcal{A}$ is a transitive algebra of $B(l^2)$ containing a weighted shift with weights $\{w_n\}$ such that*

$$\delta(n) = \sum_{k=0}^{\infty} w_{k+2} \ldots w_{k+n}/w_2 w_3 \ldots w_n$$

tends to zero as $n\to\infty$ (for $n\geqq 2$), then $\mathcal{A}=B(l^2)$.

Proof. Since the sequence $\{\delta(n)\}$ converges to zero, the corresponding sequence of weights $\{w_n\}$ converges to zero. Hence the weighted shift is compact. Thus the result follows (cf. [2]).

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