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On generalized resolvents of nondensely defined symmetric contractions

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1. In 1977 M. G. KREĬN and I. E. OVČARENKO [6] described all generalized selfadjoint contraction resolvents of a nondensely defined symmetric contraction T in Hilbert space using its minimal or maximal selfadjoint contraction extension as the fixed extension.

In this note recent results of H. Langer and the author on the extension of dual pairs of contractions together with well-known results of B. SZ.-NAGY and A. KORÁNYI [10] are used to give a description of the generalized selfadjoint contraction resolvents of T when an arbitrary selfadjoint contraction extension of T is taken as fixed. The results have immediate application to the extension problem for non-negative closed linear relations in Hilbert space.

2. Assume T is a symmetric contraction in a Hilbert space $\mathfrak{H}$, with nondense domain $\mathfrak{D}(T)$. Then results of GR. ARSENE and A. GHEONDEA [1] (cf. also [3]) on dual pairs of contractions, applied to the pair $\{T, T\}$, imply the existence of a bijection $G \to T_G$ between all canonical (i.e. remaining in $\mathfrak{H}$) contraction extensions T_G of T such that $T_G^* \supset T$ and the set of all contractions $G \in [\mathscr{D}_\Gamma]$. This bijection is given by the matrix representation

$$T_G = \begin{pmatrix} A & D_A \Gamma \\ \Gamma^* D_A & -\Gamma^* A \Gamma + D_\Gamma G D_\Gamma \end{pmatrix} \tag{1}$$

with respect to the decomposition $\mathfrak{H} = \mathfrak{D}(T) \oplus \mathfrak{D}(T)^\perp$, with some contractions $A \in [\mathfrak{D}(T)]$, $A = A^*$, $\Gamma \in [\mathfrak{D}(T)^\perp, \mathscr{D}_A]$. Here $[\mathfrak{H}_1, \mathfrak{H}_2]$ denotes the set of all bounded linear operators from all of $\mathfrak{H}_1$ into $\mathfrak{H}_2$ and we put $[\mathfrak{H}_1] := [\mathfrak{H}_1, \mathfrak{H}_1]$. If B is a contraction from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ we put $D_B := (I - B^* B)^{1/2}$ and $\mathscr{D}_B := \overline{\mathfrak{R}(D_B)}$. $\mathfrak{R}(C)$ denotes the range of the linear operator C.

The extension T_G is selfadjoint if and only if G is selfadjoint. In this case (1) gives an explicit representation of the extremal canonical selfadjoint contraction

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(c.s.c.) extensions of T. Indeed, since $-I \leqq G \leqq I$ the minimal (maximal) c.s.c.-extension T_μ (T_M resp.) is obtained from (1) by taking $G=-I$ ($G=I$ resp.). In particular, $T_M - T_\mu = 2D_\Gamma^2 P_{\mathfrak{D}(T)^\perp}$ (by $P_\mathfrak{K}$ we denote the orthogonal projector of $\mathfrak{H}$ onto a subspace $\mathfrak{K}$ of $\mathfrak{H}$). The completely undetermined case (i.e. $(T_M - T_\mu)x \neq 0$ for all $x \in \mathfrak{D}(T)^\perp \setminus \{0\}$, see [5]) thus holds if and only if D_Γ is a bijection $\mathfrak{D}(T)^\perp \to \mathscr{D}_\Gamma$.

Let $\tilde{\mathfrak{H}} \supset \mathfrak{H}$ be a Hilbert space, $\tilde{T} \in [\tilde{\mathfrak{H}}]$ an arbitrary selfadjoint contraction (s.c.) extension of T, and $\tilde{P}$ the orthogonal projector of $\tilde{\mathfrak{H}}$ onto $\mathfrak{H}$. Denote by $\Omega(-1, 1)$ the set $\mathrm{Ext}((-\infty, -1] \cup [1, \infty))$ in the extended complex plane. The operator function

$$z \to R_z(\tilde{T}) := \tilde{P}(z\tilde{T} - I)^{-1}|_{\mathfrak{H}}, \quad z \in \Omega(-1, 1),$$

with values in $\mathfrak{H}$ is called a generalized s.c.-resolvent of T. If $\tilde{T}$ is canonical ($\tilde{\mathfrak{H}} = \mathfrak{H}$) then the generalized s.c.-resolvent is called a canonical s.c.-resolvent (c.s.c.-resolvent) of T. If no ambiguity arises we use the notation R_z for $R_z(\tilde{T})$ in the sequel.

The c.s.c.-extension T_0 of T which corresponds to $G=0$ in (1) will play a special role. We set $\mathring{R}_z := (zT_0 - I)^{-1}$. The operator function

$$z \to X_0(z) := -zD_\Gamma P_{\mathfrak{D}(T)^\perp}(zT_0 - I)^{-1} P_{\mathfrak{D}(T)^\perp} D_\Gamma, \quad z \in \Omega(-1, 1),$$

with values in $[\mathscr{D}_\Gamma]$ was introduced in [9] and shown to be contractive for $|z| < 1$. We will show in Proposition 3 that there is a close connection between X_0 and the two Q-functions Q_μ and Q_M of T introduced in [5] by the relations

$$\begin{aligned} Q_\mu(z) &:= \left(C^{1/2}(T_\mu - zI)^{-1} C^{1/2} + I\right)\big|_{\mathfrak{D}(T)^\perp}, \\ Q_M(z) &:= \left(C^{1/2}(T_M - zI)^{-1} C^{1/2} - I\right)\big|_{\mathfrak{D}(T)^\perp}, \quad z \in \mathrm{Ext}[-1, 1], \end{aligned} \tag{2}$$

where $C := T_M - T_\mu (= 2D_\Gamma^2 P_{\mathfrak{D}(T)^\perp})$.

3. Let $\mathfrak{G}$ be a Hilbert space. Denote by $\mathscr{N}(\mathfrak{G})$ the set of all functions G holomorphic in $\Omega(-1, 1)$ with values in $[\mathfrak{G}]$ such that

1) $-I \leqq G(x) \leqq I$ if $-1 < x < 1$,

2) the kernel $K(s, t) := \begin{cases} (s-t)^{-1}(G(s) - G(t)), & s \neq t \\ G'(t), & s = t \end{cases} \quad (-1 < s, t < 1)$

is positive definite.

Denote by $\mathscr{N}_0(\mathfrak{G})$ the subset of $\mathscr{N}(\mathfrak{G})$ consisting of those elements of $\mathscr{N}(\mathfrak{G})$ which are independent of z.

Proposition 1. *Assume G is holomorphic in $\Omega(-1, 1)$ with values in $[\mathfrak{G}]$. Then $G \in \mathscr{N}(\mathfrak{G})$ if and only if*

1') $\|G(z)\| \leqq 1$ *if* $|z| < 1$,

2') *the kernel* $K(z, \zeta) := \begin{cases} (z-\bar{\zeta})^{-1}(G(z) - G(\zeta)^*), & z \neq \bar{\zeta} \\ G'(z), & z = \bar{\zeta} \end{cases} \quad (z, \zeta \in \Omega(-1, 1))$

is positive definite.

Proof. We must only prove that 1′) and 2′) follow from 1) and 2). Define $F(s):=s(I-sG(s))^{-1}$, $-1<s<1$. Since $F(0)=0$, $F'(0)=I$ and since for $-1<s<1$, $-1<t<1$, $s\neq t$

$$(s-t)^{-1}(F(s)-F(t))=(I-sG(s))^{-1}(I+st(s-t)^{-1}(G(s)-G(t))(I-tG(t))^{-1},$$

a well known result of B. Sz.-Nagy and A. Korányi [10, Satz C*] yields the existence of a selfadjoint contraction $\tilde{G}$ in some Hilbert space $\tilde{\mathfrak{G}}\supset\mathfrak{G}$ such that $F(s)=sP_{\mathfrak{G}}(I-s\tilde{G})^{-1}|_{\mathfrak{G}}$, $-1<s<1$. Hence, by analytic continuation

$$R_z(\tilde{G}):=P_{\mathfrak{G}}(z\tilde{G}-I)^{-1}|_{\mathfrak{G}}=(zG(z)-I)^{-1},\quad z\in\Omega(-1,1).$$

It is now straightforward to see (cf. [8], [9]) that $R_z(\tilde{G})^{-1}\in[\mathfrak{G}]$ for $z\in\Omega(-1,1)$ and that $G(z)=z^{-1}(R_z(\tilde{G})^{-1}+I)$ (the right-hand side being extended by continuity to $z=0$) is a contraction for $|z|<1$. Thus 1′) holds. Property 2′) follows from the relation

$$(z-\bar{z})^{-1}(G(z)-G(z)^*)=$$

$$=|z|^{-2}R_{\bar{z}}(\tilde{G})^{-1}P_{\mathfrak{G}}(\bar{z}\tilde{G}-I)^{-1}(I-P_{\mathfrak{G}})(z\tilde{G}-I)^{-1}P_{\mathfrak{G}}R_z(\tilde{G})^{-1}\geqq 0,\quad \operatorname{Im} z\neq 0.$$

Remark. In [5] the class $\mathfrak{R}_{\mathfrak{G}}[-1,1]$ of $[\mathfrak{G}]$-valued operator functions k in $\operatorname{Ext}[-1,1]$ with the following properties was introduced:

(i) k is holomorphic in $\operatorname{Ext}[-1,1]$,

(ii) $(\operatorname{Im} z)^{-1}\operatorname{Im} k(z)\leqq 0$ if $\operatorname{Im} z\neq 0$,

(iii) $0\leqq k(x)\leqq I$ if $x>1$ or $x<-1$.

Proposition 1 implies that there is a close connection between $\mathfrak{R}_{\mathfrak{G}}[-1,1]$ and $\mathcal{N}(\mathfrak{G})$. Namely, $k\in\mathfrak{R}_{\mathfrak{G}}[-1,1]$ if and only if $G\colon z\to 2k(z^{-1})-I$ belongs to $\mathcal{N}(\mathfrak{G})$.

Proposition 2. *Assume* $G\in\mathcal{N}(\mathscr{D}_T)$. *Then* $(I-X_0(z)G(z))^{-1}\in[\mathscr{D}_T]$ *for* $z\in\Omega(-1,1)$.

Proof. For $|z|<1$, $X_0(z)G(z)$ is a contraction; in particular, for $z=0$ it is the zero operator. Thus for $|z|<1$ the assertion follows from the maximum modulus theorem.

Next assume that $z\in C_+\cup C_-$, where C_+ (C_-) denotes the open upper (lower) half-plane of the complex plane. The operator $I-X_0(z)G(z)P_{\mathscr{D}_T}$ has the matrix representation

$$I-X_0(z)G(z)P_{\mathscr{D}_T}=\begin{pmatrix} I-X_0(z)G(z) & 0\\ 0 & I\end{pmatrix}$$

with respect to the decomposition $\mathfrak{H}=\mathscr{D}_T\oplus\mathscr{D}_T^{\perp}$. The assertion is hence equivalent to $1\notin\sigma(X_0(z)G(z)P_{\mathscr{D}_T})$. Recall that if B_1, B_2 are bounded operators on $\mathfrak{H}$, then $\sigma(B_1B_2)\setminus\{0\}=\sigma(B_2B_1)\setminus\{0\}$, so

$$-1\notin\sigma(zD_TP_{\mathfrak{D}(T)^\perp}(zT_0-I)^{-1}P_{\mathfrak{D}(T)^\perp}D_TG(z)P_{\mathscr{D}_T})$$

if and only if

$$-1 \notin \sigma\left(zP_{\mathfrak{D}(T)^\perp} D_\Gamma G(z) P_{\mathscr{D}_\Gamma} D_\Gamma P_{\mathfrak{D}(T)^\perp} (zT_0 - I)^{-1}\right).$$

Note that

$$I + zP_{\mathfrak{D}(T)^\perp} D_\Gamma G(z) P_{\mathscr{D}_\Gamma} D_\Gamma P_{\mathfrak{D}(T)^\perp} (zT_0 - I)^{-1} =$$
$$= z\left(T_0 + D_\Gamma G(z) D_\Gamma P_{\mathfrak{D}(T)^\perp} - z^{-1} I\right)(zT_0 - I)^{-1}.$$

Now the assertion follows from the fact that $T_0 + D_\Gamma G(z) D_\Gamma P_{\mathfrak{D}(T)^\perp}$ is maximal dissipative for $z \in C_+$ and therefore C_- is contained in its resolvent set.

Assume $\hat{G} \in \mathscr{N}_0(\mathscr{D}_\Gamma)$. According to (1), the corresponding c.s.c.-extension of T is $T_{\hat{G}} = T_0 + D_\Gamma \hat{G} D_\Gamma P_{\mathfrak{D}(T)^\perp}$. Introduce the corresponding Q-function $Q_{\hat{G}}$ of T with values in $[\mathfrak{D}(T)^\perp]$ by

$$Q_{\hat{G}}(z) := D_\Gamma P_{\mathfrak{D}(T)^\perp} (T_{\hat{G}} - zI)^{-1} P_{\mathfrak{D}(T)^\perp} D_\Gamma, \quad z \in \operatorname{Ext}[-1, 1]$$

(cf. [7]). Note that $Q_{\hat{G}}$ has the matrix representation

$$Q_{\hat{G}}(z) = \begin{pmatrix} D_\Gamma P_{\mathfrak{D}(T)^\perp} (T_{\hat{G}} - zI)^{-1} P_{\mathfrak{D}(T)^\perp} D_\Gamma & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathfrak{D}(T)^\perp = \mathscr{D}_\Gamma \oplus \ker(D_\Gamma)$, and that if $G_1, G_2 \in \mathscr{N}_0(\mathscr{D}_\Gamma)$ then

$$Q_{G_2}(z) = Q_{G_1}(z)\left(I + (G_1 - G_2) Q_{G_1}(z)\right)^{-1}.$$

Obviously (see (2))

$$Q_\mu(z) = \left(2Q_{-I}(z) + I\right)\big|_{\mathfrak{D}(T)^\perp}, \quad Q_M(z) = \left(2Q_I(z) - I\right)\big|_{\mathfrak{D}(T)^\perp}.$$

Proposition 3. *Assume* $\hat{G} \in \mathscr{N}_0(\mathscr{D}_\Gamma)$, $z \in \operatorname{Ext}[-1, 1]$. *Then*

$$(I - X_0(z^{-1})\hat{G})^{-1} X_0(z^{-1}) = -Q_{\hat{G}}(z)|_{\mathscr{D}_\Gamma}.$$

In particular,

$$X_0(z^{-1}) = -Q_0(z)|_{\mathscr{D}_\Gamma}, \quad (I + X_0(z^{-1}))^{-1}(I - X_0(z^{-1})) = Q_\mu(z)|_{\mathscr{D}_\Gamma},$$
$$-(I - X_0(z^{-1}))^{-1}(I + X_0(z^{-1})) = Q_M(z)|_{\mathscr{D}_\Gamma}.$$

Remark. In the completely undetermined case, $\ker D_\Gamma = \{0\}$, so $Q_{\hat{G}}(z)|_{\mathscr{D}_\Gamma} = Q_{\hat{G}}(z)$. This can always be assumed without loss of generality.

Proof. By Proposition 2, $(I - X_0(z^{-1})\hat{G})^{-1} \in [\mathscr{D}_\Gamma]$. A direct calculation gives

$$\begin{pmatrix} (I - X_0(z^{-1})\hat{G})^{-1} X_0(z^{-1}) & 0 \\ 0 & 0 \end{pmatrix} = (I - X_0(z^{-1})\hat{G} P_{\mathscr{D}_\Gamma})^{-1} X_0(z^{-1}) P_{\mathscr{D}_\Gamma} =$$
$$= -D_\Gamma P_{\mathfrak{D}(T)^\perp} (T_0 + D_\Gamma \hat{G} D_\Gamma P_{\mathfrak{D}(T)^\perp} - zI)^{-1} P_{\mathfrak{D}(T)^\perp} D_\Gamma P_{\mathscr{D}_\Gamma} = \begin{pmatrix} -Q_{\hat{G}}(z)|_{\mathscr{D}_\Gamma} & 0 \\ 0 & 0 \end{pmatrix},$$

where the matrix representations are taken with respect to the decomposition $\mathfrak{H} = \mathscr{D}_\Gamma \oplus \mathscr{D}_\Gamma^\perp$.

Being R-functions, $z \to -X_0(z^{-1})$ and Q_G are increasing on $(-\infty, -1)\cup(1, \infty)$. The first function is bounded below (by $-I$) and above (by I). Therefore, the strong limit as $t \nearrow -1$ ($t \searrow 1$) exists which is a contraction. Also the strong limits of Q_G exist under additional conditions on $\hat{G}$. We next calculate the limits.

Proposition 4. a) $s-\lim\limits_{t\to\pm\infty} Q_G(t) = s-\lim\limits_{t\to\pm\infty} X(t^{-1}) = 0$;

b) $s-\lim\limits_{t\nearrow -1} X_0(t^{-1}) = -I$, $s-\lim\limits_{t\searrow 1} X_0(t^{-1}) = I$;

c) *Assume the completely undetermined case holds. Then*

$$s-\lim_{t\nearrow -1} Q_G(t) = (I+\hat{G})^{-1} \quad \text{if} \quad -1 \notin \sigma(\hat{G}),$$

$$s-\lim_{t\searrow 1} Q_G(t) = -(I-\hat{G})^{-1} \quad \text{if} \quad 1 \notin \sigma(\hat{G}).$$

Proof. Assertion a) is obvious. Assertion b) can be derived from Proposition 3 and [5, Theorem 2.1], or by the argument below. In order to prove c), assume e.g. $-1 \notin \sigma(\hat{G})$. It is easy to see that

$$\begin{aligned} Q_G(t) &= -D_\Gamma\big(\Gamma^*(A+(I-A)(I+(t+1)(A-tI)^{-1})\Gamma + tI - D_\Gamma \hat{G} D_\Gamma\big)^{-1} D_\Gamma = \\ &= \big(I+\hat{G}-(t+1)D_\Gamma^{-1}\Gamma^*(A-tI)^{-1}\Gamma D_\Gamma^{-1} - (t+1)D_\Gamma^{-2}\big)^{-1}, \quad t<-1. \end{aligned} \tag{3}$$

Denote by E the resolution of the identity of the selfadjoint contraction $A \in [\mathfrak{D}(T)]$. Put $t := -1-\varepsilon$, $\varepsilon > 0$, and $f(\tau; \varepsilon) := \varepsilon^2(\tau+1+\varepsilon)^{-2}$, $-1 \leqq \tau \leqq 1$. Then for each $h \in \mathfrak{D}(T)^\perp$,

$$\|(t+1)(A-tI)^{-1}\Gamma h\|^2 = \int_{-1}^{1} f(\tau;\ \varepsilon)(dE_\tau \Gamma h, \Gamma h);$$

so $\lim\limits_{t\nearrow -1} \|(t+1)(A-tI)^{-1}\Gamma h\|^2 = \|E_{-1}\Gamma h\|^2$. But $E_{-1}\Gamma h = 0$, since E_{-1} is the orthogonal projector of $\mathfrak{D}(T)$ onto $\ker(A+I) = \ker(A+I)^{1/2} \subset \ker(I-A^2)^{1/2} = \mathfrak{D}(T) \ominus \mathcal{D}_A$ and $\Gamma\mathfrak{D}(T)^\perp \subset \mathcal{D}_A$. Thus, (3) implies that $(Q_G(t))^{-1}$ decreases to its strong limit $I+\hat{G} \gg 0$ as $t \nearrow -1$. The assertion follows.

4. We first prove the following theorem on the characterization of the generalized s.c.-resolvents of T.

Theorem 1. *Let T be a symmetric nondensely defined contraction in the Hilbert space $\mathfrak{H}$. Then the formula*

$$R_z = \mathring{R}_z - z\mathring{R}_z D_\Gamma G(z)\big(I - X_0(z)G(z)\big)^{-1} D_\Gamma P_{\mathfrak{D}(T)^\perp} \mathring{R}_z, \quad z \in \Omega(-1, 1) \tag{4}$$

yields a bijective correspondence between all generalized s.c.-resolvents of T and all functions $G \in \mathcal{N}(\mathcal{D}_\Gamma)$. The generalized s.c.-resolvent is canonical if and only if $G \in \mathcal{N}_0(\mathcal{D}_\Gamma)$.

Proof. Let $\tilde{T}$ be a s.c.-extension of T in $\tilde{\mathfrak{H}} \supset \mathfrak{H}$. Denote its resolvent by R_z, $z \in \Omega(-1, 1)$. Then [9, Theorem 1] implies the existence of a uniquely determined holomorphic contraction valued operator function $G: z \to G(z) \in [\mathscr{D}_\Gamma]$, $|z|<1$, such that

(5) $$R_z = \mathring{R}_z(I + zD_\Gamma G(z) D_\Gamma P_{\mathfrak{D}(T)^\perp} \mathring{R}_z)^{-1}, \quad |z| < 1.$$

If, in particular, the extension $\tilde{T}$ is canonical, then G is a constant contraction.

It is straightforward to see (cf. [8], [9] and the proof of Proposition 1) that G has an analytic continuation to $\Omega(-1, 1)$, which we also denote by G, and that $G \in \mathscr{N}(\mathscr{D}_\Gamma)$. Then (cf. the proof of Proposition 2)

$$(I + zD_\Gamma G(z) D_\Gamma P_{\mathfrak{D}(T)^\perp} \mathring{R}_z)^{-1} \in [\mathfrak{H}], \quad z \in \Omega(-1, 1),$$

so by analytic continuation the relation (5) holds in the same region. But (4) and (5) are equivalent ([9]).

Assume, conversely, $G \in \mathscr{N}(\mathscr{D}_\Gamma)$ and put for $z \in \Omega(-1, 1)$

$$S(z) := (zT_0 - I + zD_\Gamma G(z) D_\Gamma P_{\mathfrak{D}(T)^\perp})^{-1}, \quad F(z) := -zS(z).$$

Let the kernel K be given by

$$K(z, \zeta) := \begin{cases} (z-\bar{\zeta})^{-1}(F(z) - F(\zeta)^*), & z \neq \bar{\zeta} \\ F'(z), & z = \bar{\zeta}. \end{cases}$$

By Proposition 1 and the formula

(6) $$K(z, \zeta) = S(\zeta)^*(I + z\bar{\zeta}(z-\bar{\zeta})^{-1} D_\Gamma (G(z) - G(\zeta)^*) D_\Gamma P_{\mathfrak{D}(T)^\perp}) S(z), \quad z \neq \bar{\zeta},$$

this kernel is positive definite. In order to construct a s.c.-extension of T we apply a standard technique. Consider the linear set $\tilde{\mathfrak{L}}$ of all finite formal sums $\tilde{f} = \sum \varepsilon_z f_z$ $(f_z \in \mathfrak{H}, z \in \Omega(-1, 1))$ and define in $\tilde{\mathfrak{L}}$ an inner product $(\cdot, \cdot)$ by

$$(\textstyle\sum \varepsilon_z f_z, \sum \varepsilon_\zeta g_\zeta) = \sum (K(z, \zeta) f_z, g_\zeta),$$

which is nonnegative by (6). $\tilde{\mathfrak{L}}$ can be canonically embedded into a Hilbert space $\tilde{\mathfrak{H}}$ and we identify $\tilde{\mathfrak{L}}$ and its image in $\tilde{\mathfrak{H}}$. Since $(\varepsilon_0 f, \varepsilon_0 f) = (K(0, 0)f, f) = (f, f)$ we can also identify $\mathfrak{H}$ with a subset of $\tilde{\mathfrak{H}}$ by the correspondence $f \to \varepsilon_0 f$. Next we define an operator $\tilde{T}$ on $\tilde{\mathfrak{L}}_0 := \{\tilde{f} = \sum \varepsilon_z f_z \in \tilde{\mathfrak{L}} : f_0 = 0\}$ by

$$\tilde{T}(\varepsilon_z f) := z^{-1}(\varepsilon_z f - \varepsilon_0 f), \quad z \neq 0.$$

It follows from the relation

$$(\varepsilon_z f - \varepsilon_0 f, \varepsilon_z f - \varepsilon_0 f) = ((|z|^2 (z-\bar{z})^{-1} S(z)^* D_\Gamma (G(z) - G(z)^*) D_\Gamma P_{\mathfrak{D}(T)^\perp} S(z) + \\ + (S(z)^* + I)(S(z) + I)) f, f) \to 0, \quad z \to 0$$

that the domain $\tilde{\mathfrak{L}}_0$ of $\tilde{T}$ is dense in $\tilde{\mathfrak{H}}$. In order to see that $\tilde{T}$ is symmetric with defect numbers $(0, 0)$, let $z, \zeta \in \Omega(-1, 1)$, $z\zeta \neq 0$. Then for $f, g \in \mathfrak{H}$

$$(\tilde{T}\varepsilon_z f, \varepsilon_\zeta g) - (\varepsilon_z f, \tilde{T}\varepsilon_\zeta g) = (((z\bar{\zeta})^{-1}(zS(z) - \bar{\zeta}S(\zeta)^*) - \bar{\zeta}^{-1}S(z) + z^{-1}S(\zeta)^*) f, g) = 0,$$

so $\tilde{T}$ is symmetric. The denseness of $\mathfrak{R}(z\tilde{T}-I)$ follows from the relation

$$(z\tilde{T}-I)(\zeta\varepsilon_\zeta g - z\varepsilon_z g) = (z-\zeta)\varepsilon_\zeta g.$$

The closure of $\tilde{T}$, which we again denote by $\tilde{T}$, is thus selfadjoint in $\tilde{\mathfrak{H}}$. Observe that $(z\tilde{T}-I)^{-1}\varepsilon_0 f = -\varepsilon_z f$, $z\in\Omega(-1,1)$, and that for $|z|>1$

$$\|(\tilde{T}-zI)^{-1}\varepsilon_0 f\|^2 = |z|^{-2}(F'(z^{-1})f,f).$$

A minor modification of the proof of [10, Satz C] then leads to the conclusion $\|\tilde{T}\|\leqq 1$. Furthermore,

$$((z\tilde{T}-I)^{-1}\varepsilon_0 f, \varepsilon_0 g) = -(\varepsilon_z f, \varepsilon_0 g) = -(K(z,0)f, g) = (S(z)f, g),$$

hence $\tilde{P}(z\tilde{T}-I)^{-1}|_{\mathfrak{H}} = S(z)$.

It remains to show that $\tilde{T}$ is an extension of T — it is then minimal since

$$\text{c.l.s. }\{(z\tilde{T}-I)^{-1}\mathfrak{H}\colon z\in\Omega(-1,1)\} = \text{c.l.s. }\{\varepsilon_z f\colon z\in\Omega(-1,1), f\in\mathfrak{H}\} = \tilde{\mathfrak{H}}$$

— and that $\tilde{T}$ is canonical if G is constant.

In order to show the first assertion, assume $f\in\mathfrak{D}(T)$, $z\neq 0$. Then (note the relation $z^{-1}(S(z)+I) = S(z)(T_0 + D_\Gamma G(z) D_\Gamma P_{\mathfrak{D}(T)^\perp})$)

$$(\tilde{T}\varepsilon_z f - \varepsilon_0 Tf, \varepsilon_\zeta g) = (z^{-1}(K(z,\zeta)-K(0,\zeta))f - K(0,\zeta)Tf, g) =$$

$$= ((z^{-1}(S(z)+I) + \bar{\zeta}(z-\bar{\zeta})^{-1} D_\Gamma (G(z)-G(\zeta)^*) D_\Gamma P_{\mathfrak{D}(T)^\perp} S(z) + T)f, S(\zeta)g) =$$

$$= ((S(z)+I)Tf, S(\zeta)g) + ((z-\bar{\zeta})^{-1} D_\Gamma (G(z)-G(\zeta)^*) D_\Gamma P_{\mathfrak{D}(T)^\perp} S(z) f, \zeta S(\zeta) g) \to 0,$$

$$z\to 0,$$

that is, for each $\tilde{g}\in\tilde{\mathfrak{H}}$, $(\tilde{T}\varepsilon_z f, \tilde{g}) \to (Tf, \tilde{g})$, $z\to 0$. But then

$$(\tilde{T}f - Tf, \tilde{g}) = (\tilde{T}f - \tilde{T}\varepsilon_z f, \tilde{g}) + (\tilde{T}\varepsilon_z f - Tf, \tilde{g}) \to 0, \quad z\to 0,$$

for each $\tilde{g}\in\tilde{\mathfrak{H}}$. Thus $\tilde{T}f = Tf$.

The second assertion follows easily. Indeed, if $G\in\mathcal{N}_0(\mathcal{D}_\Gamma)$, then $K(z,\zeta) = S(\zeta)^* S(z)$ and

$$(\tilde{f},\tilde{f}) = \sum (K(z,\zeta)f_z, f_\zeta) = (\sum S(z)f_z, \sum S(\zeta)f_\zeta).$$

But an element $\tilde{f} = \sum \varepsilon_z f_z$ belongs to $\tilde{\mathfrak{H}}\ominus\mathfrak{H}$ if and only if $\sum S(z)f_z = 0$. This concludes the proof of the theorem.

The c.s.c.-extension T_0 and its resolvent play a special role in Theorem 1. However, it can be replaced by any canonical or even noncanonical s.c.-extension of T. In order to see this, fix a s.c.-extension $\hat{T}$ in $\hat{\mathfrak{H}}$ corresponding to the function $\hat{G}\in\mathcal{N}(\mathcal{D}_\Gamma)$ according to Theorem 1 and denote its generalized resolvent by $\hat{R}_z$:

$$\hat{R}_z := P_{\mathfrak{H}}(z\hat{T}-I)^{-1}|_{\mathfrak{H}}, \quad z\in\Omega(-1,1).$$

Introduce a function $\hat{Q}$ with values in $[\mathfrak{D}(T)^\perp]$ by

$$\hat{Q}(z) := D_\Gamma P_{\mathfrak{D}(T)^\perp} P_{\mathfrak{H}} (\hat{T} - zI)^{-1} P_{\mathfrak{D}(T)^\perp} D_\Gamma, \quad z \in \mathrm{Ext}\,[-1, 1].$$

Note that $\hat{Q} = Q_{\hat{G}}$ if $\hat{G} \in \mathcal{N}_0(\mathcal{D}_\Gamma)$.

Theorem 2. *Let T be a symmetric nondensely defined contraction in the Hilbert space $\mathfrak{H}$. Then the formula*

$$(7)\quad R_z = \hat{R}_z - z\hat{R}_z D_\Gamma (G(z) - \hat{G}(z))(I + (\hat{Q}(z^{-1})|_{\mathcal{D}_\Gamma})(G(z) - \hat{G}(z)))^{-1} D_\Gamma P_{\mathfrak{D}(T)^\perp} \hat{R}_z, \quad z \in \Omega(-1, 1)$$

yields a bijective correspondence between all generalized s.c.-resolvents of T and all functions $G \in \mathcal{N}(\mathcal{D}_\Gamma)$. The generalized s.c.-resolvent is canonical if and only if $G \in \mathcal{N}_0(\mathcal{D}_\Gamma)$.

Proof. Since (4) is equivalent to the relation

$$R_z = \mathring{R}_z (I + z D_\Gamma G(z) D_\Gamma P_{\mathfrak{D}(T)^\perp} \mathring{R}_z)^{-1},$$

from which we obtain by specialization

$$(8)\qquad \hat{R}_z = \mathring{R}_z (I + z D_\Gamma \hat{G}(z) D_\Gamma P_{\mathfrak{D}(T)^\perp} \mathring{R}_z)^{-1},$$

it is straightforward to prove (cf. [9, Theorem 2]) that the formula

$$(9)\quad R_z = \hat{R}_z - z\hat{R}_z D_\Gamma (G(z) - \hat{G}(z))(I - X_0(z) G(z))^{-1} (I - X_0(z)\hat{G}(z)) D_\Gamma P_{\mathfrak{D}(T)^\perp} \hat{R}_z, \quad z \in \Omega(-1, 1);$$

yields a bijective correspondence between all generalized s.c.-resolvents of T and all functions $G \in \mathcal{N}(\mathcal{D}_\Gamma)$, and that the generalized s.c.-resolvent is canonical if and only if $G \in \mathcal{N}_0(\mathcal{D}_\Gamma)$. It remains to show that relation (9), which still contains the special extension T_0, appearing in the function X_0, can be rewritten in the form (7).

Indeed, (8) and the definition of the function $\hat{Q}$ imply

$$\hat{Q}(z^{-1}) = D_\Gamma P_{\mathfrak{D}(T)^\perp} (T_0 + D_\Gamma \hat{G}(z) D_\Gamma P_{\mathfrak{D}(T)^\perp} - z^{-1} I)^{-1} P_{\mathfrak{D}(T)^\perp} D_\Gamma P_{\mathfrak{D}_\Gamma}, \quad z \in \Omega(-1, 1),$$

and, according to Proposition 2, $(I - X_0(z)\hat{G}(z))^{-1} \in [\mathcal{D}_\Gamma]$, $z \in \Omega(-1, 1)$. Hence, the proof of Proposition 3 carries over to yield the relation

$$(I - X_0(z)\hat{G}(z))^{-1} X_0(z) = -\hat{Q}(z^{-1})|_{\mathfrak{D}_\Gamma}, \quad z \in \Omega(-1, 1),$$

and the statement follows from the identity

$$(I - X_0(z) G(z))^{-1} (I - X_0(z)\hat{G}(z)) =$$
$$= (I - (I - X_0(z)\hat{G}(z))^{-1} X_0(z)(G(z) - \hat{G}(z)))^{-1}, \quad z \in \Omega(-1, 1).$$

Remark 1. It is easy to see that (9) implies the relation

$$\tilde{P}\tilde{T}f = \hat{P}\hat{T}f + D_\Gamma(G(0) - \hat{G}(0))D_\Gamma P_{\mathfrak{D}(T)^\perp} f, \quad f \in \mathfrak{H}.$$

In particular,

$$\tilde{P}\tilde{T}f = T_\mu f + D_\Gamma(G(0) + I)D_\Gamma P_{\mathfrak{D}(T)^\perp} f, \quad f \in \mathfrak{H}$$

(cf. [5, (5.7)]).

Remark 2. Recently E. R. CEKANOVSKIĬ [2] stated among other things the following theorem: The symmetric nondensely defined contraction T has a canonical non selfadjoint contraction extension $\tilde{T}$ such that $\tilde{T}^* \supset T$ (that is, $\tilde{T}$ is a canonical non selfadjoint extension of the dual pair $\{T, T\}$ of contractions) if and only if T has more than one c.s.c.-extension. This is a consequence of (1) and Theorem 2. Indeed, T has a unique c.s.c.-extension if and only if $D_\Gamma = 0$, which is true if and only if $\{T, T\}$ has a unique canonical contraction extension. Moreover, [9, Theorem 1] and Theorem 2 imply that if T has a unique c.s.c.-extension T', then the only generalized resolvent of the dual pair $\{T, T\}$ is the s.c.-resolvent $(zT' - I)^{-1}$. These statements carry over via the Cayley transformation to dual pairs of dissipative linear relations.

Now it is easy to recover the results of M. G. Kreĭn and I. E. Ovčarenko from Theorem 2. To this end, assume that $\hat{G} \in \mathcal{N}_0(\mathcal{D}_\Gamma)$ and that the completely undetermined case holds. If $z \in \operatorname{Ext}[-1, 1]$, the representation formula (7) can be written in the form

$$\begin{aligned}\tilde{P}(\tilde{T} - zI)^{-1}|_{\mathfrak{H}} &= (\hat{T} - zI)^{-1} - \\ &-(\hat{T} - zI)^{-1} D_\Gamma (G(z^{-1}) - \hat{G})(I + Q_G(z)(G(z^{-1}) - \hat{G}))^{-1} D_\Gamma P_{\mathfrak{D}(T)^\perp} (\hat{T} - zI)^{-1}.\end{aligned} \tag{10}$$

Assume e.g. $\hat{G} = -I$, and put $k(z) := (1/2)(G(z^{-1}) + I)$, $z \in \operatorname{Ext}[-1, 1]$. Then k belongs to $\mathfrak{R}_{\mathcal{D}_\Gamma}[-1, 1]$ according to the remark after Proposition 1. We thus obtain the formula [5, (5.1)]

$$\begin{aligned}\tilde{P}(\tilde{T} - zI)^{-1}|_{\mathfrak{H}} &= \\ &= (T_\mu - zI)^{-1} - (T_\mu - zI)^{-1} C^{1/2} k(z)(I + (Q_\mu(z) - I)k(z))^{-1} C^{1/2} (T_\mu - zI)^{-1}\end{aligned}$$

by specialization from (7).

Formula (10) can further be rewritten to yield a representation similar to the one in [7]. For each $z \in \Omega(-1, 1)$ we define a linear relation $T(z)$ by $T(z) := (G(z^{-1}) - \hat{G})^{-1}$. It has the following properties:

1) If $z \in C_+$, then $T(z)$ is a maximal dissipative closed linear relation in $\mathfrak{H}$.

2) $T(z)$ is a holomorphic function (in the sense of [7]) such that $T(\bar{z}) = T(z)^*$.

Thus [7, Proposition 1.2] implies the existence of a decomposition $\mathcal{D}_\Gamma = (\mathcal{D}_\Gamma)_0 \oplus (\mathcal{D}_\Gamma)_\infty$ independent of z such that $(\mathcal{D}_\Gamma)_0$ and $(\mathcal{D}_\Gamma)_\infty$ are reducing subspaces of $T(z)$ for all $z \in \Omega(-1, 1)$, the operator part $T(z)_0$ of $T(z)$ is a maximal dissipative operator in $(\mathcal{D}_\Gamma)_0$ if $\operatorname{Im} z > 0$, and the infinite part $T(z)_\infty$ of $T(z)$ is $T(z)_\infty := \{\{0, g\} : g \in (\mathcal{D}_\Gamma)_\infty\}$.

Put $\Gamma_z := (\hat{T}-zI)^{-1}D_{\Gamma}$. Then (10) can be written in the form

$$\tilde{P}(\tilde{T}-zI)^{-1}|_{\mathfrak{H}} = (\hat{T}-zI)^{-1} - \Gamma_z T(z)^{-1}(I+Q_G(z)T(z)^{-1})^{-1}\Gamma_{\bar{z}}^* =$$
$$= (\hat{T}-zI)^{-1} - \Gamma_z(Q_G(z)+T(z))^{-1}\Gamma_{\bar{z}}^*$$

(see [7, (1.8)]), where for $x>1$ or $x<-1$

$$-(I+\hat{G}) \leqq T(x)^{-1} = G(x^{-1}) - \hat{G} \leqq I-\hat{G}.$$

In particular, $\hat{G}=-I$ gives $0 \leqq T(x)^{-1} \leqq 2I$ (cf. [7, Theorem 4.3] and [4]).

The results can be applied in a straightforward way to solve the extension problem for a nonnegative closed linear relation S in $\mathfrak{H}$ (cf. [6]) by using the transformation $T := -I+2(S+I)^{-1}$. We leave the details to the reader.

References

[1] Gr. Arsene and A. Gheondea, Completing matrix contractions, *J. Operator Theory*, **7** (1982), 179—189.

[2] E. R. Cekanovskiĭ, Non-selfadjoint accretive extensions of positive operators and theorems of Friedrichs—Kreĭn—Phillips, *Funkcional. Anal. i Priložen.*, **14** (1980), 87—88. (Russian)

[3] M. G. Crandall, Norm preserving extensions of linear transformations on Hilbert spaces, *Proc. Amer. Math. Soc.*, **21** (1969), 335—340.

[4] M. G. Kreĭn and I. E. Ovčarenko, On the theory of generalized resolvents of nondensely defined Hermitian contractions, *Dokl. Akad. Nauk Ukrain. SSR, Ser. A*, **1976**, no. 10, 881—885. (Russian)

[5] M. G. Kreĭn and I. E. Ovčarenko, On Q-functions and s.c.-resolvents of nondensely defined Hermitian contractions, *Sibirsk. Mat. Ž.*, **18** (1977), 1032—1055. (Russian)

[6] M. G. Kreĭn and I. E. Ovčarenko, On generalized resolvents and resolvent matrices of positive Hermitian operators, *Dokl. Akad. Nauk SSSR*, **231**, no. 5 (1976), 1063—1066. (Russian)

[7] H. Langer and B. Textorius, On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space, *Pacific J. Math.*, **72** (1977), 135—165.

[8] H. Langer and B. Textorius, Generalized resolvents of contractions, *Acta Sci. Math.*, **44** (1982), 125—131.

[9] H. Langer and B. Textorius, Generalized resolvents of dual pairs of contractions, in: *Proceedings of the 6th Conference on Operator Theory in Timisoara and Herculane 1981*, Birkhäuser (Basel—Boston—Stuttgart, 1982), pp. 103—118.

[10] B. Sz.-Nagy and A. Korányi, Operatortheoretische Behandlung und Verallgemeinerung eines Problemkreises in der komplexen Funktionentheorie, *Acta Math.*, **100** (1958), 171—202.

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