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The closure of invertible operators on a Hilbert space

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1. Introduction. Let H be a separable infinite dimensional Hilbert space and let $B(H)$ be the Banach algebra of all bounded linear operators on H. Denote by $\mathbf{G}$ the group of all invertible operators in $B(H)$, then what is the condition for an operator to be in the (norm) closure $\overline{\mathbf{G}}$ or the boundary bdy $\mathbf{G}$ of $\mathbf{G}$? FELDMAN and KADISON [3] considered this problem and characterized elements in the closure of invertible operators in a weakly closed subalgebra of $B(H)$. In the setting of Banach space operators, KELLY and HOGAN [8] gave some sufficient conditions for an operator to lie in the boundary of invertible operators from a view point of conservative operators. TREESE and KELLY [10], also in the same setting, showed a characterization of such operators under the restriction that they have closed ranges. Recall that the distance $\operatorname{dist}(A, \mathbf{S})$ of an operator A to a subset $\mathbf{S}\subset B(H)$ is defined as $\inf\{\|A-S\|: S\in\mathbf{S}\}$. Now another approach to our problem is to estimate, by some familiar parameter, the distance for $\mathbf{S}=\mathbf{G}$ or some other set related to $\mathbf{G}$. In terms of essential minimum modulus, the first author [6] showed some distance formulae on $\mathbf{G}$ and certain subsets of operators with index zero. Independently, BOULDIN [2] also tried a similar approach to the problem and presented distance formulae on $\mathbf{G}$ and on the set $\mathbf{F}$ of all Fredholm operators.

In this paper we shall continue the study on the closure $\overline{\mathbf{G}}$ and the boundary bdy $\mathbf{G}$ of $\mathbf{G}$. In Section 2 we clarify operators in bdy $\overline{\mathbf{G}}$ and show that the interior int $\overline{\mathbf{G}}$ of $\overline{\mathbf{G}}$ coincides with the set of Fredholm operators with index zero. In Section 3 we characterize closed range operators in $\overline{\mathbf{G}}$, which refines results in [1] and [10]. In Section 4, as an extension of [2] or [6], we determine the distance $\operatorname{dist}(A, \mathbf{S})$ when $\mathbf{S}$ is the subset of Fredholm operators with an index or the boundary bdy $\mathbf{G}$.

Throughout this paper we assume that the Hilbert space H is separable infinite dimensional. The index ind A of an operator A is defined by $\dim\ker A-\dim\ker A^*$, where $\dim\ker B$ is the dimension of the kernel of B and $\infty-\infty$ is understood to

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be zero [9]. The minimum (resp. essential minimum) modulus $m(A)$ (resp. $m_e(A)$) of $A \in B(H)$ is defined as the number

$$\inf\{\lambda:\ \lambda \in \sigma(|A|)\} \quad (\text{resp. } \inf\{\lambda:\ \lambda \in \sigma_e(|A|)\}).$$

Here $\sigma(|A|)$ (resp. $\sigma_e(|A|)$) is the spectrum (resp. essential spectrum) of $|A| := (A^*A)^{1/2}$. Let $\mathbf{I}_n$ be the set of all operators with index n. Now, as a preliminary we state a result due to BOULDIN [2, Theorem 3] (which was essentially shown in [6, Theorem 4]).

Theorem 1.1. *Let* $A \in B(H)$.
(1) *If* $A \in \mathbf{I}_0$ *then* $\operatorname{dist}(A, \mathbf{G}) = 0$.
(2) *If* $A \notin \mathbf{I}_0$ *then* $\operatorname{dist}(A, \mathbf{G}) = \max\{m_e(A), m_e(A^*)\}$.

Concerning the index and the essential minimum modulus we want to state three more basic facts.

Lemma 1.2. *Let* $A, B \in B(H)$ *and let* $\|A - B\| < m_e(A)$. *Then* $\operatorname{ind} A = \operatorname{ind} B$ ([2, p. 513]).

Lemma 1.3. *Let* $\operatorname{ind} A = n$. *Then there is an isometry or coisometry* W *according to* $n \leqq 0$ *or* $n \geqq 0$ *such that* $A = W|A|$ *and* $\operatorname{ind} W = n$ ([9, Proof of Theorem 1.3]).

Lemma 1.4. *If* $\operatorname{ind} A \leqq 0$, *then* $m_e(A) \geqq m_e(A^*)$. *Hence, if* $A \in \mathbf{G}$ *or* $A \in \overline{\mathbf{G}}$ *then* $m_e(A) = m_e(A^*)$.

2. Operators in $\overline{\mathbf{G}}$. Let $\mathbf{F}_n = \mathbf{F} \cap \mathbf{I}_n$ be the set of all Fredholm operators with index n. Then, since $\mathbf{G} \subset \mathbf{F}_0 \subset \mathbf{I}_0$ we have, by Theorem 1.1,

$$\overline{\mathbf{G}} = \overline{\mathbf{F}}_0 = \overline{\mathbf{I}}_0. \tag{2.1}$$

First, for the boundary of this set we have:

Theorem 2.1. $\operatorname{bdy} \overline{\mathbf{G}} = \{A \in B(H):\ m_e(A) = m_e(A^*) = 0\}$.

Proof. Let $m_e(A) = m_e(A^*) = 0$. First we show $A \in \overline{\mathbf{G}}$. If $A \in \mathbf{I}_0$ then $A \in \overline{\mathbf{G}}$, say, by (2.1), and if $A \notin \mathbf{I}_0$ then by Theorem 1.1 (2) $\operatorname{dist}(A, \mathbf{G}) = 0$, so that again we have $A \in \overline{\mathbf{G}}$. Now, to see $A \in \operatorname{bdy} \overline{\mathbf{G}}$ let $\varepsilon > 0$ and suppose, without loss of generality, that $\operatorname{ind} A \leqq 0$. Then $A = W|A|$ for an isometry W with $\operatorname{ind} W \leqq 0$, by Lemma 1.3. Since $m_e(A) = 0$, we see, from [4, Theorem 1.1], that $\dim E([0, \varepsilon))$ is infinite, where $E(\cdot)$ is the spectral measure of $|A|$. For brevity, write $E_\varepsilon = E([0, \varepsilon))$ and $E_\varepsilon^\perp = 1 - E_\varepsilon$ ($E_\varepsilon^\perp$ becomes the orthogonal projection onto the subspace $E([\varepsilon, \infty))H$). Define an operator $V \in B(H)$ as

$$Vx = x \quad \text{for} \quad x \in E_\varepsilon^\perp H, \quad \text{and}$$

$$Vx_n = x_{n+1} \quad \text{for an orthonormal basis} \quad \{x_n\} \quad \text{of} \quad E_\varepsilon H.$$

Furthermore, put

$$B_\varepsilon = \int \max\{\lambda-\varepsilon, 0\}\, dE(\lambda)$$

and $C_\varepsilon = WV(B_\varepsilon+\varepsilon)$. Then, we easily see that

$$VE_\varepsilon^\perp = E_\varepsilon^\perp, \quad E_\varepsilon^\perp B_\varepsilon = B_\varepsilon, \quad \||A|-B_\varepsilon\| \leqq \varepsilon \quad \text{and} \quad m_e(C_\varepsilon) \geqq \varepsilon.$$

Since $\operatorname{ind} W \leqq 0$ (and $\operatorname{ind} V(B_\varepsilon+\varepsilon) = -1$, $W, V(B_\varepsilon+\varepsilon)$ are Fredholm operators), we see $\operatorname{ind} C_\varepsilon \leqq -1$, so that by Theorem 1.1 we have $\operatorname{dist}(C_\varepsilon, \mathbf{G}) \geqq m_e(C_\varepsilon) > 0$ or $C_\varepsilon \notin \overline{\mathbf{G}}$. But

$$\|C_\varepsilon - A\| = \|W(V(B_\varepsilon+\varepsilon)-|A|)\| = \|VB_\varepsilon - |A| - \varepsilon V\| \leqq$$
$$\leqq \|VB_\varepsilon - |A|\| + \varepsilon = \|VE_\varepsilon^\perp B_\varepsilon - |A|\| + \varepsilon = \|B_\varepsilon - |A|\| + \varepsilon \leqq 2\varepsilon.$$

Hence, since ε is arbitrary we see that A is on the boundary $\operatorname{bdy} \overline{\mathbf{G}}$. To see the converse, that is, if $A \in \operatorname{bdy} \overline{\mathbf{G}}$ then $m_e(A) = m_e(A^*) = 0$, suppose otherwise, say, $m_e(A) > 0$. Then by Lemma 1.4 $m_e(A^*) = m_e(A) > 0$, so that A is Fredholm. Besides, since $A \in \operatorname{bdy} \overline{\mathbf{G}} \subset \operatorname{bdy} \mathbf{G}$, we can find an operator $D \in \mathbf{G}$ such that $\|A-D\| < m_e(A)$. Hence $\operatorname{ind} A = \operatorname{ind} D = 0$ (say, by Lemma 1.2), so that $A \in \mathbf{F}_0$. But, since $\mathbf{F}_0$ is an open subset of $\overline{\mathbf{G}}$ we see that A is an interior point of $\overline{\mathbf{G}}$, which is a contradiction.

Remark. Denote by $\mathbf{F}_l$ (resp. $\mathbf{F}_r$) the set of all left (resp. right) semi-Fredholm operators or the set $\{A: m_e(A) > 0\}$ (resp. $\{A: m_e(A^*) > 0\}$). Then, from the proof of Theorem 2.3 (or a similar argument) we see

$$\overline{\mathbf{G}} \cap \mathbf{F}_l = \mathbf{F}_0 \quad (= \overline{\mathbf{G}} \cap \mathbf{F}_r). \tag{2.2}$$

If we denote by $\mathbf{G}_l$ (resp. $\mathbf{G}_r$) the set of all left (resp. right) invertible operators, then as (2.2) we can also see

$$\overline{\mathbf{G}} \cap \mathbf{G}_l = \mathbf{G} \quad (= \overline{\mathbf{G}} \cap \mathbf{G}_r).$$

Corollary 2.2. (1) $\operatorname{int} \overline{\mathbf{G}} = \mathbf{F}_0$, *and hence* $\mathbf{F}_0$ *is a regularly open subset in* $B(H)$.
(2) $\operatorname{bdy} \overline{\mathbf{G}} = \operatorname{bdy} \mathbf{F}_0$.
(3) $\operatorname{bdy} \mathbf{G} = \operatorname{bdy} \overline{\mathbf{G}} \cup (\mathbf{F}_0 \setminus \mathbf{G})$.

Proof. (1) Since $\mathbf{F}_0 \subset \operatorname{int} \overline{\mathbf{G}}$ is clear, we may only show the opposite inclusion. Let $A \in \operatorname{int} \overline{\mathbf{G}}$. Then by the theorem $m_e(A) > 0$ or $m_e(A^*) > 0$. Hence, in either case we have (say, by (2.2)) $A \in \mathbf{F}_0$.

(2) Clear by the theorem and (2.1).

(3) Note that $\operatorname{bdy} \mathbf{G} \supset \operatorname{bdy} \overline{\mathbf{G}}$, and that $A \in \operatorname{bdy} \mathbf{G} \setminus \operatorname{bdy} \overline{\mathbf{G}}$ if and only if $A \in \mathbf{F}_0 \setminus \mathbf{G}$.

3. Closed range operators in $\overline{\mathbf{G}}$. In this section we show some necessary and sufficient conditions for an operator to lie in $\overline{\mathbf{G}}$ or $\operatorname{bdy} \mathbf{G}$ under the restriction that the operator has closed range. For simplicity, we denote by $A \in (\mathrm{CR})$ if $A \in B(H)$ has closed range. It is well-known [1], [5] that if $A \in (\mathrm{CR})$ then there exists the unique (Moore—Penrose) generalized inverse $A^\dagger \in B(H)$ of A satisfying the following four identities;

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger \quad \text{and} \quad (A^\dagger A)^* = A^\dagger A.$$

The products $AA^\dagger$ and $A^\dagger A$ are the orthogonal projections onto the ranges $AH(=\ker^\perp A^*$, the orthogonal complement of $\ker A^*)$ and $A^*H(=\ker^\perp A)$, respectively. The next fact [7, Proposition 2.3] is useful for our discussion.

Lemma 3.1. *Let $\{A_n\}$ be a sequence of operators with closed range, and suppose that it converges to $A \in (\mathrm{CR})$ uniformly, that is, $A_n \to A$. Then the following conditions are equivalent.*

(1) $\sup_n \|A_n^\dagger\| < \infty$.

(2) $A_n A_n^\dagger \to AA^\dagger$.

(3) $A_n^\dagger A_n \to A^\dagger A$.

The equivalence (2) and (3) or (3′) of the following result was essentially shown by Beutler [1, Theorem 1].

Theorem 3.2. *Let $A \in (\mathrm{CR})$. Then the following conditions are equivalent.*

(1) $A \in \overline{\mathbf{G}}$.

(2) $A \in \mathbf{I}_0$.

(3) $A = BP$ *for an operator* $B \in \mathbf{G}$ *and an orthogonal projection* P.

(3′) $A = PB$ *for an operator* $B \in \mathbf{G}$ *and an orthogonal projection* P.

Proof. (1)⇒(2) Let $\{A_n\}$ be a sequence in $\mathbf{G}$, and let $A_n \to A$. Put $C_n = A_n A^\dagger$ and $C = AA^\dagger$. Then $C_n, C \in (\mathrm{CR})$ and $C_n \to C$. Furthermore, since $\ker^\perp C_n = AH$ we have $C_n^\dagger C_n = AA^\dagger = C = C^\dagger C$ (cf. $C = C^\dagger$). Hence, by Lemma 3.1 we have $C_n C_n^\dagger \to CC^\dagger = AA^\dagger$. Hence, for a sufficiently large n, we have

$$\|C_n C_n^\dagger - C_n^\dagger C_n\| < 1.$$

This implies $\dim\ker C_n^* = \dim\ker C_n$ or $\mathrm{ind}\, C_n = 0$. Hence $\mathrm{ind}\, A^\dagger = 0$, i.e., $\mathrm{ind}\, A = 0$.

(2)⇒(3) If $A \in \mathbf{I}_0$, then $A = U|A|$ with a unitary U. Since $P := A^\dagger A$ is an orthogonal projection such that $|A|P = |A|$, and since $B := U\{|A| + (1 - A^\dagger A)\} \in \mathbf{G}$, we see that $A = BP$ is the desired decomposition.

(3)⇒(1) Note that $\mathrm{ind}\, BP = \mathrm{ind}\, B + \mathrm{ind}\, P = 0$ for B and P in (3).

(3)⇔(3′) Note that $A \in \mathbf{I}_0 \Leftrightarrow A^* \in \mathbf{I}_0$.

In [10] Treese and Kelly characterized closed range operators in bdy $\mathbf{G}$ (in the setting of Banach space operators). From Theorem 3.2 we now deduce a similar characterization of such operators, which is to be compared with [10, Theorem].

Corollary 3.3. *Let* $A \in (\mathrm{CR})$. *Then the following conditions are equivalent.*

(1) $A \in \mathrm{bdy}\, \mathbf{G}$.

(2) $A \in \mathbf{I}_0 \setminus \mathbf{G}$.

(3) $A = BP$ *for an operator* $B \in \mathbf{G}$ *and an orthogonal projection* $P \neq 1$.

(3′) $A = PB$ *for an operator* $B \in \mathbf{G}$ *and an orthogonal projection* $P \neq 1$.

(4) $A \notin \mathbf{G}$ *and there exists a sequence* $\{B_n\}$ *in* $\mathbf{G}$ *such that* $B_n A^\dagger A \to A$.

Proof. From the theorem we easily see that (1), (2), (3) and (3′) are mutually equivalent. If (3) is assumed, then $B_n = B(P + 1/n)$ $(n = 1, 2, \ldots)$ are invertible and $B_n A^\dagger A \to AA^\dagger A = A$, that is, (4) is obtained. If we assume (4), then since $B_n A^\dagger A \in \mathbf{I}_0$ we easily see $A \in \bar{\mathbf{I}}_0 = \bar{\mathbf{G}}$, which implies $A \in \mathrm{bdy}\, \mathbf{G}$, i.e., the condition (1).

Remark. In proving the above corollary by a technique in [10], we would have to add to (4) the uniform boundedness of $\{B_n^{-1}\}$. Related to this, we observe that the sequence $\{(B_n A^\dagger A)^\dagger\}$ of generalized inverses is uniformly bounded; since $B_n A^\dagger A \to A$ and $(B_n A^\dagger A)^\dagger (B_n A^\dagger A) = A^\dagger A$, we have, by Lemma 3.1, $\sup_n \|(B_n A^\dagger A)^\dagger\| < \infty$.

4. Distance formulae related to $\mathbf{F}_n$, bdy G and bdy $\bar{\mathbf{G}}$. Recall that $\bar{\mathbf{F}}_0 = \bar{\mathbf{I}}_0 = \bar{\mathbf{G}}$, and hence that

$$\mathrm{dist}\,(A, \mathbf{F}_0) = \max\,\{m_e(A), m_e(A^*)\} \quad \text{for} \quad A \notin \mathbf{I}_0$$

by Theorem 1.1. As an extension of those facts we have:

Theorem 4.1. *Let* $A \in B(H)$.

(1) *If* $A \in \mathbf{I}_n$, *then* $\mathrm{dist}\,(A, \mathbf{F}_n) = 0$.

(2) *If* $A \notin \mathbf{I}_n$, *then* $\mathrm{dist}\,(A, \mathbf{F}_n) = \max\,\{m_e(A), m_e(A^*)\}$.

Proof. (1) If $A \in \mathbf{I}_n$ then $A = W|A|$ with an isometry (or coisometry) $W \in \mathbf{I}_n$. Let $\varepsilon > 0$ and $B = W(|A| + \varepsilon)$. Then $B \in \mathbf{F}_n$ and $\|A - B\| < \varepsilon$. Hence, $\mathrm{dist}\,(A, \mathbf{F}_n) < \varepsilon$, which implies the assertion (1).

(2) Let S be a unilateral simple shift on H, and let $B = S^n A$ or $B = AS^{*(-n)}$ according to $n \geqq 0$ or $n \leqq 0$. Then we see $\mathrm{ind}\, B \neq 0$ because of $\mathrm{ind}\; S = -1$, and

$$m_e(B) = m_e(A), \quad m_e(B^*) = m_e(A^*). \tag{4.1}$$

Furthermore, we see

$$\bar{\mathbf{F}}_n = (S^{*(n)}\mathbf{G})^- \quad \text{or} \quad \bar{\mathbf{F}}_n = (\mathbf{G}S^{(-n)})^-$$

according to $n \geqq 0$ or $n \leqq 0$. Hence, if $n \geqq 0$, then

$$\mathrm{dist}\,(A, \mathbf{F}_n) = \mathrm{dist}\,(A, S^{*(n)}\mathbf{G}) = \mathrm{dist}\,(B, S^n S^{*(n)}\mathbf{G}) = \mathrm{dist}\,(B, \mathbf{G})$$

(cf. $(S^n S^{*(n)}\mathbf{G})^- = \bar{\mathbf{G}}$). Hence, by Theorem 1.1 and (4.1) we have the desired identity in (2). For $n \leqq 0$, similarly we can obtain the identity.

Concerning the distance from an operator to the boundary bdy $\mathbf{G}$ or bdy $\overline{\mathbf{G}}$, we have:

Theorem 4.2. *Let* $A\in B(H)$. *Then*

(1) $$\operatorname{dist}(A, \operatorname{bdy}\mathbf{G}) = \begin{cases} \max\{m_e(A), m_e(A^*)\} & \text{if } A\notin\overline{\mathbf{G}}, \\ m(A)(=m(A^*)) & \text{if } A\in\overline{\mathbf{G}}. \end{cases}$$

(2) $$\operatorname{dist}(A, \operatorname{bdy}\overline{\mathbf{G}}) = \max\{m_e(A), m_e(A^*)\}.$$

Proof. (1) If $A\notin\overline{\mathbf{G}}$, then clearly

$$\operatorname{dist}(A, \operatorname{bdy}\mathbf{G}) = \operatorname{dist}(A, \mathbf{G}) = \max\{m_e(A), m_e(A^*)\}.$$

If $A\in\overline{\mathbf{G}}$, then we consider the two cases $A\in\mathbf{I}_0$ and $A\notin\mathbf{I}_0$. First, if $A\in\mathbf{I}_0$, then $A=U|A|$ for a unitary U. Let $B=U(|A|-m(A))$. Then $m(B)=0$ and $B\in\operatorname{bdy}\mathbf{G}$. Hence $\operatorname{dist}(A, \operatorname{bdy}\mathbf{G})\leqq\|A-B\|=m(A)$. To see that only the equality sign holds, suppose

$$\operatorname{dist}(A, \operatorname{bdy}\mathbf{G}) < m(A), \tag{4.2}$$

and hence also $m(A)>0$. Then $A\in\mathbf{G}_l$ or $A\in\mathbf{G}_l\cap\overline{\mathbf{G}}=\mathbf{G}$, and by (4.2) there exists an operator $C\in\operatorname{bdy}\mathbf{G}$ such that $\|A-C\|<m(A)$. Hence, since $\|A^{-1}\|=m(A)^{-1}$ (cf. [2, Theorem 1]), we have

$$\|1-A^{-1}C\| = \|A^{-1}(A-C)\| \leqq \|A^{-1}\|\,\|A-C\| < 1,$$

so that we easily see $C\in\mathbf{G}$. This is a contradiction. Next, if $A\notin\mathbf{I}_0$ then by Theorem 1.1 we see that $m_e(A)=m_e(A^*)=\operatorname{dist}(A, \mathbf{G})=0$. Hence, since $m(A)\leqq m_e(A)=0$ and since $A\in\overline{\mathbf{G}}\setminus\mathbf{I}_0\subset\operatorname{bdy}\mathbf{G}$ we again obtain the desired identity with the common value zero. It is easy to see $m(A)=m(A^*)$ for $A\in\mathbf{G}$ and hence for $A\in\overline{\mathbf{G}}$.

(2) If $A\notin\overline{\mathbf{G}}$, then clearly

$$\operatorname{dist}(A, \operatorname{bdy}\overline{\mathbf{G}}) = \operatorname{dist}(A, \overline{\mathbf{G}}) = \max\{m_e(A), m_e(A^*)\}.$$

If $A\in\overline{\mathbf{G}}$, then as (1) we consider the two cases $A\in\mathbf{I}_0$ and $A\notin\mathbf{I}_0$. If $A\in\mathbf{I}_0$, then $A=U|A|$ for a unitary U. Put $B=U(|A|-m_e(A))$. Then $B\in\operatorname{bdy}\overline{\mathbf{G}}$, because $m_e(B)=m_e(B^*)=0$ (and by Theorem 2.1). Hence, $\operatorname{dist}(A, \operatorname{bdy}\overline{\mathbf{G}})\leqq\|A-B\|=$ $=m_e(A)$. To show that the equality sign holds, suppose $\operatorname{dist}(A, \operatorname{bdy}\overline{\mathbf{G}})<m_e(A)$, and hence also $m_e(A)>0$. Then $A\in\mathbf{F}_l\cap\overline{\mathbf{G}}=\mathbf{F}_0$ (say, by (2.2)). Besides, there exists an operator $C\in\operatorname{bdy}\overline{\mathbf{G}}$ such that $\|A-C\|<m_e(A)$. Hence we see $m_e(C)\geqq m_e(A)-$ $-\|A-C\|>0$, so that $C\in\mathbf{F}_l\cap\overline{\mathbf{G}}=\mathbf{F}_0$. But this is a contradiction by Corollary 2.2 (1). If $A\notin\mathbf{I}_0$, then by Theorem 1.1 we have $m_e(A)=m_e(A^*)=\operatorname{dist}(A, \mathbf{G})=0$. This implies $A\in\operatorname{bdy}\overline{\mathbf{G}}$ and the identity in (2) holds again.

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