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# On approximation by arbitrary systems in $L^2$-spaces

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*Dedicated to Professor László Leindler on his 50th birthday*

**1. Introduction.** Let $-\infty<a<b<\infty$, $p=b-a$. Let $L^2=L^2[p]$ be the space of all square integrable functions defined on $(-\infty, \infty)$ which are $p$-periodic. The norm in $L^2[p]$ is defined by

$$\|f\|_2=\left\{\int_a^b |f(x)|^2 dx\right\}^{1/2}, \quad f\in L^2[p].$$

Let $\Phi=\{\varphi_k\}_{k=0}^{\infty}$ be a complete orthonormal system in $L^2[p]$. For $f_1, f_2, \ldots, f_n \in L^2[p]$ let us denote by $[f_1, f_2, \ldots, f_n]$ the linear span of $f_1, f_2, \ldots, f_n$. For any $f\in L^2[p]$ let

(1) $$E_n = E_n^{\Phi}(f) = \inf_{q\in[\varphi_0, \varphi_1, \ldots, \varphi_n]} \|f-q\|_2, \quad n=0,1,2,\ldots$$

be the $n$-th best approximation of $f$ with respect to the system $\Phi$. We know that $E_n^{\Phi}(f)$ can be given by the generalized Fourier coefficients of $f$ with respect to the system $\Phi$, more precisely,

$$E_n^{\Phi}(f)=\Big[\sum_{k=n+1}^{\infty} c_k^2(f)\Big]^{1/2}, \quad n=0, 1, 2, \ldots$$

where

$$c_k(f)=\int_a^b f(x)\varphi_k(x)\,dx, \quad k=0, 1, 2, \ldots.$$

In this paper we give an answer to the following question due to Prof. L. Leindler: Characterize those orthonormal systems $\Phi$ for which

$$E_n^{\Phi}(f) \leqq c\omega(f, 1/n), \quad \forall f\in L^2[p], \quad n=1, 2, \ldots$$

where $\omega(f, \delta)$ denotes the $L^2$-modulus of continuity of $f$, i.e.

$$\omega(f, \delta) = \sup_{|h|\leqq\delta} \|f(x+h)-f(x)\|_2.$$

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**2. Lemmas.** We need the following lemmas.

Lemma 1. *Let* $\varrho_n>0$ $(n=1,2,\dots)$. *Suppose that the system* $\Phi=\{\varphi_k\}_{k=0}^{\infty}$ *contains a constant function, say:* $\varphi_0\equiv C$. *The following statements are equivalent:*

a) *There exists an absolute constant* $C_1$ *such that*

(3) $$E_n^{\Phi}(f)\leqq C_1\omega(f,\varrho_n),\quad \forall f\in L^2[p].$$

b) *There exists an absolute constant* $C_2$ *such that*

(4) $$E_n^{\Phi}(F)\leqq C_2\varrho_n\|f\|_2,\quad \forall f\in L^2[p]$$

*where* $F(x)=\int_a^x f(t)\,dt$.

Proof. 1. a)$\to$b): Let $h>0$. By the formula

$$F(x+h)-F(x)=\int_0^h f(x+t)\,dt$$

we have

$$\|F(x+h)-F(x)\|_2=\left\|\int_0^h f(x+t)\,dt\right\|_2\leqq\int_0^h\|f(\cdot+t)\|_2\,dt=\int_0^h\|f\|_2\,dt=h\|f\|_2$$

hence $\omega(F,\delta)\leqq\delta\|f\|_2$. So, from a) we obtain

$$E_n(F)\leqq C_1\omega(F,\varrho_n)\leqq C_1\varrho_n\|f\|_2.$$

This proves (4).

2. b)$\to$a): We apply the transform of Steklov: Let

$$f_n(x)=\varrho_n^{-1}\int_0^{\varrho_n}f(x+t)\,dt,\quad x\in[a,b].$$

Then $f_n(x)$ is absolute continuous, therefore $f_n(x)$ is an integral function of $f_n'$:

$$f_n(x)=\int_a^x f_n'(t)\,dt+f_n(a)=\tilde f_n(x)+f_n(a).$$

Since the system $\Phi$ contains the constant function we have $E_n(f_n)=E_n(\tilde f_n)$. On the other hand, we have

$$\|f-\tilde f_n\|_2=\left\|\varrho_n^{-1}\int_0^{\varrho_n}[f(x+t)-f(x)]\,dt\right\|_2\leqq\omega(f,\varrho_n),$$

$$\|\tilde f_n'\|_2=\varrho_n^{-1}\|f(x+\varrho_n)-f(x)\|_2\leqq\varrho_n^{-1}\omega(f,\varrho_n).$$

Hence we obtain by (4):

$$E_n(f)=E_n(\tilde f_n)+\|f-\tilde f_n\|_2\leqq C_2\varrho_n\|\tilde f_n'\|_2+\omega(f,\varrho_n)\leqq(1+C_2)\omega(f,\varrho_n).$$

This proves (3).

Now, we introduce the following class of functions:

$$L_n = [\varphi_0, \varphi_1, \ldots, \varphi_n], \quad L_n^{\perp} = \{g \in L^2[p]\colon (g, q) = 0, \quad \forall q \in L_n\}, \quad n = 0, 1, 2, \ldots$$

where $(g, q) = \int_a^b g(x)q(x)\,dx$. If the system $\Phi$ is complete, then this definition is equivalent to the following:

$$L_n^{\perp} = \Big\{g = \sum_{k=n+1}^{\infty} c_k \varphi_k \colon \sum_{k=n+1}^{\infty} c_k^2 < \infty\Big\}, \quad n = 0, 1, 2, \ldots. \tag{5}$$

We notice that $L_n$ and $L_n^{\perp}$ are (linear and closed) subspaces of $L^2[p]$.

Lemma 2. (4) *is equivalent to the following:*

$$\|G\|_2 \leqq C_2 \varrho_n \|g\|_2, \quad \forall g \in L_n^{\perp}, \quad n = 0, 1, 2, \ldots \tag{6}$$

*where* $G(x) = \int_a^x g(t)\,dt$.

Proof. Let $f \in L^2[p]$ and let $S(f)$ be the generalized Fourier series of $f$ with respect to the system $\Phi$, that is

$$S(f) = \sum_{k=0}^{\infty} c_k(f)\varphi_k$$

where

$$c_k(f) = \int_a^b f(x)\varphi_k(x)\,dx, \quad k = 0, 1, 2, \ldots.$$

We have by the minimum property of an orthonormal system:

$$E_n^{\Phi}(f) = \Big\|\sum_{k=n+1}^{\infty} C_k(f)\varphi_k\Big\|_2,$$

or, equivalently,

$$E_n(f) = \sup_{\substack{g \in L_n^{\perp} \\ \|g\|_2 \leqq 1}} \int_a^b f(x)g(x)\,dx, \quad n = 0, 1, \ldots. \tag{7}$$

Now, we apply this formula for the proof of Lemma 2.

a) (6)$\to$(4): Let $f \in L_n$, $g \in L_n^{\perp}$, $\|g\|_2 \leqq 1$, and let

$$G(x) = \int_a^x g(t)\,dt, \quad F(x) = \int_a^x f(t)\,dt.$$

We have by integration by parts and (6):

$$\int_a^b F(x)g(x)\,dx = FG|_a^b - \int_a^b f(x)G(x)\,dx =$$

$$= \int_a^b f(x)G(x)\,dx \leqq \|f\|_2\|G\|_2 \leqq C_2\varrho_n\|f\|_2$$

(we notice that since $g\in L_n^\perp$ and $\varphi_0\equiv C$, we have $G(a)=G(b)=0$). From the last inequality we obtain (4) by an application of (7).

b) (4)→(6): Let $f\in L^2$, $g\in L_n^\perp$, $\|g\|_2\leqq 1$. Since

$$\int_a^b F(x)g(x)\,dx = \int_a^b G(x)f(x)\,dx,$$

from (4) and (7) we have

$$\int_a^b f(x)G(x)\,dx \leqq C_2\varrho_n\|f\|_2. \tag{8}$$

Now, let $0\neq g\in L_n^\perp$ be fixed. Let $g^*=g/\|g\|_2$; then $g^*\in L_n^\perp$ and $\|g^*\|_2=1$. Let

$$G^*(x) = \int_a^x g^*(t)\,dt.$$

From (8) we obtain:

$$\int_a^b f(x)G^*(x)\,dx \leqq C_2\varrho_n\|f\|_2, \quad \forall f\in L^2[p].$$

Hence, $\|G^*\|_2\leqq C_2\varrho_n$ from which it follows that $\|G\|_2\leqq C_2\varrho_n\|g\|_2$. This proves (6).

Now let us denote by $I$ the integral operator, that is,

$$If(x)= \int_a^x f(t)\,dt, f\in L^2[p], x\in[a, b],$$

and let $If(x)$ be a $p$-periodic function. We know that the operator $I$ is a bounded linear operator of the space $L^2$ to $L^2$. Let $I_n\colon L_n^\perp\to L^2[p]$ be the restriction of $I$ to the space $L_n^\perp$, and let $|||I_n|||$ denote the norm of $I_n$, that is,

$$|||I_n||| = \sup_{\substack{g\in L_n^\perp\\ \|g\|_2\leqq 1}} \|I_n g\|_2 = \sup_{\substack{g\in L_n^\perp\\ \|g\|_2\leqq 1}} \|Ig\|_2. \tag{9}$$

Then we have

$$\|Ig\|_2 \leqq |||I_n|||\,\|g\|_2, \quad g\in L_n^\perp, \tag{10}$$

so that (6) is always true for $C_2\varrho_n=|||I_n|||$.

Therefore we have:

Lemma 3. *Let* $\lambda_n = |||I_n|||$ $(n=0, 1, 2, \ldots)$.

a) *We have*

$$E_n(F) \leqq \lambda_n \|f\|_2, \quad \forall f \in L^2[p], \tag{11}$$

*where* $F(x) = If(x)$.

b) *The order* $\lambda_n$ *is best possible, this means that if for* $\lambda_n' > 0$:

$$E_n(F) \leqq \lambda_n \|f\|_2, \quad \forall f \in L^2[p],$$

*then* $\lambda_n' \geqq \lambda_n$ $(n=0, 1, 2, \ldots)$.

Proof. a) is proved above. Claim b) follows from the fact that if $E_n(F) \leqq \leqq \lambda_n' \|f\|_2$, $\forall f \in L^2[p]$, then by Lemma 2 we have $\|G\|_2 \leqq \lambda_n' \|g\|_2$, $\forall g \in L_n^{\perp}$, hence we obtain by the definition of the norm $|||I_n|||$ that $\lambda_n' \geqq |||I_n||| = \lambda_n$.

In the following we consider only a complete orthonormal system $\Phi = \{\varphi_0, \varphi_1, \ldots\}$ which satisfies the following conditions:

$$\varphi_0(t) \equiv C \text{ (constant)}, \tag{12}$$

$$\text{for } \; n = 0, 1, 2, \ldots, \quad I\varphi_{n+1} \in L_n^{\perp}. \tag{13}$$

We remark that the condition (13) is equivalent to the following: for $n=0, 1, 2, \ldots$, if $g \in L_n^{\perp}$ then $Ig \in L_n^{\perp}$.

Lemma 4. *Let* $\Phi = \{\varphi_0, \varphi_1, \ldots\}$ *be the complete orthonormal system satisfying* (12) *and* (13). *Let* $\psi_k = I\varphi_k$, $k=0, 1, 2, \ldots$, *where* $I$ *denotes the integral operator. Then for* $n=0, 1, 2, \ldots$ *the system* $\{\psi_k\}_{k=n+1}^{\infty}$ *is complete, linearly independent in the subspace* $L_n^{\perp}$.

Proof. a) $\{\psi_k\}_{k=n+1}^{\infty}$ is linearly independent. Suppose that $\alpha_k$ $(k = n+1, n+2, \ldots, n+m)$ are real numbers satisfying

$$\sum_{k=n+1}^{n+m} \alpha_k \psi_k = 0.$$

Then by differentiation we have

$$\sum_{k=n+1}^{n+m} \alpha_k \varphi_k = 0$$

hence $\alpha_k = 0$ $(k=n+1, \ldots, n+m)$, since $\{\varphi_k\}_{k=0}^{\infty}$ is independent.

b) $\{\psi_k\}_{k=n+1}^{\infty}$ is complete in $L_n^{\perp}$. Suppose that $g \in L_n^{\perp}$ satisfies

$$\int_0^a g(x)\psi_k(x)\,dx = 0 \quad (k \geqq n+1).$$

Let $Ig=G(x)$. Integrating by parts we obtain (by (12) we have $\psi_k(a)=\psi_k(b)=0$ for $k\geqq n+1>0$):

$$(14)\qquad 0=\int_a^b g(x)\psi_k(x)\,dx=\int_a^b G(x)\varphi_k(x)\,dx \quad (k\geqq n+1).$$

Since $g\in L_n^\perp$, by (13) we have $G\in L_n^\perp$, that is

$$\int_a^b G(x)\varphi_k(x)\,dx=0 \quad (k\leqq n)$$

and so (14) is valid for every $k=0, 1, 2, \ldots$ from which it follows by the completeness of the system $\Phi=\{\varphi_k\}_{k=0}^\infty$ that $G(x)\equiv 0$, therefore $g(x)\equiv C$ (constant). But $g\in L_n^\perp$, so by (12) we have $g(x)\equiv C=0$.

Let now $n\geqq 0$ and fixed. Let $\Phi_n=\{\psi_{n+1}, \psi_{n+2}, \ldots\}$. Since $\Phi_n$ is linearly independent (Lemma 4), by the process of Gram—Schmidt we obtain the orthonormal system $H=(h_1, h_2, \ldots)\subset L_n^\perp$ as follows. For $m=1, 2, \ldots$ let

$$(15)\qquad \Delta_m(\Phi_n)=|a_{lk}^{(n)}|_{l,k=1}^m=\begin{vmatrix} (\psi_{n+1}, \psi_{n+1})(\psi_{n+1}, \psi_{n+2})\ldots(\psi_{n+1}, \psi_{n+m}) \\ (\psi_{n+2}, \psi_{n+1})(\psi_{n+2}, \psi_{n+2})\ldots(\psi_{n+2}, \psi_{n+m}) \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ (\psi_{n+m}, \psi_{n+1})(\psi_{n+m}, \psi_{n+2})\ldots(\psi_{n+m}, \psi_{n+m}) \end{vmatrix}$$

be the $m$-th Gram—Schmidt's determinant of the system $\Phi_n$. Let $D_{m,l}^n=D_{m,l}(\Phi_n)$ be the cofactor of an element $a_{lm}^{(n)}$ $(l=1, 2, \ldots, m)$. We define the following infinite matrix:

$$(16)\qquad A(\Phi_n)=(\alpha_{lk}^{(n)})_{l,k=1}^\infty=\begin{pmatrix} \frac{1}{\sqrt{\Delta_1(\Phi_n)}} & \frac{D_{12}(\Phi_n)}{\sqrt{\Delta_1(\Phi_n)\Delta_2(\Phi_n)}} & \frac{D_{13}(\Phi_n)}{\sqrt{\Delta_2(\Phi_n)\Delta_3(\Phi_n)}} & \cdots \\ 0 & \frac{D_{22}(\Phi_n)}{\sqrt{\Delta_1(\Phi_n)\Delta_2(\Phi_n)}} & \frac{D_{33}(\Phi_n)}{\sqrt{\Delta_2(\Phi_n)\Delta_3(\Phi_n)}} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

From the matrix $A(\Phi_n)$ we define the matrix $A_m(\Phi_n)$:

$$(17)\qquad A_m(\Phi_n)=\begin{pmatrix} \alpha_{11}^{(n)} & \alpha_{12}^{(n)} & \ldots & \alpha_{1m}^{(n)} \\ 0 & \alpha_{22}^{(n)} & \ldots & \alpha_{2m}^{(n)} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \alpha_{mm}^{(n)} \end{pmatrix}=(\alpha_{lk}^{(n)})_{l,k=1}^m.$$

Let $B_m(\Phi_n)=A_m^{-1}(\Phi_n)$ be the inverse matrix of $A_m(\Phi_n)$:

$$(18)\qquad B_m(\Phi_n)=(\beta_{lk}^{(n)})_{l,k=1}^m=\begin{pmatrix} \beta_{11}^{(n)} & \beta_{12}^{(n)} & \ldots & \beta_{1m}^{(n)} \\ \beta_{21}^{(n)} & \beta_{22}^{(n)} & \ldots & \beta_{2m}^{(n)} \\ \beta_{m1}^{(n)} & \beta_{m2}^{(n)} & \ldots & \beta_{mm}^{(n)} \end{pmatrix}.$$

From the the matrix $B_m(\Phi_n)$ $(m=1, 2, \ldots)$ we define the infinite matrix:

(19) $$B(\Phi_n) = (\beta_{lk}^{(n)})_{l,k=1}^{\infty}.$$

The process of Gram—Schmidt gives the following formula:

(20) $$\Phi_n A(\Phi_n) = H, \quad HB(\Phi_n) = \Phi_n$$

where $\Phi_n A(\Phi_n)$ and $HB(\Phi_n)$ denote the usual products of matrices (infinite matrices).

Now we return to the determination of the exact value of $|||I_n|||$. Let $g \in L_n^{\perp}$. Then we have

$$g = \sum_{k=n+1}^{\infty} C_k \varphi_k, \quad \|g\|_2 = \Big(\sum_{k=n+1}^{\infty} C_k^2\Big)^{1/2}.$$

Since the operator $I$ is linear and continuous (in the metric of $L^2$), we have

$$Ig = \sum_{k=n+1}^{\infty} C_k I\varphi_k = \sum_{k=n+1}^{\infty} C_k \psi_k = \sum_{l=1}^{\infty} d_l h_l$$

where

(21) $$d = CB(\Phi_n)$$

with $C=(C_{n+1}, C_{n+2}, \ldots)$, $d=(d_1, d_2, \ldots)$. By Parseval's formula we have

(22) $$\|Ig\|_2 = \Big(\sum_{l=1}^{\infty} d_l^2\Big)^{1/2}.$$

Let $l^2$ denote the Hilbert space of all sequences $c=(c_1, c_2, \ldots)$ for which $\|c\|_{l^2} = \Big(\sum_{k=1}^{\infty} c_k^2\Big)^{1/2} < \infty$. Now, from (21), (22) we obtain

(23) $$|||I_n||| = \sup_{\substack{g \in L_n^{\perp} \\ \|g\|_2 \leq 1}} \|Ig\|_2 = \sup_{\substack{c \in l^2 \\ \|c\|_{l^2} \leq 1}} \|CB(\Phi_n)\|_{l^2}.$$

Finally, from (23), by a known theorem of functional analysis (see e.g. Л. В. Канторович—Г. П. Акилов [1], p. 193) we have

(24) $$|||I_n||| = \sup_{m \geq 1} \max_{1 \leq j \leq m} \sqrt{\lambda_j[B_m^*(\Phi_n) B_m(\Phi_m)]}$$

where $B_m^*(\Phi_n)$ denotes the adjoint matrix of $B_m(\Phi_n)$ and $\lambda_j[B_m^*(\Phi_n)B_m(\Phi_n)]$ denotes an eigenvalue of the matrix $B_m^*(\Phi_n)B_m(\Phi_n)$.

**3.** So, the formula (24), and Lemmas 1, 3 prove the following theorem.

Theorem. *Let $\Phi = \{\varphi_k\}_{k=0}^{\infty}$ be a complete orthonormal system in $L^2[p]$, which satisfies the conditions* (12) *and* (13). *Let $B_m(\Phi_n)$ be the matrix defined by* (15), (16), (17), (18), *and let $\lambda_j^{(n,m)}$ be the eigenvalues of the self-adjoint matrix $B_m^*(\Phi_n)B_m(\Phi_n)$. Let*

(25) $$\varrho_n = \varrho_n(\Phi) = \sup_{m \geq 1} \max_{1 \leq j \leq m} \sqrt{\lambda_j^{(n,m)}}, \quad n = 0, 1, 2, \ldots.$$

*Then we have*

a) $$E_n^{\Phi}(f) \leqq C_3 \omega(f, \varrho_n), \quad \forall f \in L^2[p], \quad n = 1, 2, \ldots$$

*where* $C_3$ *is an absolute constant* (we can select $C_3 = 2$; see the proof of Lemma 2);

b) $\varrho_n$ *is best possible, that is if* $E_n(f) \leqq C_4 \omega(f, \varrho_n')$, $\forall f \in L^2$, $n = 1, 2, \ldots$, *then* $\varrho_n = O(\varrho_n')$.

Remark 1. Let $\Omega(p)$ be the set of all functions $f$, which are absolute continuous in $[a, b]$ and for which $f' \in L^2[p]$, $\|f'\|_2 \leqq 1$. Let

$$E_n^{\Phi}(\Omega) = \sup_{f \in \Omega} E_n^{\Phi}(f) \quad \text{and} \quad d_n(\Omega) = \inf_{\Phi \in \mathcal{S}} E_n^{\Phi}(\Omega), \quad n = 0, 1, 2, \ldots,$$

where $\mathcal{S}$ denotes the class of orthonormal systems in $L^2[p]$; $d_n(\Omega)$ is called the $n$-th width of the set $\Omega$. If for some $\Phi^* \in \mathcal{S}$ we have $d_n(\Omega) = E_n^{\Phi^*}(\Omega)$, $n = 0, 1, 2, \ldots$, then we say that $\Phi^*$ is an extremal system for the set $\Omega$.

Let now $T$ be the trigonometric system

$$T = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \ldots \right\}.$$

We know that for a set $\Omega = \Omega(2\pi) \subset L^2[2\pi]$, the system $T$ is an extremal system in $L^2[2\pi]$, and

$$d_n[\Omega(2\pi)] = E_n^T[\Omega(2\pi)] = 1/(n+1), \quad n = 0, 1, 2, \ldots.$$

(See e.g. G. G. LORENTZ [2] p. 140.) So the system

$$T_p = \left\{ \frac{1}{\sqrt{2p}}, \frac{2\sqrt{\pi}}{p} \sin\left(\frac{p}{2\pi} t + a\right), \frac{2\sqrt{\pi}}{p} \cos\left(\frac{p}{2\pi} t + a\right), \ldots \right\}$$

is orthonormal in $L^2[p]$; it is an extremal system for the set $\Omega = \Omega(p) \subset L^2[p]$ and

(26) $$d_n[\Omega(p)] = E_n^{T_p}[\Omega(p)] = \big(1/(n+1)\big)(2\pi/p), \quad n = 0, 1, 2, \ldots.$$

We return to the definition of $\varrho_n(\Phi)$. We have

(27) $$\varrho_n(\Phi) = |||I_n||| = \sup_{\substack{g \in L_n^{\perp} \\ \|g\|_2 \leqq 1}} \|Ig\|_2 \geqq \sup_{If \in \Omega} E_n^{\Phi}(If) = E_n^{\Phi}(\Omega), \quad n = 0, 1, 2, \ldots.$$

From (26) and (27) we obtain that

(28) $$\varrho_n(\Phi) \geqq (2\pi/p)\big(1/(n+1)\big), \quad n = 0, 1, 2, \ldots.$$

Remark 2. From the above theorem and (28) it follows that for some orthonormal system $\Phi$ satisfying (12) and (13), the following statements are equivalent:

a) $$E_n^{\Phi}(f) \leqq C_5 \omega(f, 1/n), \quad f \in L^2[p], \quad n = 1, 2, \ldots,$$

b) $$(2\pi/p)\big(1/(n+1)\big) \leqq \varrho_n(\Phi) \leqq C_6(1/n), \quad n = 1, 2, \ldots,$$

where $\varrho_n(\Phi)$ is defined by (25); $C_5$ and $C_6$ denote absolute constants.

Remark 3. For the trigonometric system $T$, the following inequalities are valid (for $\varrho_n(T)=1/(n+1)$):

$$\begin{aligned} E_n^T(f) &\leqq C_7\varrho_n(T)\|f'\|, \quad \forall f\in L^2[2\pi],\ f'\in L^2[2\pi], \\ \|t_n'\| &\leqq C_8\varrho_n^{-1}(T)\|t_n\|, \quad \forall t_n\in T_n \end{aligned} \tag{29}$$

where $T_n$ denotes the set of all trigonometric polynomials of order at most $n$, and $C_7=C_8=1$. The two inequalities in (29) play an important role in the proofs of the direct and converse approximation theorems.

We can ask: is (29) true for an arbitrary system? The answer is that in general (29) is not true. Indeed, let us consider the following system. Let $n_0\geqq 1$ be a fixed integer. Let

$$T=\left\{\frac{1}{\sqrt{2\pi}},\ \frac{\cos kx}{\sqrt{\pi}},\ \frac{\sin kx}{\sqrt{\pi}}\right\}_{k=1}^{\infty}=\left\{\frac{1}{\sqrt{2\pi}};\ C_k(x),\ S_k(x)\right\}_{k=1}^{\infty}.$$

We consider the following system:

$$\begin{aligned} T^*=\{1/\sqrt{2\pi},\ &C_1, S_1, C_2, S_2, \dots, C_{n_0-1}, S_{n_0-1}, C_{n_0+1}, S_{n_0+1}, \\ &C_{n_0+2}, S_{n_0+2}, \dots, C_{n_0^2-1}, S_{n_0^2-1}, C_{n_0}, S_{n_0}, C_{n_0^2+1}, S_{n_0^2+1}, \\ &C_{n_0^2+2}, S_{n_0^2+2}, \dots, C_{n_0^4-1}, S_{n_0^4-1}, C_{n_0^2}, S_{n_0^2}, C_{n_0^4+1}, S_{n_0^4+1}, \dots\}. \end{aligned}$$

We have $\varrho_n(T^*)\sim 1/\sqrt{n}$. So the second inequality in (29) is not true for $\varrho_n^{-1}(T^*)\sim \sim\sqrt{n}$.

## References

[1] Л. В. Канторович—Г. П. Акилов, *Функциональный анализ*, Nauka (Москва, 1977).
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