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A reduction in case of compact Hamiltonian actions

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Dedicated to Professor K. Tandori on his 60th birthday

The classical results of Jacobi and Liouville on reduction of phase spaces were put in a general setting by E. Cartan which in the up-to-date formulation of R. ABRAHAM and J. MARSDEN ([1], p. 298) runs as follows: Let Q be a smooth manifold and ϱ a closed 2-form on Q then the *characteristic distribution* E of ϱ is given by the subspaces

$$E_z = \{v \mid \imath_v \varrho_z = 0, \quad v \in T_z Q\}, \quad z \in Q,$$

and the 2-form ϱ is said to be *regular* if E is a subbundle of TQ. If ϱ is regular then E proves to be an involutive distribution and thus generates the *characteristic foliation* $\mathscr{F}$ of ϱ on Q. If the quotient space $P = Q/\mathscr{F}$ admits a smooth manifold structure such that the canonical projection

$$\pi\colon Q \to Q/\mathscr{F} = P$$

is a submersion, then there is a unique sympletic form ω on P such that $\varrho = \pi^*\omega$ holds. In this case the symplectic manifold (P, ω) is called a *reduced phase space* and the above procedure is said to be a *reduction* producing it.

The existence of reductions in case of some Hamiltonian actions was observed by J. MARSDEN and A. WEINSTEIN [7]. In fact, let (P, ω) be a symplectic manifold, G a connected Lie group and

$$\Phi\colon G \times P \to P$$

a Hamiltonian action with a momentum mapping $J\colon P \to \mathfrak{g}^*$ which is equivariant with respect to Φ and to the coadjoint action Ad^* of G on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$. Assume that $\mu \in \mathfrak{g}^*$ is a regular value of the momentum mapping J then

$$Q_\mu = J^{-1}(\mu)$$

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is a smooth submanifold of P. Moreover, assume that the action of the isotropy subgroup G_μ on the manifold Q_μ is both free and proper, then the corresponding orbit space

$$P_\mu = Q_\mu / G_\mu$$

admits a smooth manifold structure such that the canonical projection

$$\pi_\mu \colon Q_\mu \to Q_\mu / G_\mu = P_\mu$$

is a submersion. Consider now the restriction ϱ_μ of the symplectic form ω to Q_μ, then the closed 2-form ϱ_μ proves to be regular, the leaves of its characteristic foliation being the orbits of G_μ on Q_μ. Moreover, there is a unique symplectic form ω_μ on P_μ such that

$$\varrho_\mu = \pi_\mu^* \omega_\mu$$

is valid. Thus the reduction procedure applies to (Q_μ, ϱ_μ) and yields the reduced phase space (P_μ, ω_μ). The above procedure is called the *Marsden—Weinstein reduction* and it has several important applications [8].

A generalization of the Marsden—Weinstein reduction is presented below in case of compact Hamiltonian actions. In fact, let (P, ω) be a symplectic manifold, G a compact connected Lie group and

$$\Phi \colon G \times P \to P$$

a Hamiltonian action with a momentum mapping $J \colon P \to \mathfrak{g}^*$ which is equivariant with respect to Φ and Ad^* and has regular elements of $\mathfrak{g}^*$ in its range. It is shown that in case of a $\mu \in \mathrm{Range}\, J$ the set

$$Q_\mu = J^{-1}(\mu)$$

is a smooth submanifold of P provided that $G(z)$ is a non-singular orbit of Φ for any $z \in Q_\mu$. Assuming that the orbits of the isotropy subgroup G_μ on Q_μ are all of the same type it is shown that the orbit space $P_\mu = Q_\mu / G_\mu$ admits a smooth manifold structure such that the canonical projection

$$\pi_\mu \colon Q_\mu \to Q_\mu / G_\mu = P_\mu$$

is a submersion. Moreover, the restriction ϱ_μ of ω to Q_μ proves to be regular and the leaves of its characteristic distribution are shown to be the orbits of G_μ on Q_μ. Thus a unique symplectic form ω_μ on P_μ with $\varrho_\mu = \pi_\mu^* \omega_\mu$ is obtained. Consequently, the reduction procedure applies to (Q_μ, ϱ_μ) and yields the reduced phase space (P_μ, ω_μ). A simple example in order to show that the generalization is essential is presented as well.

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First, a concise account of those facts is given which yield the prerequisites for the proof of the above mentioned result.

Two orbits of the action of a group are said to be of the same type if they have the same conjugacy class of isotropy subgroups. The orbit types of an action are relatively easy to survey in case of actions generated by compact connected Lie groups. Actually, a fundamental result on compact connected Lie group actions, the Principal Orbit Type Theorem, yields the following classification of the orbits of such an action: 1. There are *principal orbits,* they are all of the same type and of maximal dimension; the union of the principal orbits is an open everywhere dense subset of the manifold on which the group acts. 2. There may be *exceptional orbits;* they are also of maximal dimension but not of the same type as the principal ones. 3. There may be *singular orbits:* they are not of maximal dimension [6].

In case of the adjoint action Ad: $G\times\mathfrak{g}\to\mathfrak{g}$ of a compact connected Lie group, the regular elements of $\mathfrak{g}$ have principal orbits, there are no exceptional orbits, and the singular elements of $\mathfrak{g}$ have singular orbits.

If G is a compact Lie group then its Lie algebra $\mathfrak{g}$ is obtainable as a direct sum

$$\mathfrak{g}=\mathfrak{c}\oplus\mathfrak{u}$$

of a commutative and of a semisimple ideal. Consequently, an arbitrarily fixed interior product on $\mathfrak{c}$ and the negative of the Killing—Cartan form of $\mathfrak{u}$ yield an interior product $\langle\,,\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ which is invariant with respect to the adjoint action

$$\mathrm{Ad}\colon\ G\times\mathfrak{g}\to\mathfrak{g}.$$

The interior product $\langle\,,\rangle_{\mathfrak{g}}$ defines a vector space isomorphism $\mathfrak{g}\cong\mathfrak{g}^*$ which is equivariant with respect to the adjoint action and the coadjoint action of G. Thus the Lie algebra $\mathfrak{g}$ will be identified with its dual $\mathfrak{g}^*$ in what follows on account of the above given isomorphism. Consequently, by a momentum mapping the map

$$J\colon\ P\to\mathfrak{g}$$

will be meant subsequently which is obtained from a usual momentum mapping through the above given identification. Moreover, the equivariance of J is understood with respect to the actions Φ and Ad.

Let G be a compact connected Lie group, P a smooth manifold and

$$\Phi\colon\ G\times P\to P$$

a smooth action. It is a well-known fundamental fact, that there is a Riemannian metric $\langle\,,\rangle_P$ on P which is invariant with respect to the action Φ. Assume that there is a symplectic form ω on P which is left invariant by the action Φ; then a unique almost complex structure $\mathbf{J}\colon TP\to TP$ of P can be obtained such that

$$\langle X, Y\rangle_P=\omega(\mathbf{J}X, Y)$$

holds for any vector fields $X, Y \in \mathcal{T}(P)$ applying a basic construction ([1], pp. 172—174; [8]).

Moreover, it can be shown that J is equivariant with respect to the induced tangent action

$$T\Phi\colon G \times TP \to TP;$$

in other words, $\mathbf{J}_{z'} \circ T_z \Phi_g = T_z \Phi_g \circ \mathbf{J}_z$ holds for $z' = \Phi(g, z)$, $g \in G$, $z \in P$ where

$$\Phi_g\colon P \to P$$

is the transformation defined by $\Phi_g(z) = \Phi(g, z)$, $z \in P$, for $g \in G$ as usual [2], [9]. Consider for $z \in P$ the subspace $R_z^0 \subset T_z P$ defined by

$$R_z^0 = \{v \mid T_z \Phi_g v = v \quad \text{for} \quad g \in G_z^0, \quad v \in T_z P\}$$

where G_z^0 is the identity component of the isotropy subgroup G_z. Then the equivariance of J obviously implies that

$$\mathbf{J}_z R_z^0 = R_z^0$$

holds. Assume now that in addition to the former hypotheses the action Φ is Hamiltonian as well and that

$$J\colon P \to \mathfrak{g}$$

is an equivariant momentum mapping for Φ. Then according to earlier observations

$$\text{Kernel } T_z J = \mathbf{J}_z N_z G(z)$$

holds at any point $z \in P$ where $N_z G(z)$ is the normal space to the orbit $G(z)$ at z with respect to the Riemannian metric $\langle\, , \rangle_P$ given above [2], [9]. Consider now a point $z \in P$ such that $G(z)$ is a non-singular orbit; then $N_z G(z) \subset R_z^0$ holds and thus

$$\text{Kernel } T_z J = \mathbf{J}_z N_z G(z) \subset \mathbf{J}_z R_z^0 = R_z^0$$

holds in consequence of the preceding observations and assertions.

For some part of the subsequent arguments the fact is essential that the Riemannian metric $\langle\, , \rangle_P$ can be chosen so that it becomes Hermitian with respect to the almost complex structure J. In fact, starting with a Φ invariant Riemannian metric $\langle\, , \rangle_P$ and with J defined by $\langle\, , \rangle_P$ and ω the definition

$$2\langle X, Y \rangle_P^H = \langle \mathbf{J}X, \mathbf{J}Y \rangle_P + \langle X, Y \rangle_P, \quad X, Y \in \mathcal{T}(P)$$

yields a Hermitian metric $\langle\, , \rangle_P^H$ which is invariant with respect to the action Φ. Moreover, the equality

$$2\langle X, Y \rangle_P^H = \langle \mathbf{J}Y, \mathbf{J}X \rangle_P + \langle X, Y \rangle_P = \omega(-Y, \mathbf{J}X) + \omega(\mathbf{J}X, Y) = 2\omega(\mathbf{J}X, Y),$$

$$X, Y \in \mathcal{T}(P)$$

shows that $\langle\, ,\rangle_P^H$ and $\langle\, ,\rangle_P$ are in the same relation to ω; consequently, ω and $\langle\, ,\rangle_P^H$ define the same almost complex structure as ω and $\langle\, ,\rangle_P$. Thus there is no loss of generality by assuming that $\langle\, ,\rangle_P$ is already Hermitian with respect to J.

The following theorem concerns the originally indicated objective, a generalization of the Marsden—Weinstein reduction in case of compact Hamiltonian actions.

Theorem. *Let (P, ω) be a symplectic manifold, G a compact connected Lie group and*

$$\Phi\colon G\times P \to P$$

a Hamiltonian action with an equivariant momentum mapping $J\colon P\to\mathfrak{g}$ which has regular elements of $\mathfrak{g}$ in its range. If $\mu\in\mathfrak{g}$ is in the range of J and such that $G(z)$ is a non-singular orbit for

$$z\in Q_\mu = J^{-1}(\mu)$$

then Q_μ is a smooth submanifold of P. Furthermore, if the orbits of the isotropy subgroup G_μ on Q_μ are all of the same type then the orbit space $P_\mu = Q_\mu/G_\mu$ admits a smooth manifold structure such that the canonical projection

$$\pi_\mu\colon Q_\mu \to Q_\mu/G_\mu$$

is a submersion and ϱ_μ, the restriction of ω to Q_μ, is regular, the leaves of its characteristic foliation being orbits of G_μ. Moreover, there is a unique symplectic form ω_μ on P_μ such that

$$\varrho_\mu = \pi_\mu^*\omega_\mu$$

holds. Thus the reduction applies to (Q_μ, ϱ_μ) and yields the reduced phase space (P_μ, ω_μ).

Proof. Let P' be the union of the principal orbits of the action Φ, then the isotropy subgroups are all conjugate in points of P'. The fact, that the set of regular elements of $\mathfrak{g}$ is open, the assumption, that the range of J contains regular elements of $\mathfrak{g}$ and the fact, that P' is everywhere dense in P together imply that there is a $z\in P'$ such that $J(z)$ is a regular element of $\mathfrak{g}$. Thus the preceding observations and the equivariance of J entail that the isotropy subgroups in points of P' are all conjugate to a closed subgroup of an arbitrary maximal torus T of G. Obviously the same holds for the identity components of the isotropy subgroups in points of the exceptional orbits of the action Φ.

Consider an element μ in the range of J such that $G(z)$ is a non-singular orbit for

$$z\in Q_\mu = J^{-1}(\mu).$$

It will be shown that the isotropy subalgebra $\mathfrak{g}_z$ as function of $z\in Q_\mu$ is constant on each connected component of the set Q_μ.

In fact, consider a point $z_0 \in Q_\mu$; then since $G(z)$ is a non-singular orbit, the identity component G_z^0 of the isotropy subgroup G_z is commutative by the preceding observation. Furthermore, by an earlier result already mentioned above

$$\operatorname{Ker} T_z J = \mathbf{J}_z N_z G(z) \subset R_z^0$$

holds where $R_z^0 \subset T_z P$ is the subspace of vectors left invariant by the identity component of the isotropy subgroup. Consider the orthogonal decomposition

$$\mathfrak{g}_\mu = \mathfrak{r}_z \oplus \mathfrak{g}_z,$$

then $\mathfrak{r}_z$ is mapped into $\operatorname{Ker}(T_z J \restriction T_z G(z)) \subset R_z^0 \cap T_z G(z)$ under the canonical isomorphism

$$\mathfrak{m}_z \to T_z G(z) \quad (\mathfrak{m}_z \text{ is the orthogonal complement of } \mathfrak{g}_z \text{ in } \mathfrak{g}).$$

Since the above isomorphism is equivariant with respect to the restricted adjoint action of G_z on $\mathfrak{m}_z$ and the isotropy action of G_z on $T_z G(z)$, the following holds

$$[\mathfrak{r}_z, \mathfrak{g}_z] = \{0\}.$$

The preceding observations obviously yield now that the following is valid as well

$$[\mathfrak{g}_\mu, \mathfrak{g}_z] = [\mathfrak{r}_z \oplus \mathfrak{g}_z, \mathfrak{g}_z] \subset [\mathfrak{r}_z, \mathfrak{g}_z] + [\mathfrak{g}_z, \mathfrak{g}_z] = \{0\}.$$

Since the isotropy subalgebras of the restricted action of G_μ on Q_μ are all conjugate in $\mathfrak{g}_\mu$, and since by the equivariance of J they coincide with the isotropy subalgebras of the action of G, the assertion that $\mathfrak{g}_z$ as function of z is constant on the connected components of Q_μ follows.

Consider now an element $\mu \in \mathfrak{g}$ such that the orbits of the points $z \in Q_\mu = J^{-1}(\mu)$ are all non-singular. Let Q_μ^0 be a connected component of Q_μ, then the flat submanifold

$$\mathfrak{q}_\mu^0 = \mu + \mathfrak{g}_z$$

does not depend on the choice of z in Q_μ^0 according to the preceding observation. Fix a conic neighbourhood C of $\mathfrak{m}_z$ in $\mathfrak{g}$ such that $C \cap \mathfrak{g}_z = \{0\}$. Let W be an open and connected neighbourhood of Q_μ^0 which is disjoint from the other components of Q_μ and such that $\mathfrak{m}_x \subset C$ for $x \in W$. It will be shown now that

$$J(W) \cap \mathfrak{q}_\mu^0 = \{\mu\}$$

is valid. In fact, consider an $x \in W$ such that $J(x) = \xi \in \mathfrak{q}_\mu^0$ holds. Then there is a smooth curve $\varphi\colon [0, 1] \to W$ with $\varphi(0) = z \in Q_\mu^0$, $\varphi(1) = x$. Consider now the curve

$$\psi = J \circ \varphi\colon [0, 1] \to \mathfrak{g}.$$

Let $\mathfrak{m}_{\varphi(\tau)}$ be the orthogonal complement of $\mathfrak{g}_{\varphi(\tau)}$, then by preceding stipulations the following holds:

$$\dot{\psi}(\tau) = T_{\varphi(\tau)} J \dot{\varphi}(\tau) \in \mathfrak{m}_{\varphi(\tau)} \subset C \quad \text{for} \quad \tau \in [0, 1].$$

Consequently, $\xi-\mu=\psi(1)-\psi(0)\in C$ holds. On the other hand $\xi-\mu\in\mathfrak{g}_z$ is valid. Thus

$$\xi = \mu$$

follows by the definition of C. Next, it will be shown that the restricted map $J\restriction W$ is transversal to the submanifold $\mathfrak{q}_\mu^0$. In fact, assume that $J(x)\in\mathfrak{q}_\mu^0$ holds for some $x\in W$. Then $x\in Q_\mu^0$ and $J(x)=\mu$ by the preceding observation. Consequently, former assertions yield that

$$T_\mu\mathfrak{g} = \mathfrak{g} = \mathfrak{m}_x\oplus\mathfrak{g}_x = T_xJ(T_xW)\oplus T_\mu\mathfrak{q}_\mu^0$$

is valid which yields the transversality of $J\restriction W$. But the transversality of $J\restriction W$ entails that

$$Q_\mu^0 = (J\restriction W)^{-1}(\mathfrak{q}_\mu^0)$$

is a smooth submanifold by a fundamental theorem on transversal maps ([5], pp. 22—23).

Moreover, the same theorem yields that

$$\dim P - \dim Q_\mu^0 = \operatorname{codim} Q_\mu^0 = \operatorname{codim} \mathfrak{q}_\mu^0 = \dim\mathfrak{g} - \dim\mathfrak{g}_z = \dim G(z).$$

Consequently, all the connected components of Q_μ are of the same dimension. Thus Q_μ is a smooth submanifold of P.

The second assertion of the theorem that if the orbits of the points of Q_μ are all of the same type then the orbit space Q_μ/G_μ admits a smooth manifold structure such that

$$\pi_\mu\colon Q_\mu \to Q_\mu/G_\mu$$

is a submersion is a direct consequence of a basic theorem on orbit spaces of actions with a single orbit type ([6], pp. 6—9).

In order to prove the third assertion of the theorem, that ϱ_μ the restriction of the symplectic form ω to Q_μ is regular, consider the above defined invariant Riemannian metric $\langle\,,\rangle_P$ and the almost complex structure $\mathbf{J}$ determined by $\langle\,,\rangle_P$ and ω on P. According to former observations already mentioned above, the following holds

$$T_zQ = \operatorname{Kernel} T_zJ = \mathbf{J}_z N_z G(z) \quad \text{for} \quad z\in Q_\mu.$$

Let now ϱ_μ be the restriction of ω to the submanifold Q_μ. Then the characteristic distribution E of ϱ_μ is given by the subspaces

$$E_z = \{v \mid \iota_v\varrho_\mu = 0, \quad v\in T_zQ_\mu\}, \quad z\in Q_\mu.$$

According to former observations the following equalities are valid:

$$(\iota_v\varrho_\mu)(u) = \omega(v, u) = \langle\mathbf{J}_z v, u\rangle_P \quad \text{where} \quad u, v\in T_zQ_\mu.$$

Consequently, the subspace E_z is formed by those vectors $v \in T_z Q_\mu$ which satisfy the following condition:

$$\mathbf{J}_z v \perp T_z Q_\mu = \mathbf{J}_z N_z G(z).$$

Since by a former observation $\langle , \rangle_P$ is Hermitian therefore $\mathbf{J}_z\colon T_z P \to T_z P$ is an isometry and consequently the preceding condition is equivalent to the following one:

$$v \perp N_z G(z), \quad v \in T_z Q_\mu.$$

Consequently, the characteristic subspace E_z can be given as follows:

$$E_z = \text{Kernel } T_z J \cap T_z G(z) = T_z G_\mu(z).$$

Therefore E is integrable and its leaves are the orbits of the action of G_μ on the submanifold Q_μ; since these orbits are all of the same type by assumption, Q_μ is a fiber bundle over Q_μ/G_μ by a basic theorem ([6], pp. 6—9). Consequently, the characteristic distribution E of ϱ_μ is regular.

The existence of the symplectic form ω_μ is now a direct consequence of the fact that the natural projection

$$\pi_\mu\colon Q_\mu \to Q_\mu/G_\mu$$

is a submersion.

Remark 1. The question that in case of a Hamiltonian action $\Phi\colon G \times P \to P$ of a compact connected Lie group G with an equivariant momentum mapping $J\colon P \to \mathfrak{g}$ having regular elements of $\mathfrak{g}$ in its range, which are those elements of $\mathfrak{g}$ where the preceding theorem applies, seems to be open. In fact, if $G(z)$ is a principal orbit of Φ, then, as it was observed above, G_z is conjugate to a closed subgroup of a maximal torus T of G. Thus, provided that P is compact, a result of Guillemin and Sternberg [3] applies and yields that for any Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{g}$, the set

$$\mathfrak{t}_+ \cap J(P')$$

is the union of a finite number of open r-dimensional convex polytopes $\mathfrak{p}_1, \ldots, \mathfrak{p}_l \subset \subset \mathfrak{t}_+$ where $r = \text{rank } G - \dim G_z$. Thus $\mu \in \mathfrak{t}_+ \cap J(P')$ corresponds to the assumption of the theorem if it is not on the boundary of any one among the polytopes $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$. Moreover, the theorem applies at any point of $\mathfrak{t}_+ \cap J(P')$ provided that the conjecture of Guillemin and Sternberg that

$$\mathfrak{t}_+ \cap J(P')$$

itself is a single r-dimensional open convex polytope [3] proves to be valid. The question, which are those points of $J(P')$ on the boundary of $\mathfrak{t}_+$ where the theorem applies seems to be open, too.

Remark 2. The fact that the Marsden—Weinstein reduction in case of compact Hamiltonian actions is included in the preceding theorem can be verified as follows. Let $\mu\in\mathfrak{g}$ be a regular value of the momentum mapping J. Then $\mathfrak{g}_z=\{0\}$ for any $z\in J^{-1}(\mu)$ by a result of MARSDEN [8]; consequently $G(z)$ is non-singular orbit in case of $z\in J^{-1}(\mu)$. If μ is a regular value of J then the range of J includes a neighbourhood of μ. But the set of regular elements of $\mathfrak{g}$ is everywhere dense in $\mathfrak{g}$ and P' is everywhere dense in P; consequently, there is a $z'\in P'$ such that $J(z')$ is a regular element of $\mathfrak{g}$.

Remark 3. An example is presented below in order to show that there are cases where the Marsden—Weinstein reduction does not apply, however, the one given by the preceding theorem does so.

Let M be a Riemannian manifold with Riemannian metric $\langle\, ,\rangle_M$, G a compact connected Lie group and

$$\alpha\colon G\times M\to M$$

an isometric action. Consider the tangent bundle $P=TM$ with its canonical symplectic form ([1], pp. 182—183); then the induced action $\Phi=T\alpha$ of G on P is symplectic. Moreover, the action Φ is Hamiltonian, since an equivariant momentum mapping $J\colon P\to\mathfrak{g}^*\cong\mathfrak{g}$ is defined for Φ by

$$\langle J(v), X\rangle = \langle v, \overline{X}(z)\rangle_M, \quad v\in T_zM,\ X\in\mathfrak{g},$$

according to Noether's Theorem ([1], pp. 282—285); here of course $\overline{X}$ is the infinitesimal generator of α given by X.

Let now $(X_1, \ldots, X_n)$ be an orthonormal base of $\mathfrak{g}$ then obviously

$$J(v) = \sum_{i=1}^{n} \langle v, \overline{X}_i(z)\rangle_M X_i, \quad v\in TM,$$

holds. If in particular G is semisimple then $\langle\, ,\rangle_{\mathfrak{g}}$ the interior product of $\mathfrak{g}$ is given by the negative of the Killing—Cartan form of $\mathfrak{g}$ according to its definition.

Consider now in particular an m-dimensional Riemannian symmetric space $M=G/H$ where G is compact and semisimple and the Riemannian metric $\langle\, ,\rangle_M$ is defined by the negative of the Killing—Cartan form of $\mathfrak{g}$. Consider the canonical decomposition

$$\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}$$

and let the orthonormal base $(X_1, \ldots, X_m)$ of $\mathfrak{g}$ be compatible with this decomposition. Then

$$(\overline{X}_1(0), \ldots, \overline{X}_m(0))$$

is an orthonormal base of T_0M where $o=H\in G/H$. Consequently, the following holds:

$$J(v) = \sum_{i=1}^{m} \langle v, \bar{X}_i(0)\rangle_M X_i, \quad v \in T_0 M.$$

But then $J(T_0M) = \mathfrak{m}$ is obviously valid.

A more particular case is obtained as follows. Let A be a compact semisimple Lie group and consider the left action

$$\lambda\colon (A \times A) \times A \to A$$

of the direct product $A \times A$ on A given by $\lambda((g, h), a) = gah^{-1}$ for $a, g, h \in A$. Then A is canonically a Riemannian symmetric space for the action λ ([4], pp. 223—224). In fact, if

$$G = A \times A$$

then A can be obtained as the canonical homogeneous coset space G/H where H is the diagonal in the direct product. Consequently, the canonical decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is given now by

$$\mathfrak{m} = \{(X, -X) \mid X \in \mathfrak{a}\}, \quad \mathfrak{h} = \{(X, X) \mid X \in \mathfrak{a}\}.$$

An element (X, Y) of the semisimple Lie algebra $\mathfrak{g}$ is regular if and only if both X and Y are regular elements of $\mathfrak{a}$. Therefore, if $X \in \mathfrak{a}$ is a regular element then

$$(X, -X) \in \mathfrak{m}$$

is a regular element of $\mathfrak{g}$. But then the above observation yields that

$$\mathfrak{m} = J(T_0M) \subset J(TM) = J(P)$$

holds and consequently, $J(P)$ contains regular elements of $\mathfrak{g}$. However, Remark 1 does not apply in this case since $P = TM$ is not compact.

In order to show that the momentum mapping J considered above has no regular values, it is sufficient to see that Φ has no discrete isotropy subgroups; since the existence of a regular value of J implies the existence of trivial isotropy algebras by a result of MARSDEN [8]. Since α is transitive action, every orbit of the action $\Phi = T\alpha$ intersects the tangent space $T_0M \cong \mathfrak{m}$. Moreover, the isotropy subgroup of the action Φ at a point

$$(X, -X) \in \mathfrak{m} \cong T_0M$$

is a subgroup of H. But as a simple calculation shows the following holds:

$$\Phi((g, g), (X, -X)) = (\mathrm{Ad}(g)X, -\mathrm{Ad}(g)X) \quad \text{where} \quad X \in \mathfrak{a}, \ g \in A.$$

Consequently, the isotropy subgroup of Φ cannot be discrete at $(X, -X)$; in fact, the principal isotropy subgroups of Φ at points of $\mathfrak{m}$ are given by a suitable, maximal torus T of A as the subgroup $\{(g, g) \mid g \in T\} \subset H$. The existence of values $\mu \in J(P) \cap \mathfrak{g}$ where the theorem applies is a consequence of the above observation.

References

[1] R. Abraham and J. E. Marsden, *Foundations of Mechanics*, Second edition, The Benjamin/Cummings Publishing Company (Reading, Massachusetts, 1978).

[2] J. M. Arms, J. E. Marsden and V. Moncrief, Symmetry and bifurcations of momentum mappings, *Comm. Math. Phys.*, **78** (1981), 445—478.

[3] V. Guillemin and S. Sternberg, Convexity properties of the moment mapping, *Invent. Math.*, **67** (1982), 491—513.

[4] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press (New York, 1978).

[5] M. W. Hirsch, *Differential Topology*, Springer Verlag (New York—Heidelberg—Berlin, 1976).

[6] K. Jänich, *Differenzierbare G-Mannigfaltigkeiten*, Lecture Notes in Mathematics, vol. 59, Springer Verlag (Berlin, 1968).

[7] J. E. Marsden and A. Weinstein, Reduction of symplectic manifolds with symmetry, *Rep. Math. Phys.*, **5** (1974), 121—130.

[8] J. E. Marsden, *Lectures on Geometric Methods in Mathematical Physics*, Society for Industrial and Applied Mathematics (Philadelphia, 1981).

[9] J. Szenthe, On symplectic actions of compact Lie groups with isotropy subgroups of maximal rank, *Acta Sci. Math.*, **45** (1983), 381—388.

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