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The independence of the distributivity conditions in groupoids

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Dedicated to Professor K. Tandori on his 60th birthday

1. Introduction

In the early fifties L. Rédei raised the following problem. The associativity of a binary operation — denoted as multiplication — on a set A means the fulfilment of a set of n^3 equations where $n = \operatorname{card} A$, namely, $a(bc)=(ab)c$ for all $(a, b, c) \in A^3$. He asked whether there exists a proper subset $B \subset A^3$ such that if the equation $x(yz)=(xy)z$ holds for all $(x, y, z) \in B$, then the operation is associative on A, i.e., the groupoid $\langle A, \cdot \rangle$ is a semigroup. We say that the associativity conditions are *independent* over the set A, iff there is no such proper subset B of A^3.

This problem was solved by G. Szász [2] in 1953. He proved, that the associativity conditions are independent over any set of at least four elements, but they are not independent over sets of two or three elements.

Analogous notion of independence may also be introduced for other kinds of identities. Thus, in 1954 R. Wiegandt [3] and later R. Wiegandt and J. Wiesenbauer [4] made similar investigations on the distributivity of two binary operations.

Recently we have proved in [1], that the mediality conditions for groupoids — a groupoid is medial (entropic or Abelian according to other terminologies), if it satisfies the identity

$$(xy)(zu) = (xz)(yu)$$

— are independent over a set A if and only if A consists of at least four elements and in sets of at most three elements we have the proper subset mentioned in the original problem.

In the present note we investigate the distributivity conditions for groupoids. A groupoid $\langle A, \cdot \rangle$ is *distributive* if the following two identities hold

$$x(yz) = (xy)(xz), \tag{1}$$

$$(xy)z = (xz)(yz). \tag{2}$$

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If we have (1) (or (2)) only, we say that the groupoid $\langle A, \cdot \rangle$ is *left- (right-) distributive* or simply *semidistributive*.

We prove that the distributivity conditions are independent over a set A if and only if A consists of at least four elements, while the semidistributivity conditions are independent over A exactly when A contains at least three elements.

2. Preliminaries

Let A be any set and let $a, b, c, \ldots$ denote distinct and $x, y, z, \ldots$ arbitrary elements of A. We say that the (ordered) triplet (x, y, z) is *left- (right-) distributive* in the groupoid $\langle A, \cdot \rangle$, if $x(yz)=(xy)(xz)$ $\big((xy)z=(xz)(yz)\big)$. The triplet (x, y, z) is *distributive* in $\langle A, \cdot \rangle$, if it is both left- and right-distributive. The triplet (x, y, z) is *left- (right-) isolated over the set* A, if there exists a binary operation $\circ$ on A such that all triplets but this one are left- (right-) distributive in $\langle A, \circ \rangle$. The triplet (x, y, z) is *isolated* over A, if there exists a binary operation $*$ on A such that all triplets but this one are distributive in the groupoid $\langle A, * \rangle$. In these cases we say that the triplet (x, y, z) is *left- (right-) isolated* or *isolated* in the *groupoid* $\langle A, \circ \rangle$ or $\langle A, * \rangle$, respectively.

The triplets (x, y, z), $(x', y', z') \in A^3$ are of the same *type*, if there exists a permutation φ on A such that $x'=x\varphi$, $y'=y\varphi$ and $z'=z\varphi$.

Proposition 1. *If the triplet (x, y, z) is left-distributive (left-isolated) in the groupoid $\langle A, \cdot \rangle$, then there exists a binary operation $*$ on A such that (x, y, z) is right-distributive (right-isolated) in $\langle A, * \rangle$.*

Proof. The operation $*$ is defined by $u * v = vu$.

By this proposition in the case of semidistributivity we can restrict ourselves to left-distributivity and left-isolatedness.

Proposition 2. *If the triplets (x, y, z) and (x', y', z') over A are of the same type, moreover, (x, y, z) is distributive (left-distributive) in $\langle A, \cdot \rangle$, then there exists a binary operation $*$ on A such that (x', y', z') is distributive (left-distributive) in $\langle A, * \rangle$.*

Proof. If φ denotes the suitable permutation on A, then put

$$u * v = (u\varphi^{-1} \cdot v\varphi^{-1})\varphi.$$

Corollary. *If the triplets (x, y, z) and (x', y', z') are of the same type, moreover, (x, y, z) is isolated (left-isolated) over A, then (x', y', z') is isolated (left-isolated) over A, too.*

By the above Proposition 2 and its Corollary we have to deal with the following five types of triplets only:

(α) (a, a, a), (β) (a, a, b), (γ) (a, b, a), (δ) (a, b, b), (ε) (a, b, c).

Lemma 1. *If o is a left (right) zero element of the groupoid $\langle A, \cdot\rangle$, then for any $x, y\in A$ the triplet (o, x, y) (the triplet (x, y, o)) is distributive in $\langle A, \cdot\rangle$.*

Lemma 2. *If o is a zero element of $\langle A, \cdot\rangle$, then all triplets over A containing o are distributive.*

Lemma 3. *If e is a left unit element of $\langle A, \cdot\rangle$, then for any $x, y\in A$ the triplet (e, x, y) is left-distributive in $\langle A, \cdot\rangle$.*

Lemma 4. *Let (x, y, z) be an isolated (left-isolated) triplet in the groupoid $\langle A, \cdot\rangle$, and let $A'\supseteqq A$. Then there exists an extension $\langle A', \circ\rangle$ of $\langle A, \cdot\rangle$ such that (x, y, z) is isolated (left-isolated) in $\langle A', \circ\rangle$, too.*

Proof. For any $u, v\in A'$ define

$$u\circ v=\begin{cases} uv & \text{if } u, v\in A,\\ d & \text{otherwise,}\end{cases}$$

where d is an arbitrary fixed element of $A'\setminus A$.

3. Semidistributivity

Theorem 1. *The semidistributivity conditions are independent over a set A if and only if* card $A\geqq 3$.

To prove this theorem our first step will be the following

Lemma 5. *Let $A=\{a, b\}$. If the triplets (a, a, a), (a, b, b), (b, a, a) and (b, b, b) are left-distributive in a groupoid $\langle A, \cdot\rangle$, then this groupoid is left-distributive.*

Proof. There are sixteen binary operations on the set A. Six of them are distributive and additional three are left-distributive. If we examine the remaining seven groupoids, then we get that in each of them one or more triplets mentioned in the lemma are not left-distributive, which proves the lemma.

Proof of Theorem 1. The necessity is implied by Lemma 5. To prove the sufficiency, by Lemma 4, we have to present a suitable binary operation on the set

$A=\{a, b, c\}$ for each of the five types.

Type (α)
$$\begin{array}{c|ccc} & a & b & c \\ \hline a & c & b & b \\ b & a & b & c \\ c & a & b & c \end{array}.$$

Here b and c are left unit elements, therefore, by Lemma 3, it remains to check the triplets (a, x, y). If $x, y \in \{b, c\}$, then $a(xy)=ay=b=bb=(ax)(ay)$, moreover, $a(ax)=ab=b$ and $(aa)(ax)=cb=b$, while $a(xa)=aa=c$ and $(ax)(aa)=bc=c$. Finally, $a(aa)=ac=b$ but $(aa)(aa)=cc=c$, i.e., the triplet (a, a, a) is left-isolated over A.

Type (β)
$$\begin{array}{c|ccc} & a & b & c \\ \hline a & c & b & c \\ b & c & c & c \\ c & c & c & c \end{array}.$$

In this groupoid the only product unequal to c is $ab=b$. Therefore the triplet (a, a, b) is left-isolated.

Type (γ)
$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & c & c \\ b & b & b & b \\ c & a & b & c \end{array}.$$

Now b is a left zero and c is a left unit element, therefore the triplets (b, x, y) and (c, x, y) are left-distributive for any $x, y \in A$ by Lemmas 1 and 3. Furthermore,

$$a(xy)=\begin{cases} a & \text{if} \quad x=y=a \quad \text{or} \quad x=c, \ y=a, \\ c & \text{otherwise} \end{cases}$$

and

$$(ax)(ay)=\begin{cases} a & \text{if} \quad x=y=a \quad \text{or} \quad x=b, \ y=a \quad \text{or} \quad x=c, \ y=a, \\ c & \text{otherwise,} \end{cases}$$

which shows us, that (a, b, a) is left-isolated over A.

Type (δ)
$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & c & a \\ b & a & b & c \\ c & a & a & a \end{array}.$$

In this case a is a right zero element, moreover, b is a left unit element, therefore the triplets (x, y, a) and (b, x, y) are left-distributive for any $x, y \in A$. Since for any $x, y \in A$, $c(xy)=(cx)(cy)=a$, moreover,

$$a(xy)=\begin{cases} c & \text{if} \quad x=y=b, \\ a & \text{otherwise,} \end{cases}$$

and $(ax)(ay)=a$, we have got that the triplet (a, b, b) is left-isolated over A.

Type (ε)

	a	b	c
a	a	a	c
b	a	a	a
c	a	a	a

Here the only product different from a is $ac=c$, thus

$$x(yz) = \begin{cases} c & \text{if } x=y=a \text{ and } z=c, \\ a & \text{otherwise,} \end{cases}$$

and

$$(xy)(xz) = \begin{cases} c & \text{if } x=y=a, \quad z=c \text{ or } x=a, \quad y=b, \quad z=c \\ a & \text{otherwise} \end{cases}$$

which shows the left-isolatedness of the triplet (a, b, c).

4. Distributivity

Theorem 2. *The distributivity conditions are nonindependent over a set A, if* card $A \leqq 3$.

Proof. From Lemma 5 and its dual statement we obtain that the distributivity conditions are nonindependent over a set A, if card $A=2$. To settle the case card $A=3$ we prove the following claim:

Let $A=\{a, b, c\}$ and suppose that all the triplets (a, a, x), (x, a, a), (a, x, x) and (x, x, a) are distributive in the groupoid A, for any $x \in \{b, c\}$. Then the triplet (a, a, a) is also distributive.

It is obvious, that in the case of $aa=a$ the triplet (a, a, a) is distributive. If (a, a, a) is not left-distributive, then we have the following three possibilities:

(i) $\quad a(aa)=a$ and $(aa)(aa)=x$,

(ii) $\quad a(aa)=x$ and $(aa)(aa)=a$,

(iii) $\quad a(aa)=x$ and $(aa)(aa)=y$,

where $x, y \in \{b, c\}$ and $x \neq y$.

We shall show that from each of the above cases a contradiction can be derived.

Case (i) If $aa=x$, then $ax=a$ and $xx=x$, which leads to $a(xx)=ax=a$ and $(ax)(ax)=aa=x$, i.e., the triplet (a, x, x) is not left-distributive. If $aa=z$, where $z \neq a, x$; then $az=a$ and $zz=x$. Consider the triplet (a, a, z), which is distributive under our assumption. But $(aa)z=zz=x$ and $(az)(az)=aa=z$, thus we have a contradiction.

Case (ii). If $aa=x$, then our assumption implies $ax=x$ and $xx=a$. These contradict the left-distributivity of the triplet (a, a, x). Namely, $a(ax)=ax=x$ and $(aa)(ax)=xx=a$. If $aa=z$, where $z\neq a, x$, then we have $az=x$ and $zz=a$. From the right-distributivity of the triplet (a, a, z) we get $xx=a$ and from the left-distributivity of (a, z, z) we obtain $xx=z$, whence $a=z$, a contradiction.

Case (iii). Let $aa=x$. Then $ax=x$ and $xx=y$, which imply the nondistributivity of the triplet (a, a, x), indeed, $a(ax)=ax=x$ and $(aa)(ax)=xx=y$. If $aa=y$, then $ay=x$ and $yy=y$. The right-distributivity of the triplet (a, a, y) implies $xx=y$, while from the left-distributivity of (a, y, y) we infer that $xx=x$, i.e., $x=y$, a contradiction.

If we dualize the above procedure, i.e., consider the triplet (u, v, w) and left- (right-) distributivity instead of the triplet (w, v, u) and right- (left-) distributivity in each step respectively, then we get that the triplet (a, a, a) is right-distributive, thus it is distributive.

Theorem 3. *The distributivity conditions are independent over a set A if and only if* card $A\geqq 4$.

Proof. The necessity is the content of Theorem 2. To prove the sufficiency let $A=\{a, b, c\}$ first, and we define suitable binary operations on A which show the isolatedness of the triplets of types (β), (γ), (δ) and (ε) over A.

Type (β)

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & c \\ b & c & c & c \\ c & c & c & c \end{array}.$$

This groupoid is left-distributive, namely, for any $x, y, z\in A$

$$x(yz)=\begin{cases} a & \text{if } x=y=z=a, \\ b & \text{if } x=y=a \quad \text{and} \quad z=b, \\ c & \text{otherwise,} \end{cases}$$

$$(xy)(xz)=\begin{cases} a & \text{if } x=y=z=a, \\ b & \text{if } xy=a, \ xz=b, \ \text{i.e.,} \ x=y=a, \ z=b, \\ c & \text{otherwise.} \end{cases}$$

On the other hand

$$(xy)z=\begin{cases} a & \text{if } x=y=z=a, \\ b & \text{if } x=y=a, \ z=b, \\ c & \text{otherwise,} \end{cases}$$

$$(xz)(yz)=\begin{cases} a & \text{if } x=y=z=a, \\ c & \text{otherwise,} \end{cases}$$

which shows us, that the triplet (a, a, b) is not right-distributive, i.e., it is isolated.

Type (γ)

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & c & c \\ b & a & b & c \\ c & c & c & c \end{array}.$$

This groupoid is idempotent, therefore all triplets of the forms of (x, x, y) and (x, y, y) are distributive. Indeed,

$$x(xy) = (xx)(xy) \quad \text{and} \quad (xx)y = xy = (xy)(xy),$$

moreover,

$$x(yy) = xy = (xy)(xy) \quad \text{and} \quad (xy)y = (xy)(yy).$$

Since c is a zero element, thus by Lemma 2 all triplets containing c are distributive. It remains to check the triplets (a, b, a) and (b, a, b). After an easy computation we get that the only nondistributive triplet is (a, b, a), i.e., it is isolated over A.

Type (δ)

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & c & a & c \\ b & c & c & c \\ c & c & c & c \end{array}.$$

The only product unequal to c is $ab=a$, for this reason $x(xy)=(xy)(xz)=c$ and $(xz)(yz)=c$ for any $x, y, z, \in A$. But

$$(xy)z = \begin{cases} a & \text{if} \quad x = a \quad \text{and} \quad y = z = b, \\ c & \text{otherwise}, \end{cases}$$

which shows the isolatedness of the triplet (a, b, b).

Type (ε)

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & a \\ b & a & b & c \\ c & a & b & c \end{array}.$$

The elements b and c are left units, so the triplets (b, x, y) and (c, x, y) are left-distributive for any $x, y \in A$. Moreover, a and b are right zero elements, therefore, the triplets (x, y, a) and (x, y, b) are distributive for any $x, y \in A$. Since $a(xc)=a$ and $(ax)(ac)=a$ for arbitrary $x \in A$, the groupoid is left-distributive. For any $x \in A$ we have $(xa)c=ac=a$ and $(xc)(ac)=(xc)a=a$, $(xb)c=bc=c$ and

$$(xc)(bc) = (xc)c = \begin{cases} a & \text{if} \quad x = a, \\ c & \text{otherwise}, \end{cases}$$

and $(xc)c=(xc)(cc)$. These show, that the triplet (a, b, c) is isolated in this groupoid.

We have seen in Theorem 2 that the triplets of type (α) are nonisolated over sets of three elements, but they are isolated over four elements sets. Namely, consider the following operation on the set $A=\{a, b, c, d\}$:

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & b & d & d & d \\ b & c & d & d & d \\ c & d & d & d & d \\ d & d & d & d & d \end{array}$$

There are exactly two products different from d: $aa=b$ and $ba=c$. Since $x(yz)= =(xy)(xz)=d$ for all $x, y, z\in A$, this groupoid is left-distributive, but

$$(xy)z = \begin{cases} c & \text{if} \quad x=y=z=a, \\ d & \text{otherwise}, \end{cases}$$

and $(xz)(yz)=d$ for all $x, y, z\in A$, which lead to the isolatedness of the triplet (a, a, a) over A.

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References

[1] L. Klukovits, On the independence of the mediality conditions, in: *Contribution to General Algebra 2* (Proc. Conf. Klagenfurt, 1982), Hölder—Pichler—Tempsky (Wien) and G. B. Teubner (Stuttgart, 1983); pp. 209—214.

[2] G. Szász, Die Unabhängigkeit der Associativitätsbedingungen, *Acta Sci. Math.*, **15** (1953), 20—28.

[3] R. Wiegandt, Axiomatic investigations in rings, unpublished manuscript in Hungarian, 1954.

[4] R. Wiegandt and J. Wiesenbauer, Unabhängigkeit der Distributivitätsbedingungen, *Atti Accad. Naz. Lincei*, **58** (1975), 819—822.

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