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On homomorphic images of normal complexes in varieties of semigroups

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Following LJAPIN [2] we call a subset T of a semigroup S a normal complex if T satisfies the condition:

$$xt_1y\in T \Leftrightarrow xt_2y\in T \quad \text{for all} \quad x,y\in S^1 \quad \text{and} \quad t_1, t_2\in T.$$

A subset of a semigroup is a normal complex if and only if it is a congruence class for some congruence relation. Under a homomorphism of S, the image of a normal complex need not be a normal complex of the image of S. In connection with the investigation of M-radicals introduced by HOEHNKE [1], the question arises, which subsemigroups T of a semigroup S are homomorphic images of normal complexes under homomorphisms of semigroups from the given class onto S (STRECKER [4]). In the class of all semigroups it is easily seen that there are a semigroup S', a normal complex T' of S' and a surjection φ of S' onto S with $\varphi(T')=T$, where T is an arbitrary subsemigroup of S. In the present paper we consider an arbitrary semigroup variety V and describe, which subsemigroups are homomorphic images of normal complexes of semigroups from V. The result we obtain generalizes the corresponding one on monoids (see [3]).

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Let V be a variety of semigroups. A subsemigroup T of a semigroup $S\in V$ is called V-normal if there are a semigroup $S'\in V$ with a normal complex T' and a surjection φ: $S'\to S$ such that $\varphi(T')=T$.

Theorem. *Let V be a variety of semigroups. If V consists of completely simple semigroups, then the V-normal subsemigroups are exactly the normal subsemigroups; otherwise every subsemigroup of semigroups in V is V-normal.*

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Remark. For some varieties, every subsemigroup is normal; this is the case e.g. if V is the variety of zero semigroups or a variety of rectangular bands, but not if V contains the variety of semilattices.

The Theorem will be proved in three steps.

Proposition 1. *If V contains the two element zero semigroup (hence all zero semigroups) or the two element semilattice (hence all semilattices), then every subsemigroup T of a semigroup $S \in V$ is V-normal.*

Proof. Denote by $F_V(T)$ and $F_V(S)$ the free semigroups in V generated by the underlying sets E_T of T and E_S of S, respectively.

Suppose that V contains the zero semigroups and denote by $F_Z(S)$ the free zero semigroup generated by S. No equality of the form $wtw'=t'$ can hold in $F_V(S)$, where $w, w' \in F_V(S)$ or empty, but not both empty, $t, t' \in E_T$. For, if $wtw'=t'$ then let φ be the natural homomorphism of $F_V(S)$ onto $F_Z(S)$. It follows $\varphi(w)\varphi(t)\varphi(w')=\varphi(t')$. Here the left hand side is equal to zero, the right hand side is not, a contradiction. Therefore E_T is a normal complex of $F_V(S)$, and the natural homomorphism from $F_V(S)$ onto S maps E_T onto T.

Suppose now that V contains the semilattices and denote by $F_W(T)$ and $F_W(S)$ the free semilattices generated by E_T and E_S, respectively. $F_W(T)$ is a normal complex in $F_W(S)$. Consider the natural homomorphism φ of $F_V(S)$ onto $F_W(S)$. For any word $w \in F_V(S)$, $\varphi(w)$ contains all the letters from w. Therefore $\varphi^{-1}(F_W(T))= =F_V(T)$, hence the latter is a normal complex in $F_V(S)$. Now the natural homomorphism from $F_V(S)$ onto S maps $F_V(T)$ onto T.

Proposition 2. *Let V be a variety of semigroups containing neither zero semigroups nor semilattices. Then every semigroup in V is completely simple.*

Proof. Since the free cyclic semigroup in V admits no non-trivial zero semigroup as a homomorphic image, it must be a (finite) group. Thus all semigroups in V are unions of groups. Such a semigroup is a semilattice of completely simple semigroups, but V contains no semilattices either, hence every semigroup in V is completely simple.

Proposition 3. *The homomorphic images of normal complexes of completely simple semigroups are normal complexes of the images.*

Proof. (i) In a completely simple semigroup S, if $xy=e$ and e is an idempotent then $ex=x$ and $ye=y$, for $xS \supseteq eS$, xS is a minimal right ideal, therefore $x \in eS$.

(ii) Let $T \subset S$ be a normal complex of S, $x, y \in S$, $t, u \in T$. By u^{-1} we denote the inverse of u in the maximal subgroup of S containing u. Let φ be a homomorphism of S onto $\bar{S}$. $\bar{S}$ is also completely simple. From $\varphi(x)\varphi(t)\varphi(y)=\varphi(u)\in\bar{e}\bar{S}\bar{e}$, $\bar{e}$ idempotent, it follows that

$$\bar{e}=(\varphi(u))^{-1}\varphi(x)\varphi(t)\varphi(y)=\varphi(x)\varphi(t)\varphi(y)(\varphi(u))^{-1},$$

and by (i) we have $\bar{e}\varphi(x)=\varphi(x)$ and $\varphi(y)\bar{e}=\varphi(y)$. Let $xtyu^{-1}=g\in eSe$. Again by (i), $ex=x$, $u^{-1}e=u^{-1}$, further $\varphi(g)=\bar{e}$ and $(\varphi(g))^{-1}=\bar{e}=\varphi(e)$. Let $u\in e'Se'$, then $\varphi(e')=\bar{e}$ and from $u^{-1}e=u^{-1}$ it follows $e'e=e'$. Now we have $g^{-1}xtyu^{-1}=e$ and therefore $g^{-1}xtye'=eu$ and $e'g^{-1}xtye'=e'eu=e'u=u$. Since T is a normal complex, it follows $e'g^{-1}xt'ye'\in T$ for all $t'\in T$. Applying φ we obtain

$$\varphi(x)\varphi(t')\varphi(y)=\varphi(xt'y)=\bar{e}\varphi(xt'y)\bar{e}=\varphi(e'g^{-1}xt'ye')\in\varphi(T).$$

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