Acta Sci. Math., 49 (1985), 63—70

Amalgamated free products of n-groups

JACEK MICHALSKI

1. Introduction

The aim of this paper is to give a description for the amalgamated free products of n-groups. The present paper is the second in a series of papers devoted to the study of constructions of some special limits in the category $\mathbf{Gr}_n$ of n-groups (in [7] two constructions of free products were given). Both these papers are based on the results of [6]. The relation of the functors Φ and Ψ to inductive and projective limits given in [6] is used to investigate the above-mentioned inductive limits of n-groups.

In these constructions the notion of a free covering group (cf. [10]) plays a central rôle. The paper [3] (cf. also [5]) contains a generalization of this notion, namely a free covering $(k+1)$-group of an $(n+1)$-group (where $n=sk$). The assignment of free covering $(k+1)$-groups to $(n+1)$-groups is functorial. This leads to the functor Φ_s: $\mathbf{Gr}_{n+1}\to\mathbf{Gr}_{k+1}$ (in fact a class of functors depending on which construction of the free covering $(k+1)$-group we consider), which is left adjoint to the forgetful functor Ψ_s: $\mathbf{Gr}_{k+1}\to\mathbf{Gr}_{n+1}$ (cf. [3], [6]). Here (contrary to [7]) the meaning of Φ_s is the same as in [6]. In Proposition 1, by $\Phi_q(\tilde{H})$ and $\Phi_q(\tilde{G}_t)$ we mean the respective free covering $(k+1)$-groups of the $(qk+1)$-groups $\tilde{H}$ and $\tilde{G}_t$, disregarding their constructions.

In a category with initial objects any free product with an amalgamated initial object is isomorphic to the corresponding free product. The construction of free products of n-groups was given in [7]. However, the construction of amalgamated free products given here exploits the non-emptiness of amalgamated sub-$(n+1)$-groups. Therefore we always assume that a polyadic group is nonempty (like in [7]).

The terminology and notation of this paper is the same as in [5], [6], [7]. We recall only that $(\mathfrak{C}_{s,k+1}, \varphi)$ denotes the cyclic $(k+1)$-group of order s (cf. [10], [3]) and the letters f and g denote the $(n+1)$-group and $(k+1)$-group operations, respec-

Received September 22, 1982, and in revised form February 6, 1984.

tively, in the $(n+1)$-groups and $(k+1)$-groups under consideration. The symbol $f_{(s)}$ is understood as in [7] (in particular $f_{(0)}(x)=x$). Throughout the paper we assume $n=sk$, $s=mq$, $n>1$.

Let us introduce some new abbreviations of the notation. In place of $f(\ldots, x_1, \ldots, x_r, \ldots, x_1, \ldots, x_r, \ldots)$ with $x_1, \ldots, x_r$ repeated t times we write $f(\ldots, [\![x_1, \ldots, x_r]\!]^t, \ldots)$. In particular, instead of $f(\ldots, x, \ldots, x, \ldots)$ with x repeated t times we write briefly $f(\ldots, [\![x]\!]^t, \ldots)$.

2. Preliminaries

We start with recalling the construction of an amalgamated free product in the category $\mathbf{Gr}_2$ of groups (cf. [2], [11]).

Consider a nonempty family of monomorphisms $\{\gamma_t\colon B\to A_t\}_{t\in T}$ in $\mathbf{Gr}_2$, where the groups A_t are mutually disjoint. Let e denote the neutral element of A_t for each $t\in T$, and $B_t=\gamma_t(B)$. Form a set D_t consisting of exactly one representative for any left coset xB_t of the subgroup B_t and such that $eB_t\cap D_t=\{e\}$. Thus every element $a\in A_t$ can be expressed uniquely in the form $a=\hat{a}\gamma_t(b)$ where $\hat{a}\in D_t$, $b\in B$. The sequences of the form $\hat{a}_1\ldots\hat{a}_r b$ where $\hat{a}_i\in D_{t_i}-B_{t_i}$, $b\in B$, $t_i\neq t_{i+1}$, $r=0, 1, 2, \ldots$, will be called words. In the set A of words define a binary operation which from any two words forms a word in the following manner. Juxtapose the words to get a "long word" and perform all the necessary cancellations. The set A with this operation is the free product of A_t with amalgamated subgroup B. Henceforth by the amalgamated free product of groups we always mean the group described above.

Lemma 1. *Let $[L'; \{\gamma_t\colon \Phi_n(G_t)\to L'\}_{t\in T}]$ be the free product of a nonempty family of groups $\{\Phi_n(G_t)\}_{t\in T}$ with amalgamated subgroup $\Phi_n(H)$ (i.e., the inductive limit of the family of monomorphisms $\{\Phi_n(\varepsilon_t)\colon \Phi_n(H)\to\Phi_n(G_t)\}_{t\in T}$) where $\langle\Phi_n(H), \tau_H, \zeta_H\rangle$ and $\{\langle\Phi_n(G_t), \tau_t, \zeta_t\rangle\}_{t\in T}$ are the free covering groups of the $(n+1)$-groups H and $\{G_t\}_{t\in T}$, respectively. Then the morphism $\zeta\colon L'\to\mathfrak{C}_{n,2}$ defined by $\zeta(\hat{a}_1\ldots\hat{a}_r b)=\varphi_{(r)}(\zeta_{t_1}(a_1), \ldots, \zeta_{t_r}(a_r), \zeta_H(b))$ (where $\hat{a}_1\ldots\hat{a}_r b\in L'$, $b\in\Phi_n(H)$, $\hat{a}_i\in\Phi_n(G_{t_i})$ for $i=1, \ldots, r$) is an epimorphism and a pair $\langle L', \tau\rangle$, where τ is the inclusion of $\zeta^{-1}(0)$ into L', is the free covering group of the $(n+1)$-group $L=\zeta^{-1}(0)$. Furthermore, $[L; \{\alpha_t\colon G_t\to L\}_{t\in T}]$, where $\tau\alpha_t=\Psi_n(\gamma_t)\tau_t$, is the free product of the $(n+1)$-groups $\{G_t\}_{t\in T}$ with an amalgamated sub-$(n+1)$-group H.*

Proof. The proof of this lemma is analogous to that of Lemma 1 of [7].

The following two lemmas concern the decomposition of an $(n+1)$-group $\mathfrak{G}=(G,f)$ into left cosets of a nonempty sub-$(n+1)$-group H (cf. [1], [10]). As usual, for the construction of the free covering group one fixes an element $c \in G$ (cf. [3], [5]). Since the element c is arbitrary, we may assume $c \in H$.

Lemma 2. *Let H be a sub-$(n+1)$-group of an $(n+1)$-group $\mathfrak{G}=(G, f)$ and $a \in G$. Then every element of the form (a, l) (with $l=0, 1, \dots, n-1$) in the group $\mathfrak{G}^{*n}=(G^{*n}, f^*)$ belongs to $(a, 0)H^{*n}$.*

Proof. Let $a \in G$ and $l=0, 1, \dots, n-1$. Then

$$f^*((a, 0), (c, l-1)) = (f(a, c, [\![c]\!]^{l-1}, \bar{c}, [\![c]\!]^{n-1-l}), l) = (a, l) \in (a, 0)H^{*n},$$

since $(c, l-1) \in H^{*n}$.

Lemma 3. *Two elements (a_1, l_1) and (a_2, l_2) of $\mathfrak{G}^{*n}$ belong to the same left coset of the subgroup H^{*n} if and only if there exists an element $b \in H$ such that $a_1 = f(a_2, b, [\![c]\!]^{n-1})$.*

Proof. Let $(a_1, l_1) \in (a_2, l_2)H^{*n}$. In view of Lemma 2 $(a_1, 0) \in (a_2, 0)H^{*n}$, whence $(a_1, 0) = f^*((a_2, 0), (b, n-1)) = (f(a_2, b, [\![c]\!]^{n-1}, \bar{c}, [\![c]\!]^{n-1}), 0) = (f(a_2, b, [\![c]\!]^{n-1}), 0)$ for some $b \in H$. Thus $a_1 = f(a_2, b, [\![c]\!]^{n-1})$.

Conversely, let $a_1 = f(a_2, b, [\![c]\!]^{n-1})$. Thus $(a_1, 0) = f^*((a_2, 0), (b, n-1))$, whence $(a_1, 0) \in (a_2, 0)H^{*n}$. Then, by Lemma 2, $(a_1, l_1) \in (a_2, l_2)H^{*n}$.

3. A construction of amalgamated free products

Consider a nonempty family of monomorphisms $\{\varepsilon_t : H \to G_t\}_{t \in T}$ where H and G_t are nonempty $(n+1)$-groups. Choose an arbitrary but fixed element $c \in H$. Let $c_t = \varepsilon_t(c)$. Decompose every G_t into left cosets of the sub-$(n+1)$-group $H_t = \varepsilon_t(H)$ (i.e., elements a' and a'' belong to the same coset if and only if there exists an element $b_t \in H_t$ such that $a' = f(a'', b_t, [\![c_t]\!]^{n-1})$) and choose one element in every coset distinct from H_t. The representative of the coset aH_t (where $a \in G_t - H_t$) is denoted by $\hat{a}$. Therefore $a = f(\hat{a}, b_t, [\![c_t]\!]^{n-1})$ for some $b_t \in H_t$. By a word we shall mean a sequence of the form $\hat{a}_1 \dots \hat{a}_r b \overset{l}{c}$, where $r = 0, 1, \dots$ and for $i = 1, \dots, r$ we have $a_i \in G_{t_i} - H_{t_i}$, $b \in H$, $t_i \neq t_{i+1}$, $l = 0, 1, \dots, n-1$, $r + l \equiv 0 \pmod{n}$. Now we define an $(n+1)$-ary operation f on the set L of all words. Given $n+1$ words, form by juxtaposition a "long word" and perform the following cancellations: If in the "long word" there appear neighbouring expressions of the form

1. $b_1 \overset{l_1}{c}$ and $b_2 \overset{l_2}{c}$, where $b_1, b_2 \in H$, then we replace them by $b \overset{\varphi(l_1, l_2)}{c}$, where $b = f_{(\cdot)}(b_1, [\![c]\!]^{l_1}, b_2, [\![c]\!]^{l_2}, \bar{c}, [\![c]\!]^{n-1-\varphi(l_1, l_2)})$. If $b = \bar{c}$ and $\varphi(l_1, l_2) = n-1$, then we cancel the resulting expression $\bar{c} \overset{n-1}{c}$, unless it remains at the end of the "long word".

2. $\hat{a}_1$ and $\hat{a}_2$, where $\hat{a}_1, \hat{a}_2 \in G_t - H_t$, then depending on the element $a = f(\hat{a}_1, \hat{a}_2, \bar{c}_t, [\![c_t]\!]^{n-2})$ we replace them by

(a) $\hat{a}b\overset{0}{c}$, where $b \in H$ is the solution of the equation $a = f(\hat{a}, \varepsilon_t(b), \bar{c}_t, [\![c_t]\!]^{n-2})$, if $a \notin H_t$;

(b) $a'\overset{1}{c}$, where $\varepsilon_t(a') = a$, if $a \in H_t$.

3. $b_1\overset{l}{c}$ and $\hat{a}_1$, where $\hat{a}_1 \in G_t - H_t$, then we replace them by $\hat{a}b\overset{l}{c}$, where $a = f_{(\cdot)}(\varepsilon_t(b_1), [\![c_t]\!]^l, \hat{a}_1, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)})$ and b is the solution of the equation $a = f_{(\cdot)}(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^l, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)})$.

After a finite number of steps the "long word" becomes a word. Note that the resulting word does not depend on the order of the cancellations performed.

Define the family of morphisms $\{\alpha_t \colon G_t \to L\}_{t \in T}$ by the formula:

1. $\alpha_t(a) = \hat{a}b\overset{n-1}{c}$, where b is the solution of the equation $a = f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^{n-1})$, if $a \in G_t - H_t$;

2. $\alpha_t(a) = a'\overset{0}{c}$, where $\varepsilon_t(a') = a$, if $a \in H_t$.

Theorem 1. *The $(n+1)$-groupoid L is an $(n+1)$-group. The $(n+1)$-group L together with the family of morphisms $\{\alpha_t \colon G_t \to L\}_{t \in T}$ is the free product of the $(n+1)$-groups $\{G_t\}_{t \in T}$ with an amalgamated sub-$(n+1)$-group H.*

Proof. We use the same notation as in Lemma 1. Let $\langle \Phi_n(H), \tau_H, \zeta_H \rangle$ and $\{\langle \Phi_n(G_t), \tau_t, \zeta_t \rangle\}_{t \in T}$ be the free covering groups of the $(n+1)$-groups H and $\{G_t\}_{t \in T}$, respectively, with distinguished elements $c \in H$ and $c_t = \varepsilon_t(c) \in G_t$.

As was mentioned above, the elements of the free product L' of the groups $\Phi_n(G_t)$ with an amalgamated subgroup $\Phi_n(H)$ are words of the form $\hat{a}_1^* \ldots \hat{a}_r^* b^*$, where $\hat{a}_i^* \in \Phi_n(G_{t_i}) - \Phi_n(H_{t_i})$, $b^* \in \Phi_n(H)$, $t_i \neq t_{i+1}$, $r = 0, 1, 2, \ldots$. According to Lemma 2, the elements $\hat{a}_i^*$ can be chosen to be of the form $\hat{a}_i^* = (\hat{a}_i, 0)$. On the other hand, by Lemma 3, the decomposition of $\Phi_n(G_t)$ into left cosets of the subgroup $\Phi_n(H_t)$ coincides for elements of the form $(a, 0)$ with the decomposition of the $(n+1)$-group G_t into left cosets of the sub-$(n+1)$-group H_t. Therefore every element of L' is of the form $w = (\hat{a}_1, 0) \ldots (\hat{a}_r, 0)(b, l)$ where $b \in H$, $l = 0, \ldots, n-1$ and $\hat{a}_i \in G_{t_i} - H_{t_i}$; $t_i \neq t_{i+1}$ for $i = 1, \ldots, r$. According to Lemma 1 the morphism $\zeta \colon L' \to \mathfrak{C}_{n,2}$ defined by $\zeta(w) = \varphi_{(r)}(\zeta_{t_1}(\hat{a}_1, 0), \ldots, \zeta_{t_r}(\hat{a}_r, 0), \zeta_H(b, l))$ is an epimorphism. Let $L = \zeta^{-1}(0)$ and let $\tau \colon L \to L'$ be the inclusion of L into L'. Then $w \in L$ if and only if $r + l \equiv 0 \pmod{n}$ (since $\zeta(w) = \varphi_{(r)}(0, \ldots, 0, l) \equiv r + l \pmod{n}$). The $(n+1)$-group operation f on L is simply the long product obtained from the group operation f^* on L'. To simplify words of the form $(a, 0)$ in the $(n+1)$-group L we write simply a. Then $\tau(a) = (a, 0) \in L$. Let $w = (\hat{a}_1, 0) \ldots (\hat{a}_r, 0)(b, l) \in L$. Then (cf. [5])

$$w = f_{(\cdot)}^*((\hat{a}_1, 0), \ldots, (\hat{a}_r, 0), (b, 0), [\![(c, 0)]\!]^l) =$$
$$= f_{(\cdot)}^*(\tau(\hat{a}_1), \ldots, \tau(\hat{a}_r), \tau(b), [\![\tau(c)]\!]^l) = \tau(\hat{a}_1 \ldots \hat{a}_r b\overset{l}{c}).$$

Thus it is convenient to define L as the set of all sequences of the form $\hat{a}_1 \ldots \hat{a}_r b\overset{l}{c}$ where $b \in H$, $r=0, 1, 2, \ldots$, $l=0, \ldots, n-1$, $r+l \equiv 0 \pmod n$ and $a_i \in G_{t_i} - H_{t_i}$, $t_i \neq \neq t_{i+1}$ for $i=1, \ldots, r$. The $(n+1)$-ary operation f on L is given by juxtaposition of $n+1$ words and performing all possible cancellations:

1. If there appear neighbouring expressions $b_1 \overset{l_1}{c}$ and $b_2 \overset{l_2}{c}$, where $b_1, b_2 \in H$, then

$$\tau(\ldots b_1 \overset{l_1}{c} b_2 \overset{l_2}{c} \ldots) = f^*_{(\cdot)}(\ldots, \tau(b_1), [\![\tau(c)]\!]^{l_1}, \tau(b_2), [\![\tau(c)]\!]^{l_2}, \ldots) =$$
$$= f^*_{(\cdot)}(\ldots, (f(b_1, [\![c]\!]^{l_1}, b_2, [\![c]\!]^{l_2}, \bar{c}, [\![c]\!]^{n-1-\varphi(l_1, l_2)}), \varphi(l_1, l_2)), \ldots) =$$
$$= f^*_{(\cdot)}(\ldots, \tau(f(b_1, [\![c]\!]^{l_1}, b_2, [\![c]\!]^{l_2}, \bar{c}, [\![c]\!]^{n-1-\varphi(l_1, l_2)})), [\![\tau(c)]\!]^{\varphi(l_1, l_2)}, \ldots) =$$
$$= \tau(\ldots f(b_1, [\![c]\!]^{l_1}, b_2, [\![c]\!]^{l_2}, \bar{c}, [\![c]\!]^{n-1-\varphi(l_1, l_2)}) \overset{\varphi(l_1, l_2)}{c} \ldots).$$

If we obtain the expression $\bar{c} \overset{n-1}{c}$ not at the end of the "long word", then as in the proof of Theorem 2 of [7] one can show that it may be cancelled.

2. If there appear neighbouring expressions $\hat{a}_1$ and $\hat{a}_2$, where $\hat{a}_1, \hat{a}_2 \in G_t - H_t$, then

$$\tau(\ldots \hat{a}_1 \hat{a}_2 \ldots) = f^*_{(\cdot)}(\ldots, \tau(\hat{a}_1), \tau(\hat{a}_2), \ldots) =$$
$$= f^*_{(\cdot)}(\ldots, (f(\hat{a}_1, \hat{a}_2, \bar{c}_t, [\![c_t]\!]^{n-2}), 1), \ldots) = f^*_{(\cdot)}(\ldots, (a, 1), \ldots)$$

where $a = f(\hat{a}_1, \hat{a}_2, \bar{c}_t, [\![c_t]\!]^{n-2}) \in G_t$. Consider two cases:

(a) Let $a \notin H_t$. Then $(a, 1) = f^*((\hat{a}, 0), (\varepsilon_t(b), 0)) = (f(\hat{a}, \varepsilon_t(b), \bar{c}_t, [\![c_t]\!]^{n-2}), 1)$, thus $a = f(\hat{a}, \varepsilon_t(b), \bar{c}_t, [\![c_t]\!]^{n-2})$ and therefore $b \in H$ given by the equality $(a, 1) = f^*((\hat{a}, 0), (\varepsilon_t(b), 0))$ is the solution of the equation $a = f(\hat{a}, \varepsilon_t(b), \bar{c}_t, [\![c_t]\!]^{n-2})$. Hence

$$\tau(\ldots \hat{a}_1 \hat{a}_2 \ldots) = f^*_{(\cdot)}(\ldots, (\hat{a}, 0), (b, 0), \ldots) = f^*_{(\cdot)}(\ldots, \tau(\hat{a}), \tau(b), \ldots) = \tau(\ldots \hat{a} b \overset{0}{c} \ldots).$$

(b) Let $a \in H_t$. Then

$$\tau(\ldots \hat{a}_1 \hat{a}_2 \ldots) = f^*_{(\cdot)}(\ldots, (a, 1), \ldots) = f^*_{(\cdot)}(\ldots, \tau(a), \tau(c), \ldots) = \tau(\ldots a' \overset{1}{c} \ldots)$$

where $\varepsilon_t(a') = a$.

3. If there appear neighbouring expressions $b_1 \overset{l}{c}$ and $\hat{a}_1$; where $\hat{a}_1 \in G_t - H_t$; then

$$\tau(\ldots b_1 \overset{l}{c} \hat{a}_1 \ldots) = f^*_{(\cdot)}(\ldots, (\varepsilon_t(b_1), l), (\hat{a}_1, 0), \ldots) =$$
$$= f^*_{(\cdot)}(\ldots, (f(\varepsilon_t(b_1), [\![c_t]\!]^l, \hat{a}_1, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)}), \varphi(l, 0)), \ldots) = f^*_{(\cdot)}(\ldots, (a, \varphi(l, 0)), \ldots)$$

where $a = f_{(\cdot)}(\varepsilon_t(b_1), [\![c_t]\!]^l, \hat{a}_1, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)})$. Then

$$(a, \varphi(l, 0)) = f^*((\hat{a}, 0), (\varepsilon_t(b), l)) = (f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^l, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)}), \varphi(l, 0));$$

thus $a=f_{(\cdot)}(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^l, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)})$ and therefore $b\in H$ given by the equality $(a, \varphi(l, 0))=f^*((\hat{a}, 0), (\varepsilon_t(b), l))$ is the solution of the equation $a=$ $=f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^l, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)})$. Hence

$$\tau(\dots b_1 \overset{l}{c} \hat{a}_1 \dots) = f^*_{(\cdot)}(\dots, (\hat{a}, 0), (\varepsilon_t(b), l), \dots) =$$

$$= f^*_{(\cdot)}(\dots, \tau(\hat{a}), \tau(b), [\![\tau(c)]\!]^l, \dots) = \tau(\dots \hat{a}b\overset{l}{c}\dots).$$

The uniqueness of the resulting word is implied by the uniqueness of the form of a word in the amalgamated free product of groups.

According to Lemma 1, $\tau\alpha_t(a)=\Psi_n(\gamma_t)(a, 0)$. Consider two cases:

1. Let $a\in G_t-H_t$. Then $(a, 0)=f^*((\hat{a}, 0), (\varepsilon_t(b), n-1))=(f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^{n-1}), 0)$; thus $a=f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^{n-1})$ and therefore $b\in H$ given by the equality $(a, 0)=$ $=f^*((\hat{a}, 0), (\varepsilon_t(b), n-1))$ is the solution of the equation $a=f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^{n-1})$. Hence $\gamma_t(a, 0) = (a, 0)(b, n-1)$, so $\tau\alpha_t(a) = (\hat{a}, 0)(b, n-1) = \tau(\hat{a}b\overset{n-1}{c})$.

2. Let $a\in H_t$. Then $\gamma_t(a, 0)=(a, 0)$, whence $\tau\alpha_t(a)=\tau(a, 0)=a'\overset{0}{c}$ where $\varepsilon_t(a')=a$.

This completes the proof of Theorem 1.

4. Some properties of amalgamated free products

In view of Theorem 3 of [6], if every $(n+1)$-group G_t and also the $(n+1)$-group H are derived from $(k+1)$-groups, then the amalgamated free product is also derived from a $(k+1)$-group. The converse is also true except for the following two cases: when T has only one element or when at most one of the monomorphisms is not an isomorphism. Then the amalgamated free product is isomorphic either to G_t or to H. It may happen that in this case the amalgamated free product (being isomorphic to one of the $(n+1)$-groups G_t) is derived from a $(k+1)$-group; none the less the $(n+1)$-group H (as a sub-$(n+1)$-group of that $(n+1)$-group G_t) need not be derived from any $(k+1)$-group. For this reason we have to make some additional assumptions.

Theorem 2. *Let L be the free product of $(n+1)$-groups $\{G_t\}_{t\in T}$ with an amalgamated sub-$(n+1)$-group H, where more than one monomorphism $\varepsilon_t\colon H\to G_t$ is not an isomorphism. Then the $(n+1)$-group L is derived from a $(k+1)$-group if and only if every $(n+1)$-group G_t and the $(n+1)$-group H are also derived from $(k+1)$-groups.*

Proof. We use the notation of Theorem 1. Let L be an $(n+1)$-group derived from a certain $(k+1)$-group and let the word $w=\hat{a}_1\dots\hat{a}_r b\overset{l}{c}$ be skew to the element

$\overset{0}{cc}=\alpha(c)\in\alpha(H)$ (where $\alpha=\alpha_t\varepsilon_t$) in that $(k+1)$-group. In view of Corollary 2 of [9] the element w is s-skew to $\overset{0}{cc}$ in the $(n+1)$-group L.

Suppose that $r\neq 0$. Then $a_1\in G_{t_1}$ for some $t_1\in T$. Take any element of the form $\hat{a}\bar{c}\overset{n-1}{c}$ where $a\in G_t-H_t$ and $t\neq t_1$. From the definition of an s-skew element (cf. [9]) it follows that $w[\![\overset{0}{cc}]\!]^{k-1}[\![\hat{a}\bar{c}\overset{n-1}{c}]\!]^{n-k-1}=\hat{a}\bar{c}\overset{n-1}{c}w[\![\overset{0}{cc}]\!]^{k-1}[\![\hat{a}\bar{c}\overset{n-1}{c}]\!]^{n-k}$. After performing all the necessary cancellations the reduced word on the left side of the equality starts with $\hat{a}_1$, the reduced word on the right side starts with $\hat{a}$. This contradicts the uniqueness of the form of a reduced word, since $\hat{a}_1\neq\hat{a}$ $(t_1\neq t)$.

Thus the word w is of the form $w=\overset{0}{bc}$. Since $w=\overset{0}{bc}=\alpha(b)\in\alpha(H)$, by Proposition 3 of [9] the sub-$(n+1)$-group $\alpha(H)$ of the $(n+1)$-group L is also a sub-$(k+1)$-group of the creating $(k+1)$-group of L. Hence the $(n+1)$-group H (isomorphic to the $(n+1)$-group $\alpha(H)$) is also derived from a $(k+1)$-group. On the other hand $\alpha(H)\subset\gamma_t(G_t)$; so every $(n+1)$-group $\gamma_t(G_t)$ is derived from a $(k+1)$-group.

Conversely, let $(n+1)$-groups $\{G_t\}_{t\in T}$ and H be derived from $(k+1)$-groups. Then, by Theorem 3 of [6], the $(n+1)$-group L is also derived from a $(k+1)$-group, which completes the proof of Theorem 2.

In [6] we proved a general theorem on the inductive limits of covering $(k+1)$-groups of $(n+1)$-groups. This theorem applied to the case of the free product yields Theorem 4 of [7].

In a category complete with respect to inductive limits the free product is a particular case of the free product with an amalgamated subobject (taking an initial object for the subobject). This is the case for the category $\mathbf{Gr}_2$ and also for the categories $\mathbf{Gr}_n$ with $n>2$. Therefore in $\mathbf{Gr}_2$ the construction of a free product is a particular case of the construction of the free product with an amalgamated subgroup (in this case a one-element group). Note that the situation is quite different when we pass to $\mathbf{Gr}_n$ for $n>2$. In the construction of a free product with an amalgamated sub-n-group presented here it is important that this sub-n-group is nonempty. Hence the construction is not a generalization of the construction of the free product. In particular, Theorem 4 of [7] is not applicable to the description of an amalgamated free product of covering $(k+1)$-groups of $(n+1)$-groups.

Proposition 1. *Let $\{\varepsilon_t\colon H\to G_t\}_{t\in T}$ and $\{\varepsilon_t'\colon H'\to G_t'\}_{t\in T}$ be nonempty families of monomorphisms, where $\langle H',\lambda_H,\zeta_H\rangle$ and $\{\langle G_t',\lambda_t,\zeta_t\rangle\}_{t\in T}$ are covering $(k+1)$-groups of indices q_H and $\{q_t\}_{t\in T}$ of the $(n+1)$-groups H and $\{G_t\}_{t\in T}$, respectively, and in addition $\Psi_s(\varepsilon_t')\lambda_H=\lambda_t\varepsilon_t$ for each $t\in T$. Then for each $t\in T$ we have $q_t=q_H$ and the free product of the $(k+1)$-groups $\{G_t'\}_{t\in T}$ with an amalgamated sub-$(k+1)$-group H' is a covering $(k+1)$-group of index q_H of the free product of the $(n+1)$-groups $\{G_t\}_{t\in T}$ with an amalgamated sub-$(n+1)$-group H.*

Proof. The commutativity of the diagram

$$\begin{array}{ccc} \Psi_s(H') & \xrightarrow{\Psi_s(\varepsilon'_t)} & \Psi_s(G'_t) \\ \uparrow{\scriptstyle \lambda_H} & & \uparrow{\scriptstyle \lambda_t} \\ H & \xrightarrow{\varepsilon_t} & G_t \end{array}$$

together with Theorem 4 of [5] implies the existence of morphisms $\xi_t\colon \mathfrak{C}_{q_H,k+1}\to\mathfrak{C}_{q_t,k+1}$ such that $\xi_t\zeta_H=\zeta_t\varepsilon'_t$. Since the morphisms $\varepsilon'_t\colon H'\to G'_t$ are (by assumption) monomorphisms, the morphisms ξ_t are isomorphisms (cf. Corollary 4 of [8]). Hence $q_H=q_t$. For simplicity we shall write q instead of q_H. In view of Corollary 3 of [5], the $(n+1)$-groups H and $\{G_t\}_{t\in T}$ are derived from $(qk+1)$-groups $\tilde{H}$ and $\tilde{G}_t$, respectively, where in addition (see the remark on the definition of the functor Φ in the Introduction) $H'=\Phi_q(\tilde{H})$, $G'_t=\Phi_q(\tilde{G}_t)$, $\lambda_H=\Psi_m(\tau_{\tilde{H}})$, $\lambda_t=\Psi_m(\tau_t)$ (here $s=mq$). Let $[\tilde{L}; \{\tilde{\alpha}_t\colon \tilde{G}_t\to\tilde{L}\}_{t\in T}]$, $[L; \{\alpha_t\colon G_t\to L\}_{t\in T}]$, $[L'; \{\alpha'_t\colon G'_t\to L'\}_{t\in T}]$ be the amalgamated free products. The functors Ψ_m and Φ_q preserve and reflect amalgamated free products (cf. [6]). Hence $L=\Psi_m(\tilde{L})$, $L'=\Phi_q(\tilde{L})$. Let $\langle L', \tau_{\tilde{L}}\rangle$ be the free covering $(k+1)$-group of the $(qk+1)$-group $\tilde{L}$. Thus, by Corollary 4 of [5], $\langle L', \lambda_L\rangle$ (where $\lambda_L=\Psi_m(\tau_{\tilde{L}})$) is a covering $(k+1)$-group of index q of the $(n+1)$-group L, which completes the proof of Proposition 1.

References

[1] W. Dörnte, Untersuchungen über einen verallgemeinerten Gruppenbegriff, *Math. Z.*, **29** (1929), 1—19.

[2] A. G. Kuroš, *Group theory*, Nauka (Moscow, 1967). (Russian)

[3] J. Michalski, On some functors from the category of n-groups, *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, **27** (1979), 437—441.

[4] J. Michalski, Inductive and projective limits of n-groups, *Bull. Acad. Polon. Sci. Sér. Sci. Math.*, **27** (1979), 443—446.

[5] J. Michalski, Covering k-groups of n-groups, *Arch. Math. (Brno)*, **17** (1981), 207—226.

[6] J. Michalski, On the category of n-groups, *Fund. Math.*, **122** (1984), 187—197.

[7] J. Michalski, Free products of n-groups, *Fund. Math.*, **123** (1984), 11—20.

[8] J. Michalski, Morphisms in the category of covering k-groups of n-groups, *Demonstratio Math.*, **16** (1983), 977—990.

[9] J. Michalski, On s-skew elements in polyadic groups, *Arch. Math. (Brno)*, **19** (1983), 215—218.

[10] E. Post, Polyadic groups, *Trans. Amer. Math. Soc.*, **48** (1940), 208—350.

[11] Z. Semadeni, A. Wiweger, *Einführung in die Theorie der Kategorien und Funktoren*, Teubner (Leipzig, 1979).

TEACHERS TRAINING COLLEGE
UL. DAWIDA 1A
50-527 WROCŁAW, POLAND