Varieties of algebras as a lattice with an additional operation

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1. Introduction

Let $\mathfrak{k}$ be a non-trivial associative and commutative ring with 1. In the present paper we are concerned with varieties (equational classes) of $\mathfrak{k}$-algebras that are not necessarily associative and not necessarily with 1. These are classes of algebras closed under the formation of subalgebras, homomorphic images and Cartesian products; equivalently, classes of all algebras satisfying given sets of polynomial identities. Two basic properties of free groups enhanced the theory of group varieties: a subgroup of a free group is free, and a fully invariant subgroup of a fully invariant subgroup of a free group is fully invariant. Given two group varieties $\mathscr{U}$, $\mathscr{V}$, $\mathscr{U}\cdot\mathscr{V}$ is the class of all groups that are Schreier-extensions of a group in $\mathscr{U}$ by a group in $\mathscr{V}$. It turns out that $\mathscr{U}\cdot\mathscr{V}$ is a variety. Under this multiplication, the groupoid of group varieties is a free monoid with zero. This was shown independently by B. H. NEUMANN, HANNA NEUMANN and P. M. NEUMANN [15] and by A. L. ŠMELKIN [21]. A similar result holds for the groupoid of Lie algebra varieties over a field of characteristic 0; this is due to V. A. PARFENOV [18]. A subalgebra of a free associative algebra need not be free, P. M. COHN [4]. A T-ideal of a T-ideal of a free associative algebra may not be a T-ideal, A. I. MAL'CEV [13], A. A. ISKANDER [11]. It turns out that the groupoid of ring varieties is not associative and certainly not relatively free. It is not even power associative. The groupoid of varieties of $\mathfrak{k}$-algebras contains infinite submonoids. This groupoid has some sort of decomposition. The minimal varieties are determined. If $\mathfrak{k}$ has exactly 2 idempotent ideals, then a family of identities that is attainable on all power associative algebras is equivalent to $x=x$ or $x=y$.

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The word "algebra" will mean "$\mathfrak{k}$-algebra". The word "variety" will mean "variety of $\mathfrak{k}$-algebras". An algebra is called power-associative if every subalgebra generated by one element is associative. By a theorem of A. A. ALBERT [1], [2], if $\mathfrak{k}$ is a field of characteristic not 2, 3 or 5, then an algebra is power associative if it satisfies $(xx)x=x(xx)$ and $((xx)x)x=(xx)(xx)$. Let $\mathscr{A}i\mathfrak{k}$, $i=0, 1, 2, 3$, denote, respectively, the varieties of all algebras, all power-associative algebras, all associative algebras and all associative and commutative algebras. If $\mathscr{V}$ is a variety, we denote by $L\mathscr{V}$ the set of all subvarieties of $\mathscr{V}$. Under class inclusion $L\mathscr{V}$ is a complete modular lattice. Under an additional operation $L\mathscr{V}$ is a partially ordered groupoid with zero ($\mathscr{V}$) and 1 ($\mathscr{E}$); where $\mathscr{E}$ is the trivial variety of one-element algebras.

For an account of the variety theory, the reader may consult [3], [5], [10], [14], [16], [17].

Definition 1. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C} \in 0\mathfrak{k}$. Then $\mathfrak{C}$ is an extension of $\mathfrak{A}$ by $\mathfrak{B}$ if $\mathfrak{C}$ possesses an ideal isomorphic to $\mathfrak{A}$ whose factor is isomorphic to $\mathfrak{B}$. If $\mathscr{U}$, $\mathscr{V}$, $\mathscr{K}$ are classes of algebras, then $\mathscr{U} \cdot_{\mathscr{K}} \mathscr{V}$ is the class of all algebras of $\mathscr{K}$ that are extensions of an algebra of $\mathscr{U}$ by an algebra of $\mathscr{V}$.

We will write $\mathscr{U} \cdot_i \mathscr{V}$ for $\mathscr{U} \cdot_{\mathscr{A}i\mathfrak{k}} \mathscr{V}$, $i=0, 1, 2, 3$.

Ring extensions were introduced by C. J. EVERETT [8]. It is the analogue of O. SCHREIER's group extensions [20]. The concept of class multiplication for groups may be found in HANNA NEUMANN [16]. A. I. MAL'CEV [13] generalized class multiplication and proved the following theorem for algebraic systems.

Theorem 1. *If* $\mathscr{U}$, $\mathscr{V}$, $\mathscr{K}$ *are varieties, then* $\mathscr{U} \cdot_{\mathscr{K}} \mathscr{V}$ *is a subvariety of* $\mathscr{K}$. $\langle L\mathscr{K}; \cdot_{\mathscr{K}} \rangle$ *is a partially ordered groupoid with zero and* 1; $\mathscr{K}$ *is the zero-element and the trivial variety* $\mathscr{E}$ *is* 1. *If* $\mathscr{A}, \mathscr{B}, \mathscr{C} \in L\mathscr{K}$, *then* $\mathscr{A} \cdot_{\mathscr{K}} (\mathscr{B} \cdot_{\mathscr{K}} \mathscr{C}) \subseteqq (\mathscr{A} \cdot_{\mathscr{K}} \mathscr{B}) \cdot_{\mathscr{K}} \mathscr{C}$. *If* $\mathscr{A} \subseteqq \mathscr{B}$, *then* $\mathscr{C} \cdot_{\mathscr{K}} \mathscr{A} \subseteqq \mathscr{C} \cdot_{\mathscr{K}} \mathscr{B}$ *and* $\mathscr{A} \cdot_{\mathscr{K}} \mathscr{C} \subseteqq \mathscr{B} \cdot_{\mathscr{K}} \mathscr{C}$.

Although the lattice of group varieties has a complicated structure, the groupoid of group varieties has a very simple structure: a free monoid with zero. However, from A. A. ISKANDER [11], $\langle L\mathscr{A}2\mathbf{Z}; \cdot \rangle$ ($\mathbf{Z}$ is the ring of integers) contains a denumerable set of idempotents. Thus, it is far from being free. We will show that $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i \rangle$ are not power-associative and, under some restrictions on $\mathfrak{k}$, a decomposition is valid in $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i \rangle$.

Theorem 2. *Let* $i=0, 1, 2$. *Then* $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i \rangle$ *is not power-associative; in fact,* $(\mathscr{C} \cdot_i \mathscr{C}) \cdot_i \mathscr{C} \neq \mathscr{C} \cdot_i (\mathscr{C} \cdot_i \mathscr{C})$, *where* $\mathscr{C} = \mathscr{A}3\mathfrak{k}$. *If* $\mathfrak{k}$ *is a field of characteristic* 0, *then* $\langle L\mathscr{A}3\mathfrak{k}; \cdot_i \rangle$ *is isomorphic to the multiplicative monoid of natural numbers.*

However, we will show that $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i \rangle$ contains infinite associative subgroupoids.

Definition 2. Let $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}\neq\mathscr{E}$, $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$. $\mathscr{V}$ is i-indecomposable if $\mathscr{V}=\mathscr{U}\cdot_i\mathscr{W}$, $\mathscr{U}$, $\mathscr{W}\in L\mathscr{A}i\mathfrak{k}$ implies $\mathscr{U}=\mathscr{E}$ or $\mathscr{V}=\mathscr{E}$. $\mathscr{V}$ is i-pseudo-indecomposable if $\mathscr{V}$ contains a non-trivial algebra satisfying $xy=0$ and $\mathscr{V}=\mathscr{U}\cdot_i\mathscr{W}$, $\mathscr{U}$, $\mathscr{W}\in L\mathscr{A}i\mathfrak{k}$ implies $\mathscr{U}$ or $\mathscr{W}$ does not contain any non-trivial algebras satisfying $xy=0$, $i=0, 1, 2, 3$.

Theorem 3. *Let $\mathfrak{k}$ be a Dedekind domain, $i=0, 1, 2, 3$. If $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$, then either $\mathscr{V}$ does not contain any non-trivial algebras satisfying $xy=0$ or $\mathscr{V}$ is a product of a finite number of i-pseudo-indecomposable varieties; if $\mathfrak{k}$ is a field of characteristic* 0, *then $\mathscr{V}=\mathscr{E}$ or $\mathscr{V}$ is a product of a finite number of i-indecomposable varieties.*

An equationally complete variety is a variety whose lattice of subvarieties contains exactly 2 elements; i.e., it is a minimal non-trivial variety. A. TARSKI [24] determined the equationally complete associative ring varieties: they are those determined by $px=0$, $xy=0$ for some prime p or by $px=0$, $x-x^p=0$ for some prime p. The following theorem determines the minimal varieties in $L\mathscr{A}i\mathfrak{k}$, $i=1, 2, 3$:

Theorem 4. *The equationally complete varieties of $L\mathscr{A}i\mathfrak{k}$, $i=1, 2, 3$, are exactly the equationally complete varieties of $L\mathscr{A}3\mathfrak{k}$. They are the varieties determined by one of the following sets of identities:*

(1) *for some maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$, $ax=0$ for all $a\in\mathfrak{m}$, $xy=0$;*

(2) *for some maximal ideal $\mathfrak{m}$ of finite index in $\mathfrak{k}$, $ax=0$ for all $a\in\mathfrak{m}$ and $x-x^n=0$ where $n=|\mathfrak{k}/\mathfrak{m}|$.*

Thus the minimal varieties of $L\mathscr{A}i\mathfrak{k}$, $i=1, 2, 3$, are those generated by $\mathfrak{k}/\mathfrak{m}$ for some maximal ideal of $\mathfrak{k}$ of finite index in $\mathfrak{k}$ or by the zero algebra on $\mathfrak{k}/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of $\mathfrak{k}$.

Definition 3. Let I be a set of polynomial identities. An algebra $\mathfrak{R}$ is I-indecomposable if A is an ideal of $\mathfrak{R}$ such that $\mathfrak{R}/A$ satisfies I implies $A=\mathfrak{R}$. I is attainable on $\mathscr{K}\subseteq\mathscr{A}0\mathfrak{k}$ if for every $\mathfrak{R}\in\mathscr{K}$, the least ideal of $\mathfrak{R}$ whose factor satisfies I is I-indecomposable.

This concept is due to T. TAMURA [22] where he determined the sets of identities attainable on the class of all semigroups. As shown by T. TAMURA and F. M. YAQUB [23], the sets $\{xy=yx\}$, $\{px=0, x=x^p\}$, p is prime, are not attainable on the class of all associative rings. It was shown by A. A. ISKANDER [11] that a family of identities that is attainable on the variety of all associative rings or on the variety of all commutative and associative rings is equivalent to $x=x$ or $x=y$. In general

Theorem 5. *Let $\mathfrak{k}$ contain no idempotent ideals other than $\mathfrak{o}$, $\mathfrak{k}$, and suppose $i=1, 2, 3$. If I is a set of polynomial identities that is attainable on $\mathscr{A}i\mathfrak{k}$, then I is equivalent on $\mathscr{A}i\mathfrak{k}$ to $x=x$ or $x=y$.*

M. V. Volkov [25] introduced and successfully used the concept of "S-joined varieties", where S is a submonoid of the multiplicative monoid of $\mathfrak{k}$ containing no zero-divisors, to gain information about the lattice of subvarieties of a variety $\mathscr{V}$ by studying the corresponding lattice of varieties of $\mathfrak{k}'$-algebras, where $\mathfrak{k}'$ is the ring of fractions of $\mathfrak{k}$ relative to S. In the present paper, we study a slightly more general case and show that the S-joined subvarieties of a variety $\mathscr{V}$ form a subgroupoid of $\langle L\mathscr{V}; \cdot_{\mathscr{V}}\rangle$.

2. Relatively free algebras and T-ideals

Before we prove Theorems 2, 3, 4 and 5, we will need some preliminaries and prove some other results.

For every cardinal number $n>0$, $X(n)$ is a set of cardinality n and $F(n, \mathscr{V})$ is the free algebra of $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$ whose free generating set is $X(n)$. Let $X=\{x_0, x_1, \ldots\}$ be a denumerable set. $F\mathscr{V}$ will denote the free algebra of $\mathscr{V}$ whose free generating set is X; $Fi=F\mathscr{A}i\mathfrak{k}$, $i=0, 1, 2, 3$. Let $G0, G1, G2$ and $G3$ be, respectively, the free groupoid, the free power-associative groupoid, the free semigroup and the free commutative semigroup whose set of free generators is X. The following lemma is in the literature:

Lemma 6. *The $\mathfrak{k}$-module structure of Fi is the free unital $\mathfrak{k}$-module with basis Gi. The multiplication in Fi is defined by $(au)(bv)=(ab)(uv)$, $a(bv)=(ab)v$ and distributivity, where $a, b\in\mathfrak{k}$, $u, v\in Gi$, $i=0, 2, 3$.*

For $i=0$, cf. J. M. Osborn [17], p. 167. For $i=2$, cf. P. M. Cohn [6], p. 30 and [7], p. 63. $i=3$ is similar.

If $f\neq 0, f\in F0$, $d(f)$ denotes the degree of f, i.e., the highest among the lengths of elements of $G0$ with non-zero coefficients in f. $o(f)$ denotes the order of f, i.e., the least among the lengths of elements of $G0$ with non-zero coefficients in f. $f(x_1, \ldots, x_n)$ will mean that the elements of X occurring at least once in f are among $x_1, \ldots, x_n$. f is called homogeneous of degree r in x_i if every element of $G0$ with non-zero coefficient in f has exactly r entries of x_i; f is called homogeneous if it is homogeneous in every $x_i\in X$. f is called multilinear if f is homogeneous of degree at most 1 in every x_i. Every variety of algebras is determined by a set of identities. If $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$, then the set of all $f\in F0$, such that $f=0$ is an identity in $\mathscr{V}$, is a T-ideal of $F0$; that is an ideal of $F0$ closed under all endomorphisms of $F0$, cf. [6], [17]. In fact, if $\mathscr{W}\in L\mathscr{V}$, then the identities $f=0$ of $\mathscr{W}$ relative to $\mathscr{V}$ form a T-ideal of $F\mathscr{V}$. The correspondence between $L\mathscr{V}$ and the T-ideals of $F\mathscr{V}$ is an antiisomorphism of the lattice $\langle L\mathscr{V}; \wedge, \vee\rangle$ onto the lattice of T-ideals of $F\mathscr{V}$. Script capital letters will denote classes or varieties of algebras; the corresponding Latin capitals will denote the T-ideals of $F0$ determined by them. Algebras will be denoted by German capitals and ideals of $\mathfrak{k}$ will be denoted by lower case German letters. Homomorphisms will be denoted by lower case Greek letters and will be applied to the right.

If $\mathfrak{R}\in\mathscr{A}0\mathfrak{k}$, $A\subseteq F0$, $B\subseteq\mathfrak{k}$, then $B\mathfrak{R}$ is the set of all finite sums of elements of $\mathfrak{R}$ of the type bx, $b\in B$, $x\in\mathfrak{R}$ and $A(\mathfrak{R})$ is the set of all elements of $\mathfrak{R}$ that are equal to $f(r_1, \ldots, r_n)$ where $r_1, \ldots, r_n\in\mathfrak{R}$ and $f\in A$.

Lemma 7. *If $\mathfrak{R}\in\mathscr{A}0\mathfrak{k}$, $\mathfrak{a}$ is an ideal of $\mathfrak{k}$, V is a T-ideal of $F0$, then $\mathfrak{a}\mathfrak{R}$ is an ideal of $\mathfrak{R}$ and $V(\mathfrak{R})$ is a T-ideal of $\mathfrak{R}$. $V(\mathfrak{R})$ is the least ideal of $\mathfrak{R}$ whose factor belongs to $\mathscr{V}$. $F\mathscr{V}\cong F0/V$.*

Cf. [5], [10], [14], [17].

The following lemma is a special case of a result of A. I. MAL'CEV [13]:

Lemma 8. *Let $\mathscr{K}, \mathscr{U}, \mathscr{V}, \mathscr{W}\in L\mathscr{A}0\mathfrak{k}$, $\mathscr{W}\subseteq\mathscr{K}$. Then $(\mathscr{U}\cap\mathscr{W})\cdot_{\mathscr{W}}(\mathscr{V}\cap\mathscr{W})=(\mathscr{U}\cdot_{\mathscr{K}}\mathscr{V})\cap\mathscr{W}$. Furthermore $\mathfrak{R}\in\mathscr{U}\cdot_{\mathscr{K}}\mathscr{V}$ iff $\mathfrak{R}\in\mathscr{K}$ and $V(\mathfrak{R})\in\mathscr{U}$.*

Lemma 9. *If A is a basis of identities for $\mathscr{U}\in L\mathscr{A}0\mathfrak{k}$, $A\subseteq F0$ and $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$, then $A(V)$ is a basis of identities for $\mathscr{U}\cdot_0\mathscr{V}$. The T-ideal of $F0$ determined by $\mathscr{U}\cdot_0\mathscr{V}$ is the ideal of $F0$ generated by $U(V)$.*

The T-ideal of $F0$ determined by $\mathscr{U}\cdot_0\mathscr{V}$ will be denoted by $U\circ V$.

Proof. By Lemma 8, $\mathfrak{R}\in\mathscr{U}\cdot_0\mathscr{V}$ iff $V(R)\in\mathscr{U}$, i.e., iff $A(V(\mathfrak{R}))=0$. Thus $A(V)$ is a basis of identities for $\mathscr{U}\cdot_0\mathscr{V}$. Hence $U(V)$ is a basis for $\mathscr{U}\cdot_0\mathscr{V}$. However, the ideal of $F0$ generated by $U(V)$ is a T-ideal of $F0$ since it is the set of all finite sums of $w, fw, wg, (fw)g, f(wg), (f(gw))h, \ldots$ where $w\in U(V), f, g, h, \ldots\in F0$.

Proposition 10. *If $\mathscr{U}, \mathscr{V}\in L\mathscr{A}0\mathfrak{k}$ are defined by multilinear identities, then $\mathscr{U}\cdot_0\mathscr{V}$ is definable by multilinear identities. If $\mathscr{U}$ is defined by a finite set of multilinear identities, $\mathscr{V}$ is finitely based, $\mathscr{U}, \mathscr{V}\in L\mathscr{A}2\mathfrak{k}$, then $\mathscr{U}\cdot_2\mathscr{V}$ is finitely based.*

Proof. Suppose $0\neq g\in F0$ and the number of elements of X occurring in g is r. Let $\bar{g}=g(x_1, \ldots, x_r)$. Let $D(g)$ be the set of all elements of $F0$ obtained from $\bar{g}$ by a finite number of applications of the following: If $h(x_1, \ldots, x_m)\in D(g)$ and every x_i, $1\leqq i\leqq m$, occurs in h, then $x_{m+1}h, hx_{m+1}\in D(g)$. Suppose $h_1, \ldots, h_r$ are non-zero elements of $F0$. $g*(h_1, \ldots, h_r)$ is the set of all $h(x_1, \ldots, x_n)=g(h'_1, \ldots, h'_r)$ such that h'_i is obtained from an element of $D(h_i)$ by renaming the elements of X so that h'_i and h'_j have no elements of X in common if $i\neq j$, and n is the number of elements of X occurring in $h'_1, \ldots, h'_r$. If $A\subseteq U$ is a basis for $\mathscr{U}$, $B\subseteq V$ is a basis for $\mathscr{V}$ and every element in A is multilinear, then $D=\bigcup\{g*(h_1, \ldots, h_r)\colon g\in A, h_1, \ldots, h_r\in B\}$ is a basis for $\mathscr{U}\cdot_0\mathscr{V}$. This is true since every element of $\mathscr{V}$ is a finite sum of $h(f_1, \ldots, f_t)$, $h\in D(g)$, $g\in B$, $f_1, \ldots, f_t\in F0$. If g is multilinear, then $g(v_1, \ldots, v_r)$, where $v_1, \ldots, v_r\in V$, is a finite sum of elements of the form

$g(h_1(f_1, ..., f_l), ..., h_r(f_s, ..., f_t))$, where $h_1, ..., h_r \in \bigcup \{D(g): g \in B\}$, $f_1, ..., f_t \in F0$, i.e., every element in $A(V)$ is a finite sum of $h(f_1, ..., f_t)$, where $h \in D, f_1, ..., f_t \in F0$. Since $A(V)$ is a basis for $\mathcal{U} \cdot_0 \mathcal{V}$ (by Lemma 9), $D \subseteq A(V)$, D is a basis for $\mathcal{U} \cdot_0 \mathcal{V}$. If every element in B is multilinear, then $D(g)$ contains only multilinear elements for every $g \in B$ and D is composed of multilinear elements. If $\mathcal{U}, \mathcal{V} \in L\mathcal{A}2\mathfrak{k}$, then in $F2$, $D'(g) = \{\bar{g}, x_{r+1}\bar{g}, \bar{g}x_{r+1}, x_{r+1}\bar{g}x_{r+2}\}$. Thus D' can be chosen finite in case A is finite and B is finite. Thus $\mathcal{U} \cdot_2 \mathcal{V}$ is finitely based; $D' = \bigcup \{g *'(h_1, ..., h_r): h_1, ..., h_r \in B, g \in A\}$, where $*'$ is similar to $*$, using $D'(h_i)$ instead of $D(h_i)$.

For example, $\mathcal{C} \cdot_2 \mathcal{C}$ has the following basis:

$$[[x_1, x_2], [x_3, x_4]],$$

$$[[x_1, x_2], [x_3, x_4]x_5], \; [[x_1, x_2], x_5[x_3, x_4]], \; [[x_1, x_2], x_5([x_3, x_4]x_6)],$$

$$[[x_1, x_2]x_3, [x_4, x_5]x_6], \; [[x_1, x_2]x_3, x_6[x_4, x_5]], \; [[x_1, x_2]x_3, x_6([x_4, x_5]x_7)],$$

$$[x_3[x_1, x_2], x_6[x_4, x_5]], \; [x_3[x_1, x_2], x_6([x_4, x_5]x_7)], \; [x_3([x_1, x_2]x_4), x_5([x_6, x_7]x_8)],$$

$$x_1(x_2x_3) - (x_1x_2)x_3,$$

where $\mathcal{C} = \mathcal{A}3\mathfrak{k}$, $[x_1, x_2] = x_1x_2 - x_2x_1$.

3. Proof of Theorem 2, first part

By Lemma 8, $\langle L\mathcal{A}2\mathfrak{k}; \cdot_2 \rangle$ is a homomorphic image of $\langle L\mathcal{A}i\mathfrak{k}; \cdot_i \rangle$, $i = 0, 1$. Thus, it is sufficient to show that $(\mathcal{C} \cdot_2 \mathcal{C}) \cdot_2 \mathcal{C} \neq \mathcal{C} \cdot_2 (\mathcal{C} \cdot_2 \mathcal{C})$. This will be done by showing that $((C \circ C) \circ C)(F) \neq (C \circ (C \circ C))(F)$, where F is the free associative algebra on 2 generators a, b. We will show that

$$p = [a[[a, b], [a, b]a], [[a, b], [a, b]a]] \in (C \circ (C \circ C))(F)$$

but $p \notin ((C \circ C) \circ C)(F)$. Let T be the free semigroup on $\{a, b\}$. By Lemma 6, the $\mathfrak{k}$-module structure of F is a free unital module over $\mathfrak{k}$ with basis T. Thus, every element of F is a unique $\mathfrak{k}$-linear combination of elements of T. Let $N = ((C \circ C) \circ C)(F)$, $L = C(F) = [F, F]$ and $M = C(L)$. L is the ideal of F generated by $[f, g] = fg - gf$, $f, g \in F$. M is the ideal of L generated by all $[u, v]$, $u, v \in L$. By Lemma 9, N is the ideal of F generated by $(C \circ C)(C(F))$; i.e., N is the ideal of F generated by $C(C(C(F)))$. Thus N is the ideal of F generated by $[M, M]$, i.e., N is the ideal of F generated by $[u, v]$, $u, v \in M$. Let $c = ab - ba = [a, b]$. Elements of L are $\mathfrak{k}$-linear combinations of $[u, v]$, $s[u, v]$, $[u, v]t$, $s[u, v]t$; $s, t, u, v \in T$. By induction on the length of uv, $[u, v]$ is a $\mathfrak{k}$-linear combination of c, sc, ct, sct; $s, t \in T$; i.e., every element in L is a $\mathfrak{k}$-linear combination of sct, where $s, t \in T \cup \{1\}$, $1c = c = c1$. Elements of M are $\mathfrak{k}$-linear combinations of $w[sct, ucv]z$, where $s, t, u, v \in T \cup \{1\}$, $w, z \in L \cup \{1\}$.

The elements of M of least degree are of degree 5 and they are $\mathfrak{k}$-linear combinations of

(i) $[c, ac], \quad [c, ca], \quad [c, bc], \quad [c, cb].$

Elements of M of order 6 and degree 6 are $\mathfrak{k}$-linear combinations of

(ii)
$$\begin{array}{llll} [c, a^2c], & [c, aca], & [c, ca^2], & [ac, ca], \\ [c, b^2c], & [c, bcb], & [c, cb^2], & [bc, cb], \\ [c, abc], & [c, bac], & [c, acb], & [c, bca], \\ [c, cab], & [c, cba], & [ac, bc], & [ac, cb], \\ [ca, bc], & [ca, cb]. & & \end{array}$$

The ideal N of F generated by $[M, M]$ is generated by all $[u, v]$, $u, v \in M$. The elements of least degree in N are of degree 10. The elements of N of degree 10 are $\mathfrak{k}$-linear combinations of

(iii)
$$\begin{array}{lll} [[c, ac], [c, ca]], & [[c, ac], [c, cb]], & [[c, ac], [c, bc]], \\ [[c, ca], [c, bc]], & [[c, ca], [c, cb]], & [[c, bc], [c, cb]]. \end{array}$$

The elements of N of order 11 and degree 11 are $\mathfrak{k}$-linear combinations of ad, da, bd, db and $[g, h]$ where d belongs to the set (iii), i.e., d is of degree 10, g belongs to the set (i), i.e., g is of degree 5, and h belongs to the set (ii), i.e., h is of degree 6.

$F(2, \mathscr{C} \cdot_2 (\mathscr{C} \cdot_2 \mathscr{C})) \cong F/K$, where $K = (C \circ (C \circ C))(F)$. Thus K is the ideal of F generated by $C((C \circ C)(F))$. That is K is the ideal of F generated by $[(C \circ C)(F), (C \circ C)(F)] = [\overline{M}, \overline{M}]$, where $\overline{M}$ is the ideal of F generated by M. Now $a[c, ca] \in \overline{M}$, $[c, ca] \in M \subseteqq \overline{M}$. Hence $p = [a[c, ca], [c, ca]] \in K$. We will be through if we show that $p \notin N$. Since p is homogeneous of degree 7 in a and 4 in b, and by Lemma 6, F is a free $\mathfrak{k}$-module whose basis is T, $p \in N$ iff p is a $\mathfrak{k}$-linear combination of

$$\begin{array}{ll} u1 = [[c, ac], [c, a^2c]], & u2 = [[c, ac], [c, ca^2]], \\ u3 = [[c, ac], [c, aca]], & u4 = [[c, ac], [ac, ca]], \\ u5 = [[c, ca], [c, a^2c]], & u6 = [[c, ca], [c, ca^2]], \\ u7 = [[c, ca], [c, aca]], & u8 = [[c, ca], [ac, ca]], \\ u9 = a[[c, ac], [c, ca]], & u10 = [[c, ac], [c, ca]]a. \end{array}$$

The homogeneous elements of F of degree 7 in a and 4 in b with 0 form a free $\mathfrak{k}$-submodule P of rank $\binom{11}{4} = 330$. The basis of P is the set of all words of T of length 11 in which exactly 7 entries are a. Let R be the submodule of P spanned by $\{ui \colon 1 \leqq i \leqq 10\}$, and let S be the submodule of P spanned by $R \cup \{p\}$. $p \in N$ iff $p \in R$, i.e., iff $S = R$. Let B be a subset of the basis of P, then if $p \in N$, the images of

R and S under the module homomorphism $\sum \{a_i s_i: s_i \in \text{basis of } P\} \to \sum \{a_i s_i: s_i \in B\}$ coincide. Table I below gives the coefficients of $p, u1, \ldots, u10$ as $\mathfrak{k}$-linear combinations of elements of T in

$$x1 = ba^2ba^3ba^2b, \quad x2 = ba^4ba^2bab,$$
$$x3 = ba^3baba^3b, \quad x4 = baba^3ba^3b,$$
$$x5 = baba^5bab, \quad x6 = a^2b^2a^4bab,$$
$$x7 = ababa^3ba^2b, \quad x8 = a^3b^2aba^3b,$$
$$x9 = ba^2ba^3b^2a^2, \quad x10 = a^2bababa^3b.$$

Table I

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10
p	−2	0	−1	−1	0	0	1	0	−2	2
u1	1	−1	0	0	0	−1	−2	1	0	−2
u2	2	1	0	1	1	0	−2	0	0	1
u3	−3	0	−2	0	0	0	4	0	0	0
u4	0	1	0	0	0	−1	−2	0	−1	2
u5	−2	0	0	1	−1	0	2	−1	0	1
u6	−1	0	0	−2	0	0	1	0	−1	0
u7	3	0	2	1	0	0	−3	0	1	−1
u8	0	0	0	−1	0	0	1	0	2	−1
u9	0	0	0	0	0	−1	−2	1	0	0
u10	0	0	0	0	0	0	0	0	2	0

Table II

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10
$-u6$	1	0	0	2	0	0	−1	0	1	0
$u4$	0	1	0	0	0	−1	−2	0	−1	2
$-u3+3u6$	0	0	2	−6	0	0	−1	0	−3	0
$-u8$	0	0	0	1	0	0	−1	0	−2	1
$u2-u4+2u6-3u8$	0	0	0	0	1	1	−1	0	−7	2
$-u9$	0	0	0	0	0	1	2	−1	0	0
$u1+u3+u4+u6+u7-u8-2u9$	0	0	0	0	0	0	1	−1	−3	0
$-2u1-u3-2u4-2u6-u7+3u8+4u9$	0	0	0	0	0	0	0	2	9	−2
$u10$	0	0	0	0	0	0	0	0	2	0
$4p+u3-2u6+3u7+3u8-u10$	0	0	0	0	0	0	0	0	1	2
$8p+2u3-4u6+6u7+6u8-3u10$	0	0	0	0	0	0	0	0	0	4
$u2-u3-u4+u5-u7+u8+u9-u10$	0	0	0	0	0	0	0	0	0	0

The image of R into the submodule $\mathfrak{k}x3$ is $(2\mathfrak{k})x3$; the image of S is $\mathfrak{k}x3$. If 2 is not invertible in $\mathfrak{k}$, then $2\mathfrak{k}\neq\mathfrak{k}$ and $R\neq S$. If 2 is invertible in $\mathfrak{k}$, from Table II we get bases for the images of R and S in the free $\mathfrak{k}$-module $\sum\{\mathfrak{k}xi\colon 1\leqq i\leqq 10\}$.

The image of S into $\sum\{\mathfrak{k}xi\colon 1\leqq i\leqq 10\}$ is the whole submodule, i.e., it is a free $\mathfrak{k}$-module of rank 10. The image of R is a submodule generated by 9 elements. If $R=S$, we get a free $\mathfrak{k}$-module of two distinct ranks: 9 and 10. This is impossible since $\mathfrak{k}$ is a nontrivial commutative and associative ring with 1 and by reduction to $\mathfrak{k}/\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$, we get a vector space with two dimensions: 9 and 10, cf. P. M. Cohn [6], p. 6. This concludes the proof of the first part of Theorem 2.

4. Multinilpotent varieties

In this section we prove the second part of Theorem 2 and some results of interest in their own right.

Let $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$. If $\mathscr{V}\supseteqq\mathscr{A}i\mathfrak{k}$, then $d(i,\mathscr{V})=\infty$, otherwise $d(i,\mathscr{V})$ is the least degree of elements of $V(Fi)$, $\mathfrak{c}(i,n,\mathscr{V})$ is the ideal of $\mathfrak{k}$ generated by the coefficients of elements of $V(Fi)$ of degree n, $\mathfrak{c}(i,\mathscr{V})=\mathfrak{c}(i,d(i,\mathscr{V}),\mathscr{V})$, $i=0,1,2,3$.

Since V contains with every element of $F0$ all its linearizations, i.e.,

$$f(x_1,\dots,x_j+x_{n+1},\dots,x_n)-f(x_1,\dots,x_j,\dots,x_n)-f(x_1,\dots,x_{n+1},\dots,x_n),$$

$1\leqq j\leqq n$, cf. J. Goldman and S. Kass [9] and J. M. Osborn [17], $d(i,\mathscr{V})$ is achieved by multilinear identities.

Lemma 11. *If* $\mathscr{V},\mathscr{W}\in L\mathscr{A}i\mathfrak{k}$, *then* $d(i,\mathscr{V}\cdot_i\mathscr{W})=d(i,\mathscr{V})\,d(i,\mathscr{W})$. *Thus, if* $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$, $\mathscr{W}\neq\mathscr{A}i\mathfrak{k}$, *then* $\mathscr{V}\cdot_i\mathscr{W}\neq\mathscr{A}i\mathfrak{k}$, *i.e.,* $\langle L\mathscr{A}i\mathfrak{k};\cdot_i\rangle$ *has no zero-divisors. Furthermore,* $\mathfrak{c}(i,n,\mathscr{V})=\mathfrak{o}$ *iff* $d(i,\mathscr{V})>n$. *If* $\mathscr{V}\subseteqq\mathscr{W}$, *then* $d(i,\mathscr{V})\leqq d(i,\mathscr{W})$ *and* $\mathfrak{c}(i,n,\mathscr{V})\supseteqq\mathfrak{c}(i,n,\mathscr{W})$, $n\geqq 1$, $i=0,1,2,3$.

Proof. By Lemma 9, $(V\circ W)(Fi)$ is the ideal of Fi generated by $V(W(Fi))$, in the sense of the proof of Lemma 9. Thus the elements of least degree in $(V\circ W)(Fi)$ belong to $V(W(Fi))$. Let $f\in V(W(Fi))$. Then $f=g(w_1,\dots,w_n)$, where $g(x_1,\dots,x_n)\in V$, $w_1,\dots,w_n\in W$. $o(f)\geqq o(g)\min\{o(w_1),\dots,o(w_n)\}\geqq d(i,\mathscr{V})\cdot d(i,\mathscr{W})$. If g is multilinear of degree $d(i,\mathscr{V})$, each of $w_1,\dots,w_n$ are multilinear of degree $d(i,\mathscr{W})$, $w_1,\dots,w_n$ involves exactly $nd(i,\mathscr{W})$ elements of X, then $n=d(i,\mathscr{V})$, f is multilinear and $o(f)=d(f)=d(i,\mathscr{V})\,d(i,\mathscr{W})$. If $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$, $\mathscr{W}\neq\mathscr{A}i\mathfrak{k}$, then $d(i,\mathscr{V})$, $d(i,\mathscr{W})<\infty$, and $d(i,\mathscr{V}\cdot_i\mathscr{W})=d(i,\mathscr{V})\,d(i,\mathscr{W})<\infty$. $\mathscr{V}\subseteqq\mathscr{W}$ iff $V\supseteqq W$, from which the rest of the lemma follows.

Definition 4. A variety $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$ is i-multinilpotent if $V(Fi)=\sum\{\mathfrak{a}_n Fi^n\colon n\geqq 1\}$ where $\mathfrak{a}_1,\mathfrak{a}_2,\dots$ are ideals of $\mathfrak{k}$ and Fi^n is the set of all finite sums of all possible products of n elements of Fi, $i=0,1,2,3$.

It is clear that $Fi^{n+1} \subseteq Fi^n$, $n \geqq 1$. Thus we can assume $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ an ascending chain of ideals of $\mathfrak{k}$.

Lemma 12. *Let Mi be the set of all i-multinilpotent varieties. Then Mi is a complete sublattice of $\langle L\mathscr{A}i\mathfrak{k}; \wedge, \vee\rangle$, $i=0, 1, 2, 3$.*

Proof. Let $\mathscr{V}a$, $a \in I$, be i-multinilpotent varieties. Then there are ascending chains of ideals of $\mathfrak{k}$: $(\mathfrak{a}a_n)$, $n \geqq 1$, $a \in I$, such that $Va(Fi) = \sum \{\mathfrak{a}a_n Fi^n\colon n \geqq 1\}$, $a \in I$.

$$\left(\sum \{Va\colon a \in I\}\right)(Fi) = \sum \{Va(Fi)\colon a \in I\} =$$

$$= \sum \left\{\sum \{\mathfrak{a}a_n Fi^n\colon n \geqq 1\}\colon a \in I\right\} = \sum \left\{\sum \{\{\mathfrak{a}a_n\colon a \in I\} Fi^n\colon n \geqq 1\right\}.$$

Thus, the intersection of any family of i-multinilpotent varieties is i-multinilpotent.

$$\left(\bigcap\{Va\colon a \in I\}\right)(Fi) = \bigcap\{Va(Fi)\colon a \in I\} =$$

$$= \bigcap\left\{\sum \{\mathfrak{a}a_n Fi^n\colon n \geqq 1\}\colon a \in I\right\} \supseteqq \sum \left\{\bigcap\{\mathfrak{a}a_n\colon a \in I\} Fi^n\colon n \geqq 1\right\}.$$

If $f \in Va(Fi)$ for all $a \in I$, $f = f_1 + \ldots + f_r$ where each f_j is of order and of degree n_j, $n_1 < n_2 < \ldots < n_r$, then $f_j \in Fi^{n_j}$ and $f_j \in \mathfrak{a}a_{n_j} Fi^{n_j}$ for all $a \in I$, $1 \leqq j \leqq r$. Hence $f_j \in \bigcap \bigcap \{\mathfrak{a}a_{n_j}\colon a \in I\} Fi^{n_j}$, $1 \leqq j \leqq r$, i.e., $f \in \sum \left\{\bigcap \{\mathfrak{a}a_n\colon a \in I\} Fi^n\colon n \geqq 1\right\}$. Thus, the join of any family of i-multinilpotent varieties is i-multinilpotent.

Lemma 13. *Let $\mathscr{V}$ be i-multinilpotent, $\mathscr{V}, \mathscr{W} \in L\mathscr{A}i\mathfrak{k}$. Then $(V \circ W)(Fi) = V(W(Fi))$, $i=2, 3$.*

Proof. Since $(V \circ W)(Fi)$ is the ideal of Fi generated by $V(W(Fi))$ (from Lemma 9), we need to show that $V(W(Fi))$ is an ideal of Fi. $W(Fi)$ is an ideal of Fi. Hence $W(Fi)^n$ is an ideal of Fi and consequently $\mathfrak{a}_n W(Fi)^n$ is an ideal of Fi, where $\mathfrak{a}_n$ is an ideal of $\mathfrak{k}$, $n \geqq 1$. If $V(Fi) = \sum \{\mathfrak{a}_n Fi^n\colon n \geqq 1\}$, then $V(W(Fi)) = \sum \{\mathfrak{a}_n W(Fi)^n\colon n \geqq 1\}$ is an ideal of Fi.

Corollary 14. *If $\mathscr{U}, \mathscr{V}, \mathscr{W} \in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}$ is i-multinilpotent, then $(\mathscr{U} \cdot_i \mathscr{V}) \cdot_i \mathscr{W} = \mathscr{U} \cdot_i (\mathscr{V} \cdot_i \mathscr{W})$, $i=2, 3$.*

Proof. $(U \circ (V \circ W))(Fi)$ is the ideal of Fi generated by $U((V \circ W)(Fi))$ (from Lemma 9). From Lemma 13, $(V \circ W)(Fi) = V(W(Fi))$. Thus $U((V \circ W)(Fi)) = U(V(W(Fi)))$. $((U \circ V) \circ W)(Fi)$ is the ideal of Fi generated by $(U \circ V)(W(Fi))$. This is also the ideal of Fi generated by $U(V(W(Fi)))$. Hence $((U \circ V) \circ W)(Fi) = (U \circ (V \circ W))(Fi)$.

Since $\mathscr{A}i\mathfrak{k}$ and $\mathscr{E}$ are i-multinilpotent, Mi generates a submonoid with zero of $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i\rangle$, $i=2, 3$.

By Lemma 12, if $\mathscr{V} \in L\mathscr{A}i\mathfrak{k}$, the join of all i-multinilpotent varieties contained in $\mathscr{V}$ is i-multinilpotent. We will denote the largest i-multinilpotent variety contained in $\mathscr{V}$ by $\mathscr{V}'$.

Lemma 15. *Suppose* $\mathscr{V}, \mathscr{W} \in L\mathscr{A}i\mathfrak{k}$, $V(Fi)=\sum\{\mathfrak{a}_n Fi^n\colon n\geqq 1\}$, $W(Fi)= =\sum\{\mathfrak{b}_n Fi^n\colon n\geqq 1\}$, $(\mathfrak{a}_n)$, $(\mathfrak{b}_n)$ *are ascending chains of ideals of* $\mathfrak{k}$. *If* $\mathscr{U}=(\mathscr{V}\cdot_i\mathscr{W})$, *then* $U(Fi)=\sum\{\mathfrak{c}_n Fi^n\colon n\geqq 1\}$, *where* $\mathfrak{c}_n=\sum\{\mathfrak{a}_r\mathfrak{b}_{t1}\dots\mathfrak{b}_{tr}\colon t1+\dots+tr=n\}$, $i=0, 1, 2, 3$.

Proof. $(V\circ W)(Fi)$ is the ideal of Fi generated by $V(W(Fi))$.

$$V(W(Fi)) = V\left(\sum\{\mathfrak{a}_n W(Fi)^n\colon n\geqq 1\}\right) =$$

$$= \mathfrak{a}_1\sum\{\mathfrak{b}_n Fi^n\colon n\geqq 1\}+\mathfrak{a}_2\sum\{\mathfrak{b}_n Fi^n\colon n\geqq 1\}^2+\dots$$

$$= \sum\left\{\sum\{\mathfrak{a}_r\mathfrak{b}_{t1}\dots\mathfrak{b}_{tr}Fi^{t1}\dots Fi^{tr}\colon t1+\dots+tr=n\}\colon n\geqq 1\right\}\subseteqq\sum\{\mathfrak{c}_n Fi^n\colon n\geqq 1\}.$$

If $\sum\{\mathfrak{d}_n Fi^n\colon n\geqq 1\}\supseteqq (V\circ W)(Fi)$, then $\sum\{\mathfrak{a}_r\mathfrak{b}_{t1}\dots\mathfrak{b}_{tr}\colon t1+\dots+tr=n\}\subseteqq\mathfrak{d}_n$, $n\geqq 1$.

Proposition 16. *Let* $i=2, 3$. *Then* Mi *is a submonoid with zero of* $\langle L\mathscr{A}i\mathfrak{k};\ \cdot\rangle$, *and* $\langle Mi;\ \wedge, \vee, \cdot\rangle$ *is isomorphic to the partially ordered monoid of ascending chains of ideals of* $\mathfrak{k}$, *where* $(\mathfrak{a}_n)\leqq(\mathfrak{b}_n)$ *iff* $\mathfrak{a}_n\supseteqq\mathfrak{b}_n$ *for all* $n\geqq 1$ *and* $(\mathfrak{a}_n)\cdot(\mathfrak{b}_n)=(\mathfrak{c}_n)$, *where* $\mathfrak{c}_n=\sum\{\mathfrak{a}_r\mathfrak{b}_{t1}\dots\mathfrak{b}_{tr}\colon t1+\dots+tr=n\}$.

Proof. For $i=2, 3$, Fi is associative. Thus $Fi^m Fi^n=Fi^{m+n}$. From the proof of Lemma 15, the product of i-multinilpotent varieties is i-multinilpotent. Proposition 16 then follows from Lemmas 12 and 15.

Proof of Theorem 2, second part. If $\mathfrak{k}$ is a field of characteristic 0, every identity is equivalent to multilinear identities, cf. J. M. Osborn [17], p. 181. Hence, in $\mathscr{A}3\mathfrak{k}$ every variety is 3-multinilpotent. In fact every variety in $\mathscr{A}3\mathfrak{k}$ is either $\mathscr{A}3\mathfrak{k}$ or defined by $x_1\dots x_n=0$ for some $n\geqq 1$. By Proposition 16, $\langle L\mathscr{A}3\mathfrak{k};\ \cdot\rangle$ is isomorphic to the monoid of ascending chains of ideals of $\mathfrak{k}$; this is isomorphic to the multiplicative monoid of natural numbers.

5. Subgroupoids of varieties and minimal varieties

Lemma 17. *If* $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}'\neq\mathscr{E}$, *then* $V\subseteqq\mathfrak{m}F0+F0^2$ *for some maximal ideal* $\mathfrak{m}$ *of* $\mathfrak{k}$, $i=0, 1, 2, 3$.

Proof. Since $V(Fi)\subseteqq V'(Fi)=\mathfrak{a}_1 Fi+\mathfrak{a}_2 Fi^2+\dots$ and $\mathscr{V}'\neq\mathscr{E}$, then $\mathfrak{a}_1\neq\mathfrak{k}$. Thus $V(Fi)\subseteqq\mathfrak{a}_1 Fi+Fi^2\subseteqq\mathfrak{m}Fi+Fi^2$ for any maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$ containing $\mathfrak{a}_1$. Since $F0/\mathfrak{m}F0+F0^2\cong Fi/\mathfrak{m}Fi+Fi^2$, $V\subseteqq\mathfrak{m}F0+F0^2$.

Definition 5. A set P of non-trivial algebras is verbally closed if for every $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathfrak{R}\in P$, $V(\mathfrak{R})\in P$ or $\mathfrak{R}/V(\mathfrak{R})\in P$. $N(i, P)$ is the set of all subvarieties of $\mathscr{A}i\mathfrak{k}$ containing no members of P.

Any family of algebras with precisely 2 T-ideals (i.e., T-simple) is verbally closed. Any family of simple algebras is verbally closed.

Lemma 18. *Let* $M \subseteq L\mathscr{A}i\mathfrak{k}$. *Then* M *is a subgroupoid of* $\langle L\mathscr{A}i\mathfrak{k};\ \cdot_i\rangle$ *and a lattice ideal of* $\langle L\mathscr{A}i\mathfrak{k};\ \wedge, \vee\rangle$ *iff* $M=N(i, P)$ *for some verbally closed set of nontrivial algebras* P, $i=0, 1, 2, 3$.

Proof. Let P be verbally closed, $\mathscr{V}, \mathscr{W}\in N(i, P)$, $\mathscr{U}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{U}\subseteq\mathscr{V}$. Then $\mathscr{U}\in N(i, P)$. Since $\mathscr{V}\vee\mathscr{W}\subseteq\mathscr{V}\cdot_i\mathscr{W}$, $N(i, P)$ is a lattice ideal of $\langle L\mathscr{A}i\mathfrak{k};\ \wedge, \vee\rangle$ if $\mathscr{V}\cdot_i\mathscr{W}\in N(i, P)$. $\mathfrak{R}\in\mathscr{V}\cdot_i\mathscr{W}$ iff $W(\mathfrak{R})\in\mathscr{V}$, $\mathfrak{R}/W(\mathfrak{R})\in\mathscr{W}$ and $\mathfrak{R}\in\mathscr{A}i\mathfrak{k}$. Thus $\mathscr{V}\cdot_i\mathscr{W}$ does not contain any member $\mathfrak{R}$ of P, otherwise $W(\mathfrak{R})\in P$ or $\mathfrak{R}/W(\mathfrak{R})\in P$ contradicting $\mathscr{V}\in N(i, P)$, $\mathfrak{W}\in N(i, P)$. Conversely, let M be a subgroupoid of $\langle L\mathscr{A}i\mathfrak{k};\ \cdot_i\rangle$ and a lattice ideal of $\langle L\mathscr{A}i\mathfrak{k};\ \wedge, \vee\rangle$. Let K be the set of all non-trivial algebras obtained from $\{F\mathscr{V}\colon \mathscr{V}\in L\mathscr{A}i\mathfrak{k}\}$ by a finite number of applications of: If $\mathfrak{R}\in K$, $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $V(\mathfrak{R})\neq 0$, then $V(\mathfrak{R})\in K$ and if $\mathfrak{R}\neq V(\mathfrak{R})$, $\mathfrak{R}/V(\mathfrak{R})\in K$. Let P be the set of all algebras $\mathfrak{R}$ of K such that $\operatorname{var}\mathfrak{R}$, i.e., the variety generated by $\mathfrak{R}$, does not belong to M. We claim that $M=N(i, P)$. Let $\mathscr{V}\in M$. If $\mathfrak{R}\in\mathscr{V}$, then $\operatorname{var}\mathfrak{R}\subseteq\mathscr{V}$. Hence, $\operatorname{var}\mathfrak{R}\in M$ as M is a lattice ideal of $\langle L\mathscr{A}i\mathfrak{k};\ \wedge, \vee\rangle$. Thus, $\mathfrak{R}\notin P$, i.e., $M\subseteq N(i, P)$. Let $\mathscr{V}\in N(i, P)$. Then $F\mathscr{V}\notin P$. Since $\mathscr{V}=\operatorname{var} F\mathscr{V}$, $\mathscr{V}\in M$. It remains to check that P is verbally closed. Let $\mathfrak{R}\in\mathscr{A}i\mathfrak{k}$, $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$. If neither $V(\mathfrak{R})$ nor $\mathfrak{R}/V(\mathfrak{R})$ belongs to P, then $\operatorname{var} V(\mathfrak{R})$, $\operatorname{var}\mathfrak{R}/V(\mathfrak{R})\in M$. But

$$\mathfrak{R}\in\operatorname{var} V(\mathfrak{R})\cdot_i\operatorname{var}\mathfrak{R}/V(\mathfrak{R}).$$

Hence $\operatorname{var}\mathfrak{R}\subseteq\operatorname{var} V(\mathfrak{R})\cdot_i\operatorname{var}\mathfrak{R}/V(\mathfrak{R})$. Since M is a subgroupoid and a lattice ideal, $\operatorname{var}\mathfrak{R}\in M$, i.e., $\mathfrak{R}\notin P$.

Lemma 19. *The following conditions on a variety* $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $i=0, 1, 2, 3$, *are equivalent:*

(1) $x_1+f(x_1)\in V$ *for some* $f\in F_0^2$.

(2) $\mathscr{V}'=\mathscr{E}$, *i.e.,* $\mathscr{V}$ *does not contain any nontrivial i-multinilpotent varieties.*

(3) $\mathscr{V}\in N(i, \{O(\mathfrak{m})\colon \mathfrak{m}$ *is a maximal ideal of* $\mathfrak{k}\})$, *where* $O(\mathfrak{m})$ *is the algebra with zero multiplication on* $\mathfrak{k}/\mathfrak{m}$ *as a* $\mathfrak{k}$-*module.*

Proof. Let $x_1+f(x_1)\in V$, $f\in F0^2$. If $\mathscr{V}'\neq\mathscr{E}$, then $V\subseteq\mathfrak{m}F0+F0^2$ (by Lemma 17), for some maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$. Thus $x_1+f(x_1)\in\mathfrak{m}F0+F0^2$, i.e., $x_1\in\mathfrak{m}F0$ — a contradiction. If $\mathscr{V}'=\mathscr{E}$, then $O(\mathfrak{m})\notin\mathscr{V}$. In fact, if $\mathscr{W}=\operatorname{var} O(\mathfrak{m})=$ $=$the variety generated by $O(\mathfrak{m})$, then $W=\mathfrak{m}F0+F0^2$, $\mathscr{W}\subseteq\mathscr{V}$. Hence $V\subseteq\mathfrak{m}F0+$ $+F0^2$ and $V'\subseteq\mathfrak{m}F0+F0^2$, i.e., $\mathscr{V}'\neq\mathscr{E}$. Finally, if $O(\mathfrak{m})\notin\mathscr{V}$ for any maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$, then $V+F0^2=F0$. Otherwise, $V+F0^2$ is i-multinilpotent, $V+F0^2\neq$ $\neq F0$. Hence $V\subseteq V+F0^2\subseteq\mathfrak{m}F0+F0^2$ for some maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$. Thus $F0/\mathfrak{m}F0+F0^2\in\mathscr{V}$. This implies $O(\mathfrak{m})\in\mathscr{V}$ since the subalgebra of $F0/\mathfrak{m}F0+F0^2$ generated by $x_1+\mathfrak{m}F0+F0^2$ is isomorphic to $O(\mathfrak{m})$. Now $x_1\in F0=V+F0^2$. Hence, there are $v \in V$, $f \in F0^2$ such that $x_1 = v - f$. By substituting 0 for all $x_i \neq x_1$, we can assume $f = f(x_1)$. Thus $x_1 + f(x_1) \in V$, $f \in F0^2$.

Corollary 20. *The set of varieties $\mathscr{V} \in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}' = \mathscr{E}$ is a subgroupoid of $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i \rangle$ and a lattice ideal of $\langle L\mathscr{A}i\mathfrak{k}; \wedge, \vee \rangle$, $i = 0, 1, 2, 3$.*

This follows from Lemmas 18 and 19.

Corollary 21. *Let $\mathscr{U}, \mathscr{V} \in L\mathscr{A}i\mathfrak{k}$. Then $(\mathscr{U} \cdot_i \mathscr{V})' = \mathscr{E}$ iff $\mathscr{U}' = \mathscr{V}' = \mathscr{E}$, $i = 0, 1, 2, 3$.*

This follows from Corollary 20 and Lemma 19 since $\mathscr{U}' \vee V' \subseteq (\mathscr{U} \vee \mathscr{V})' \subseteq \subseteq (\mathscr{U} \cdot_i \mathscr{V})'$.

Let G be a commutative non-trivial ring with 1 and let α be a homomorphism of $\mathfrak{k}$ into G preserving 1. Then G has a natural $\mathfrak{k}$-algebra structure: $ag = (a\alpha)g$, $a \in \mathfrak{k}$, $g \in G$. This $\mathfrak{k}$-algebra structure on G will be denoted by $\mathfrak{G}\alpha$.

Lemma 22. *Let G, H be commutative non-trivial rings with 1 and α, β homomorphisms of $\mathfrak{k}$ into G, H, respectively, preserving 1. Then $\mathfrak{G}\alpha$ is isomorphic to a subalgebra of $\mathfrak{H}\beta$ iff there is an injective ring homomorphism γ of G into H such that $\alpha\gamma = \beta$, γ preserves 1.*

Proof. If γ is an injective ring homomorphism of G into H and $\alpha\gamma = \beta$, then γ is an injective homomorphism of $\mathfrak{k}$-algebras. Conversely, if there is an injective homomorphism γ of $\mathfrak{G}\alpha$ into $\mathfrak{H}\beta$ and γ preserves 1, then γ is a ring homomorphism and $a\alpha\gamma = (a\alpha)\gamma = ((a1)\alpha)\gamma = (a\alpha \cdot 1\alpha)\gamma = a((1\alpha)\gamma) = a(1\beta) = a\beta$ for every $a \in \mathfrak{k}$.

Lemma 23. *Let $\mathscr{V} \in L\mathscr{A}2\mathfrak{k}$, $\mathscr{V}' = \mathscr{E}$, $\mathscr{V} \neq \mathscr{E}$. Then $\mathscr{V}$ satisfies $x - x^m = 0$ for some $m > 1$. There are a finite number of non-isomorphic finite fields G_j, $1 \leqq j \leqq n$, and sets Ij of homomorphisms of $\mathfrak{k}$ into G_j preserving 1, $1 \leqq j \leqq n$, such that $\mathscr{V}$ is generated by $\{\mathfrak{G}j\alpha \colon \alpha \in Ij,\ 1 \leqq j \leqq n\}$.*

Proof. Let $\mathfrak{R} = F(1, \mathscr{V})$. $\mathfrak{R}$ is associative and commutative. Since $\mathscr{V}' = \mathscr{E}$, by Lemma 19, $\mathscr{V}$ satisfies $x + f(x) = 0$ where $f(x)$ is of order $\geqq 2$. Thus $\mathscr{V}$ satisfies $x = x^2 h(x)$ where $h(x) \in \mathfrak{k}[x]$, the ring of polynomials in x over $\mathfrak{k}$. Thus $\mathfrak{R}$ is a commutative and associative von Neumann regular ring. Hence $\mathfrak{R}$ is a ring subdirect product of fields. If G is one of these fields, there is a ring homomorphism γ of R onto G. G inherits a $\mathfrak{k}$-algebra structure: $ag = (ag_1)\gamma$, $a \in \mathfrak{k}$, $g_1 \in \mathfrak{R}$, $g_1\gamma = g \in G$. There is a homomorphism α of $\mathfrak{k}$ into G preserving 1: $a\alpha = ae$ where e is the identity of G. $\mathfrak{G}\alpha$ is a homomorphic image of $\mathfrak{R}$ as $\mathfrak{k}$-algebras. Thus $\mathfrak{G}\alpha$ satisfies $x = x^2 h(x)$. Thus G is finite since all its elements are roots of $x^2 h(x) - x = 0$, $|G| \leqq \text{degree}\, x^2 h(x) = = \text{degree}\, f$. There is only a finite number of non-isomorphic fields satisfying $x = x^2 h(x)$. Let $G1, \ldots, Gn$ be all the finite fields such that if G is a field and G is a homomorphic image of $\mathfrak{R}$, then G is an isomorphic copy of $G1, \ldots,$ or Gn and $Gi \not\cong Gj$ if $i \neq j$, $1 \leqq i, j \leqq n$, and let Ij be the set of all homomorphisms α of $\mathfrak{k}$ into

Gj such that $\mathfrak{G}j\alpha$ is a homomorphic image of $\mathfrak{R}$, $1 \leqq j \leqq n$. Then there is $m > 1$ such that $x = x^m$ in $G1 \times \ldots \times Gn$. Thus $\mathscr{V}$ satisfies $x - x^m = 0$. Thus, by Jacobson's Theorem $\mathscr{V}$ is commutative and FV is a ring subdirect sum of fields satisfying $x - x^m = 0$, i.e., finite fields. If H is one of these fields, then as above, there is a homomorphism α of $\mathfrak{k}$ into H such that $\mathfrak{H}\alpha$ is a homomorphic image of $F\mathscr{V}$. Thus $\mathfrak{H}\alpha \in \mathscr{V}$. But $\mathfrak{H}\alpha$ is generated by one element. Hence, $\mathfrak{H}\alpha$ is a homomorphic image of $F(1, \mathscr{V}) = \mathfrak{R}$. Thus $H \cong Gj$ for some $1 \leqq j \leqq n$, and $\mathfrak{H}\alpha \cong \mathfrak{G}j\beta$ for some $\beta \in Ij$. Thus $\mathscr{V}$ is generated by $\{\mathfrak{G}j\alpha \colon \alpha \in Ij,\ 1 \leqq j \leqq n\}$. That $\mathfrak{H}\alpha \cong \mathfrak{G}j\beta$ follows from Lemma 22.

It may be noted that although the non-isomorphic fields in Lemma 23 are finitely many, the non-isomorphic algebras $\mathfrak{G}j\alpha$, $\alpha \in Ij$, $1 \leqq j \leqq n$, can be infinitely many. For instance, if $\mathfrak{k}$ is an infinite Boolean ring, $\mathscr{V}$ is the variety of associative $\mathfrak{k}$-algebras satisfying $x + x^2 = 0$, then $F(1, \mathscr{V})$ is ring isomorphic to an infinite subdirect power of $\mathbf{Z}_2$, the prime field of 2 elements; $F(1, \mathscr{V}) \cong \mathfrak{k}$. However, $\mathfrak{k}$ is a subdirect product of $\{\mathfrak{k}/\mathfrak{m} \colon \mathfrak{m}$ is a maximal ideal of $\mathfrak{k}\}$. $\mathfrak{k}/\mathfrak{m}$ is ring isomorphic to $\mathbf{Z}_2$, but $\mathfrak{k}/\mathfrak{m} \cong \mathfrak{k}/\mathfrak{m}'$ as $\mathfrak{k}$-algebras iff $\mathfrak{m} = \mathfrak{m}'$.

Corollary 24. *Let* $\mathscr{V} \in L\mathscr{A}1\mathfrak{k}$, $\mathscr{V}' = \mathscr{E}$, $\mathscr{V} \neq \mathscr{E}$. *Then* $\mathscr{V}$ *satisfies* $x - x^m = 0$ *for some* $m > 1$. *There are a finite number of non-isomorphic finite fields* $G1, \ldots, Gn$ *and sets* Ij *of homomorphisms of* $\mathfrak{k}$ *into* Gj *preserving* 1, $1 \leqq j \leqq n$, *such that* $F(1, \mathscr{V})$ *is a subdirect product of* $\{\mathfrak{G}j\alpha \colon \alpha \in Ij,\ 1 \leqq j \leqq n\}$.

This follows from Lemma 23 since $\operatorname{var} F(1, \mathscr{V}) \in L\mathscr{A}2\mathfrak{k}$ and $\operatorname{var} F(1, \mathscr{V})' \subseteq \subseteq \mathscr{V}' = \mathscr{E}$.

Lemma 25. *Let* $\mathfrak{R} \in \mathscr{A}2\mathfrak{k}$, *I an ideal of* $\mathfrak{R}$ *and J an ideal of I. If J or* I/J *satisfies* $x + f(x) = 0$ *for some* $f \in F0^2$, *then J is an ideal of* $\mathfrak{R}$.

Proof. Let J satisfy $x + f(x) = 0$. Hence by Lemma 23, J satisfies $x = x^m$ for some $m > 1$. Let $a \in \mathfrak{R}$, $b \in J$. Then $ab = ab^m = (ab^{m-1})b$. But $ab^{m-1} \in I$. Hence $ab \in J$. Similarly $ba \in J$. Let I/J satisfy $x + f(x) = 0$. Hence I/J satisfies $x - x^m = 0$ for some $m > 1$. If $a \in \mathfrak{R}$, $b \in J$, $c \in I$, then $c - c^m \in J$, $ac, ca \in I$. Thus $ab \in I$, and $ab - (ab)^m \in J$. $(ab)^m = ((ab)^{m-1}a)b$. But $(ab)^{m-1}a \in I$. Hence $(ab)^m \in J$ and $ab \in J$. Similarly, $ba \in J$.

Corollary 26. *Let* $\mathscr{U}, \mathscr{V}, \mathscr{W} \in L\mathscr{A}i\mathfrak{k}$. *If* $\mathscr{U}' = \mathscr{E}$ *or* $\mathscr{V}' = \mathscr{E}$, *then* $(\mathscr{U} \cdot_i \mathscr{V}) \cdot_i \mathscr{W} = = \mathscr{U} \cdot_i (\mathscr{V} \cdot_i \mathscr{W})$, $i = 2, 3$.

Proof. By Theorem 1, $\mathscr{U} \cdot_i (\mathscr{V} \cdot_i \mathscr{W}) \subseteq (\mathscr{U} \cdot_i \mathscr{V}) \cdot_i \mathscr{W}$. Let $\mathfrak{R} \in (\mathscr{U} \cdot_i \mathscr{V}) \cdot_i \mathscr{W}$. Then $\mathfrak{R} \in \mathscr{A}2\mathfrak{k}$, there is an ideal I of $\mathfrak{R}$ and an ideal J of I such that $\mathfrak{R}/I \in \mathscr{W}$, $I \in \mathscr{U} \cdot_i \mathscr{V}$, $J \in \mathscr{U}$, $I/J \in \mathscr{V}$. Since $\mathscr{U}' = \mathscr{E}$ or $\mathscr{V}' = \mathscr{E}$, by Lemma 19, I/J or J satisfies $x + +f(x) = 0$ for some $f \in F0^2$. Hence, by Lemma 25, J is an ideal of $\mathfrak{R}$. Thus $I/J \in \mathscr{V}$ and $\mathfrak{R}/J \in \mathscr{V} \cdot \mathscr{W}$, i.e., $\mathfrak{R} \in \mathscr{U} \cdot_i (\mathscr{V} \cdot_i \mathscr{W})$.

Lemma 27. *Let $\mathfrak{R}\in\mathscr{A}\mathfrak{A}$ and S an ideal of $\mathfrak{R}$ satisfying $x+f(x)=0$ for some $f\in F0^2$. Then $\mathfrak{R}$ is isomorphic to a subdirect product of $\mathfrak{R}/S$ and an algebra satisfying all the identities of S. If $\mathfrak{R}$ is finitely generated, then S is a direct summand of $\mathfrak{R}$.*

Proof. By Lemma 23, S satisfies $x-x^m=0$ for some $m>1$ and S is commutative. In fact S is central in $\mathfrak{R}$. Let $a\in\mathfrak{R}$, $b\in S$. Then $ab=ab^m=(ab)b^{m-1}=$ $=b^{m-1}(ab)=(b^{m-1}a)b=b(b^{m-1}a)=b^m a=ba$. Let $A=\mathrm{Ann}\, S$, i.e.,

$$A = \{x\colon\ x\in\mathfrak{R},\ xS = 0\}.$$

A is an ideal of $\mathfrak{R}$, $A=\bigcap\{\mathrm{Ann}\, b\colon\ b\in S\}$, $\mathrm{Ann}\, b$ is an ideal of $\mathfrak{R}$. $A\cap S=0$, since $b\in A\cap S$ implies $b=b^m=bb^{m-1}=0$. Thus $\mathfrak{R}$ is isomorphic to a subdirect product of $\mathfrak{R}/S$ and $\mathfrak{R}/A$. If $b\in S$, b^{m-1} is a central idempotent and $b^{m-1}\mathfrak{R}=b\mathfrak{R}$. Thus $\mathfrak{R}\cong b\mathfrak{R}\oplus\mathrm{Ann}\, b$. Hence $\mathfrak{R}/\mathrm{Ann}\, b\cong b\mathfrak{R}\subseteq S$. But $\mathfrak{R}/A$ is a subdirect product of $\mathfrak{R}/\mathrm{Ann}\, b\cong b\mathfrak{R}$. Thus $\mathfrak{R}/A$ satisfies all the identities of S. If $\mathfrak{R}$ is finitely generated, then $\mathfrak{R}/A$ is finitely generated. As $A=\mathrm{Ann}\, S$, there are $b1, \dots, bm\in S$ such that $b1+A, \dots, bm+A$ generate $\mathfrak{R}/A$. Hence $b1, \dots, bm$ generate S. If $ei=(bi)^{m-1}$, then $e1, \dots, em$ are central idempotents, $S=e1R+\dots+emR$. There is an orthogonal set of idempotents $f1, \dots, fr$ such that $S=f1R\oplus\dots\oplus frR$. Thus S has an identity element $e=f1+f2+\dots+fr$, e is a central idempotent $\mathfrak{R}=e\mathfrak{R}\oplus\mathrm{Ann}\, e=S\oplus\mathrm{Ann}\, e=$ $=S\oplus A$.

Corollary 28. *Let $\mathscr{U}, \mathscr{V}\in L\mathscr{A}\mathfrak{k}$, $\mathscr{U}'=\mathscr{E}$. Then $\mathscr{U}\vee\mathscr{V}=\mathscr{U}\cdot_i\mathscr{V}$, $i=2, 3$.*

If $\mathfrak{R}\in\mathscr{U}\cdot_i\mathscr{V}$, there is an ideal I of $\mathfrak{R}$ such that $\mathfrak{R}/I\in\mathscr{V}$ and $I\in\mathscr{U}$. The corollary follows from Lemmas 19 and 27.

Corollary 29. *If $\mathscr{U}\in L\mathscr{A}\mathfrak{k}$, $\mathscr{U}'=\mathscr{E}$, then $\mathscr{U}\cdot_i\mathscr{U}=\mathscr{U}$, $i=2, 3$.*

This follows from Corollary 28.

Corollary 30. *Let $\mathscr{V}\in L\mathscr{A}1\mathfrak{k}$, $\mathscr{V}'=\mathscr{E}$ and let $\mathscr{V}^{(1)}$ be the variety defined by all one-variable identities of $\mathscr{V}$. Then $\mathscr{V}^{(1)}\cdot_1\mathscr{V}^{(1)}=\mathscr{V}^{(1)}$.*

Proof. $\mathscr{V}^{(1)}\in L\mathscr{A}1\mathfrak{k}$ since every member of $\mathscr{V}^{(1)}$ generated by one element belongs to $\mathscr{V}$. $\mathscr{V}^{(1)\prime}=\mathscr{E}$ since $\mathscr{V}$ and also $\mathscr{V}^{(1)}$ satisfy $x+f(x)=0$ for some $f\in F0^2$ (by Lemma 19). Let $\mathfrak{R}=F(1, \mathscr{V}^{(1)}\cdot_1\mathscr{V}^{(1)})$. By Corollary 24, $\mathfrak{R}$ is a subdirect product of $\mathfrak{G}j\alpha$, $\alpha\in Ij$, $1\leqq j\leqq n$, $G1, \dots, Gn$ are finite fields. Since $\mathfrak{G}j\alpha\in\mathscr{V}^{(1)}\cdot_1\mathscr{V}^{(1)}$ and $\mathfrak{G}j\alpha$ is simple $\mathfrak{G}j\alpha\in\mathscr{V}^{(1)}$, i.e., $\mathfrak{G}j\alpha\in\mathscr{V}$. Thus $\mathfrak{R}\in\mathscr{V}$. Hence, $\mathscr{V}^{(1)}\cdot_1\mathscr{V}^{(1)}$ satisfies all the one-variable identities of $\mathscr{V}$. Thus $\mathscr{V}^{(1)}\subseteqq\mathscr{V}^{(1)}\cdot_1\mathscr{V}^{(1)}\subseteqq\mathscr{V}^{(1)}$.

Lemma 31. *Let G, H be finite fields, α, β homomorphisms of $\mathfrak{k}$ into G, H, respectively, preserving 1. Then $\mathfrak{H}\beta\in\mathrm{var}\,\mathfrak{G}\alpha$ iff $\ker\alpha=\ker\beta$ and H is isomorphic to a subfield of G.*

Proof. If $\ker\alpha=\ker\beta$ and H is isomorphic to a subfield of G, then $\mathfrak{H}\beta$ is isomorphic to a subalgebra of $\mathfrak{G}\alpha$ since H contains $\mathfrak{k}\beta\cong\mathfrak{k}\alpha$. Conversely, if $\mathfrak{H}\beta\in\operatorname{var}\mathfrak{G}\alpha$, $|G|=p^n$, then H satisfies $x-x^{p^n}=0$ and $ax=0$ for all $a\in\ker\alpha$. Thus H is of order p^m, $m|n$. As $\ker\alpha$ and $\ker\beta$ are maximal ideals of $\mathfrak{k}$, $\mathfrak{H}\beta$ satisfies $ax=0$ for all $a\in\ker\alpha+\ker\beta$, H is non-trivial, $\ker\alpha=\ker\beta$ and H is isomorphic to a subfield of G.

Proposition 32. *The set Ti of all varieties* $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}'=\mathscr{E}$ *is a submonoid of* $\langle L\mathscr{A}i\mathfrak{k};\ \cdot_i\rangle$ *and a lattice ideal of* $\langle L\mathscr{A}i\mathfrak{k};\ \wedge, \vee\rangle$. *On Ti, the lattice join* $\vee$ *and the variety multiplication* $\cdot_i$ *coincide. The lattice* $\langle Ti, \wedge, \vee\rangle$ *is isomorphic to the lattice of left ideals with a finite number of right components of* $\langle\{(\mathfrak{m}, p^n)\colon \mathfrak{m}$ *is a maximal ideal such that* $\mathfrak{k}/\mathfrak{m}$ *is a subfield of a finite field of order* $p^n\};\ \leqq\rangle$, $(\mathfrak{m}, p^n)\leqq(\mathfrak{m}', q^{n'})$ *iff* $\mathfrak{m}=\mathfrak{m}'$ *and* $p=q$, $n|n'$, $i=2, 3$.

Proof. That $\langle Ti, \cdot\rangle$ is a submonoid of $\langle L\mathscr{A}i\mathfrak{k};\ \cdot_i\rangle$ follows from Corollaries 20 and 26. Also from Corollary 20, Ti is a lattice ideal. By Corollary 28, $\mathscr{U}\cdot_i\mathscr{V}=\mathscr{U}\vee\mathscr{V}$. By Lemma 23, if $\mathscr{V}\in Ti$, $\mathscr{V}\neq\mathscr{E}$, then $\mathscr{V}=\operatorname{var}\{\mathfrak{G}j\alpha\colon \alpha\in Ij,\ 1\leqq j\leqq n\}$. By Lemma 31, $\mathfrak{H}\beta\in\mathscr{V}$ iff $\mathfrak{H}\beta$ is isomorphic to a subalgebra of $\mathfrak{G}j\alpha$ for some $\alpha\in Ij$, $1\leqq j\leqq n$. Thus $\mathscr{V}\in Ti$ is determined by the set of all pairs $(\mathfrak{m}, p^n)$ such that G is a field of p^n elements and $\mathfrak{k}/\mathfrak{m}$ is a subfield of G, $\mathfrak{G}\alpha\in\mathscr{V}$, where α is the natural homomorphism of $\mathfrak{k}$ onto $\mathfrak{k}/\mathfrak{m}\subseteq G$. The set of all such pairs for a given $\mathscr{V}$ satisfies $(\mathfrak{m}, p^n)\leqq(\mathfrak{m}', q^{n'})$, $(\mathfrak{m}', q^{n'})$ is in the set implies $(\mathfrak{m}, p^n)$ is in the set. Thus it is a left ideal. Since every $\mathscr{V}$ involves only a finite number of non-isomorphic fields, the set of right components in the set of pairs is finite.

Proof of Theorem 4. Let $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$ be equationally complete and $\mathscr{V}'\neq\mathscr{E}$. Then $\mathscr{V}=\mathscr{V}'$. By Lemma 17, $V\subseteq\mathfrak{m}F_0+F_0^2$ for some maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$. Hence, $V=\mathfrak{m}F_0+F_0^2$. V is a maximal T-ideal of F_0, $\mathscr{V}$ satisfies $ax=0$ for all $a\in\mathfrak{m}$, and $xy=0$. This is the type of equationally complete varieties $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$, $\mathscr{V}'\neq\mathscr{E}$. If $\mathscr{V}\in L\mathscr{A}1\mathfrak{k}$, $\mathscr{V}'=\mathscr{E}$, $\mathscr{V}$ is equationally complete, then $\mathscr{V}=\operatorname{var}F(1, \mathscr{V})$, $\operatorname{var}F(1, \mathscr{V})\in L\mathscr{A}3\mathfrak{k}$. By Lemma 23, $\mathscr{V}=\operatorname{var}\{\mathfrak{G}j\alpha\colon \alpha\in Ij,\ 1\leqq j\leqq n\}$. Hence $\mathscr{V}=\operatorname{var}\mathfrak{G}\alpha$, for some finite field G and a homomorphism α of $\mathfrak{k}$ into G preserving 1. Thus $\mathscr{V}=\operatorname{var}\mathfrak{k}/\mathfrak{m}$ for some maximal ideal of finite index in $\mathfrak{k}$, since $\mathfrak{G}\alpha$ contains a subalgebra isomorphic to $\mathfrak{k}/\ker\alpha$. By Lemma 31, $\operatorname{var}\mathfrak{k}/\mathfrak{m}$ does not contain any non-trivial proper subvarieties. Thus $\mathscr{V}$ is determined by the identities $ax=0$ for all $a\in\mathfrak{m}$, $x-x^{p^r}=0$ where $p^r=|\mathfrak{k}/\mathfrak{m}|$.

6. Varieties of algebras over rings with exactly 2 idempotent ideals

Throughout Section 6, we assume that if $\mathfrak{a}$ is an ideal of $\mathfrak{k}$, and $\mathfrak{a}^2=\mathfrak{a}$, then $\mathfrak{a}=\mathfrak{o}$ or $\mathfrak{a}=\mathfrak{k}$.

Lemma 33. *Let* $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}\cdot_i\mathscr{V}=\mathscr{V}$. *Then* $\mathscr{V}=\mathscr{A}i\mathfrak{k}$ *or* $\mathscr{V}'=\mathscr{E}$; $i=0, 1, 2, 3$.

Proof. If $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$, then $d(i,\mathscr{V})<\infty$ and $d(i,\mathscr{V})=d(i,\mathscr{V}\cdot_i\mathscr{V})=d(i,\mathscr{V})^2$ (by Lemma 11). Thus, $d(i, V)=1$, i.e., there are non-trivial polynomials of degree 1 in V and $V(Fi)\subseteqq V'(Fi)=\mathfrak{a}_1Fi+\mathfrak{a}_2Fi^2+\dots$ where $\mathfrak{a}_1\neq\mathfrak{o}$. Hence $(V\circ V)(Fi)\subseteqq\subseteqq(V'\circ V')(Fi)\subseteqq\mathfrak{a}_1^2Fi+\mathfrak{a}_1\mathfrak{a}_2Fi^2+\dots$ (by Lemma 15). Thus $V(Fi)=(V\circ V)(Fi)\subseteqq\subseteqq\mathfrak{a}_1^2Fi+\mathfrak{a}_1\mathfrak{a}_2Fi^2+\dots\subseteqq V'(Fi)$. But $\mathfrak{a}_1^2Fi+\mathfrak{a}_1\mathfrak{a}_2Fi^2+\dots$ is i-multinilpotent, whence $V'(Fi)=\mathfrak{a}_1^2Fi+\mathfrak{a}_1\mathfrak{a}_2Fi^2+\dots=\mathfrak{a}_1Fi+\mathfrak{a}_2Fi^2+\dots$. Hence $\mathfrak{a}_1^2=\mathfrak{a}_1$. But $\mathfrak{a}_1\neq\mathfrak{o}$. Hence $\mathfrak{a}_1=\mathfrak{k}$, i.e., $V'(Fi)=Fi$, i.e., $\mathscr{V}'=\mathscr{E}$.

Corollary 34. *Let* $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$. *Then* $\mathscr{V}\cdot_i\mathscr{V}=\mathscr{V}$ *iff* $\mathscr{V}'=\mathscr{E}$, $i=2, 3$.

This follows from Corollary 29 and Lemma 33.

It may be noted that if $\mathfrak{k}$ has an ideal $\mathfrak{a}\neq\mathfrak{o}$, $\mathfrak{a}\neq\mathfrak{k}$, $\mathfrak{a}^2=\mathfrak{a}$, then the variety $\mathscr{V}$ of all $\mathfrak{k}$-algebras satisfying $ax=0$ for all $a\in\mathfrak{a}$ is idempotent, i.e., $\mathscr{V}\cdot_0\mathscr{V}=\mathscr{V}$, $\mathscr{V}'=\mathscr{V}\neq\mathscr{E}$.

Proof of Theorem 5. A set $I\subseteqq F0$ is attainable on a variety $\mathscr{V}$ iff the T-ideal of $F0$ generated by I is attainable on $\mathscr{V}$. It was shown by A. I. MAL'CEV [13], that if I is attainable on $\mathscr{V}$, then the variety $\mathscr{U}\in L\mathscr{V}$ determined by I satisfies $\mathscr{U}\cdot_{\mathscr{V}}\mathscr{U}=\mathscr{U}$, or equivalently $(U\circ U)(F\mathscr{V})=U(F\mathscr{V})$. If $\mathscr{U}\cdot_i\mathscr{U}=\mathscr{U}$, then $\mathscr{U}=\mathscr{A}i\mathfrak{k}'$ or $\mathscr{U}'=\mathscr{E}$ by Lemma 33. Let $i=1, 2, 3$. Then $\mathscr{U}\cap\mathscr{A}2\mathfrak{k}$ is generated by $\{\mathfrak{G}j\alpha\colon \alpha\in Ij,\ 1\leqq j\leqq n\}$, by Lemma 23, if $\mathscr{U}\neq\mathscr{E}$, $\mathscr{U}\neq\mathscr{A}i\mathfrak{k}$. Let $\mathfrak{m}$ be $\ker\alpha$ for some $\alpha\in Ij$, $1\leqq j\leqq n$. Let $\mathfrak{R}$ be the ideal of $(\mathfrak{k}/\mathfrak{m})[x]$ generated by x. $U(\mathfrak{R})==\bigcap\{Vj\alpha(\mathfrak{R})\colon \alpha\in Ij,\ 1\leqq j\leqq n\}$, $\mathscr{V}j\alpha=\operatorname{var}\mathfrak{G}j\alpha$. $Vj\alpha(\mathfrak{R})\neq\mathfrak{R}$ iff $\mathfrak{m}=\ker\alpha$. Also, $G1, \dots, Gn$ are finitely many and each Gj is a finite field, there is only finitely many $\mathfrak{G}j\alpha$ such that $Vj\alpha(\mathfrak{R})\neq\mathfrak{R}$ for some $\alpha\in Ij$, $1\leqq j\leqq n$. $Vj\alpha(\mathfrak{R})\neq 0$ for any $\alpha\in Ij$, $1\leqq j\leqq n$. Thus $U(\mathfrak{R})$ is a proper non-trivial ideal of $\mathfrak{R}$. Hence, there is a polynomial $h(x)\in\mathfrak{R}$, $h(x)\neq x$, $h(x)\neq 0$, such that $U(\mathfrak{R})=h(x)(\mathfrak{k}/\mathfrak{m})[x]$. By the methods of the proof of A. A. ISKANDER's [11], Theorem 15, p. 237, replacing the prime field $\mathbf{Z}_p$ by $\mathfrak{k}/\mathfrak{m}$ one can show that $U(U(\mathfrak{R}))\neq U(\mathfrak{R})$. Thus U is not attainable on $\mathfrak{R}$. Hence, if I is attainable on $\mathscr{A}i\mathfrak{k}$, $\mathscr{U}$ is the variety of $L\mathscr{A}i\mathfrak{k}$ determined by I, then $\mathscr{U}=\mathscr{A}i\mathfrak{k}$, i.e., I is equivalent to $x=x$ on $\mathscr{A}i\mathfrak{k}$, or $\mathscr{U}=\mathscr{E}$, i.e., I is equivalent to $x=y$ on $\mathscr{A}i\mathfrak{k}$.

7. Varieties of algebras over Dedekind domains

Throughout Section 7, unless otherwise stated, $\mathfrak{k}$ is a Dedekind domain.

Proposition 35. *The following conditions on a variety $\mathcal{V} \in L\mathcal{A}i\mathfrak{k}$ are equivalent:*

(1) *$\mathcal{V}$ satisfies $x^n=0$ for some natural number $n>0$.*

(2) *$\mathcal{V} \in N(i, \{\mathfrak{k}/\mathfrak{m}$: $\mathfrak{m}$ is a maximal ideal of $\mathfrak{k}\})$, $i=1, 2, 3$.*

Proof. Since a field does not contain any non-zero nilpotent elements, (1) implies (2). Let $\mathcal{V} \in L\mathcal{A}i\mathfrak{k}$, $\mathfrak{k}/\mathfrak{m} \notin \mathcal{V}$ for any maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$. As $F(1, \mathcal{V}) \in \mathcal{A}3\mathfrak{k}$, the factor algebra $F(1, \mathcal{V})/\mathfrak{N}$, where $\mathfrak{N}$ is the nilradical, is a subdirect product of rings without zero-divisors. Thus, if $\mathfrak{N} \neq F(1, \mathcal{V})$, the algebra $F(1, \mathcal{V})$ has a non-trivial factor algebra $\mathfrak{R}$ without zero-divisors, and it is not difficult to show that $\mathfrak{R}$ can be chosen such that for its "characteristic" $\mathfrak{p} \lhd \mathfrak{k}$ one has either $\mathfrak{R} \cong x(\mathfrak{k}/\mathfrak{p})[x]$ or $\mathfrak{R} \cong x(\mathfrak{k}/\mathfrak{p})[x]/f(x)$ where f is primitive irreducible, and $\mathfrak{R}$ obviously has in both cases field factors, which, in their turn, must have a subfield of the prescribed form.

Proposition 36. *If $\mathfrak{k}$ is a principal ideal ring or a Dedekind domain and $\mathcal{V} \in L\mathcal{A}i\mathfrak{k}$ is i-multinilpotent, then $\mathcal{V} = \mathcal{U} \cdot_i \mathcal{W}$, where $U(Fi) = \mathfrak{a}Fi$, and $\mathcal{W}$ is a nilpotent variety that is i-multinilpotent, $i=0, 1, 2, 3$.*

Proof. $V(Fi) = \mathfrak{a}_1 Fi + \mathfrak{a}_2 Fi^2 + \dots$ where $(\mathfrak{a}_n)$ is an ascending chain of ideals of $\mathfrak{k}$. Since $\mathfrak{k}$ is Noetherian there is n such that $\mathfrak{a} = \mathfrak{a}_n = \mathfrak{a}_m$, for all $m > n$. If $\mathfrak{k}$ is a principal ideal ring, $\mathfrak{a}_r = a_r\mathfrak{k}$, $\mathfrak{a} = a\mathfrak{k}$, $\mathfrak{a}_r \subseteq \mathfrak{a}$ implies $a_r = ab_r$, $a, b_r, a_r \in \mathfrak{k}$. Hence $\mathfrak{a}_r = \mathfrak{a}(b_r\mathfrak{k}) = \mathfrak{a}\mathfrak{b}_r$ for all $r \leqq n$, $b_n = 1$. If $\mathfrak{k}$ is a Dedekind domain, $\mathfrak{a}_r = \mathfrak{m}1^{s1} \dots \mathfrak{m}t^{st}$ and $\mathfrak{a}_r \subseteq \mathfrak{a} = \mathfrak{m}1^{u1} \dots \mathfrak{m}t^{ut}$ implies $s1 \geqq u1, \dots, st \geqq ut$. Thus $\mathfrak{a}_r = \mathfrak{b}_r\mathfrak{a}$ where $\mathfrak{b}_r = \mathfrak{m}1^{v1} \dots \mathfrak{m}t^{vt}$, $v1 = s1 - u1, \dots, vt = st - ut$. Hence,

$$V(Fi) = \mathfrak{a}\mathfrak{b}_1 Fi + \mathfrak{a}\mathfrak{b}_2 Fi^2 + \dots + \mathfrak{a}Fi^n =$$

$$= \mathfrak{a}(\mathfrak{b}_1 Fi + \mathfrak{b}_2 Fi^2 + \dots + Fi^n) = (\mathfrak{a}F0)(\mathfrak{b}_1 Fi + \mathfrak{b}_2 Fi^2 + \dots + Fi^n).$$

Proof of Theorem 3. From Definition 2 and Lemma 17, $\mathcal{V} \in L\mathcal{A}i\mathfrak{k}$ is i-pseudo-indecomposable iff $\mathcal{V} \neq \mathcal{A}i\mathfrak{k}$, $\mathcal{V}' \neq \mathcal{E}$ and $\mathcal{V} = \mathcal{U} \cdot_i \mathcal{W}$, $\mathcal{U}, \mathcal{W} \in L\mathcal{A}i\mathfrak{k}$ implies $\mathcal{U}' = \mathcal{E}$ or $\mathcal{W}' = \mathcal{E}$, $i=0, 1, 2, 3$. We will write $\mathcal{V}_1 \cdot_i \mathcal{V}_2 \cdot_i \mathcal{V}_3$ to mean one of the products $(\mathcal{V}_1 \cdot_i \mathcal{V}_2) \cdot_i \mathcal{V}_3$, $\mathcal{V}_1 \cdot_i (\mathcal{V}_2 \cdot_i \mathcal{V}_3)$. In general, $\mathcal{V}_1 \cdot_i \mathcal{V}_2 \cdot_i \dots \cdot_i \mathcal{V}_n$ will mean any of the products obtained by the introduction of suitable parentheses.

Lemma 37. *Let $\mathcal{V} \in L\mathcal{A}i\mathfrak{k}$, $\mathcal{V} \neq \mathcal{A}i\mathfrak{k}$ and $\mathcal{V} = \mathcal{V}_1 \cdot_i \mathcal{V}_2 \cdot_i \dots \cdot_i \mathcal{V}_n$. Then the number of $\mathcal{V}_j$ such that $d(i, \mathcal{V}_j) > 1$ is at most equal to the number of primes (including repetitions) in the prime factorization of $d(i, \mathcal{V})$; the number of $\mathcal{V}_j$ such that $\mathcal{V}_j' \neq \mathcal{E}$ and $d(i, \mathcal{V}_j) = 1$ is at most equal to the number of maximal ideals (including repetitions) in the factorization of $\mathfrak{c}(i, \mathcal{V})$ as a product of maximal ideals, $i=0, 1, 2, 3$.*

Proof. By Lemma 11, $d(i, \mathcal{U} \cdot_i \mathcal{W}) = d(i, \mathcal{U}) d(i, \mathcal{W})$. By induction on n, $d(i, \mathcal{V}) = d(i, \mathcal{V}_1) \ldots d(i, \mathcal{V}_n)$. Hence the number of $\mathcal{V}_j$ such that $d(i, \mathcal{V}_j) > 1$ cannot exceed the number of primes in the factorization of $d(i, \mathcal{V})$. To prove the rest of the lemma, we show first that for any variety $\mathcal{W} \in L\mathcal{A}i\mathfrak{k}$, $d(i, \mathcal{W}) = d(i, \mathcal{W}')$. In fact, $d(i, \mathcal{W}) = d$ iff $W(Fi) \subseteq Fi^d$ and $W(Fi) \nsubseteq Fi^{d+1}$. This is true since if $d(i, \mathcal{W}) = d$, $W(Fi)$ contains elements of degree d and no elements of degree less than d. Thus $W(Fi) \subseteq F^d$, $W(Fi) \nsubseteq Fi^{d+1}$ (due to linearization). Conversely, if $W(Fi) \nsubseteq Fi^{d+1}$, then $W(Fi)$ contains elements of degree $\leqq d$; if $W(Fi) \subseteq Fi^d$, then $W(Fi)$ does not contain any elements of degree less than d (again W is closed under linearization). Thus $d(i, \mathcal{W}) = d$. Now, $W'(Fi) \supseteq W(Fi)$, $W(Fi) \nsubseteq Fi^{d+1}$. So $W'(Fi) \nsubseteq Fi^{d+1}$. Also $Fi^d \supseteq W(Fi)$, $Fi^d = F0^d(Fi)$. If $\mathcal{U}$ is the subvariety of $\mathcal{A}i\mathfrak{k}$ whose T-ideal in Fi is Fi^d, then $\mathcal{U}$ is i-multinilpotent. Thus $\mathcal{W}' \supseteq \mathcal{U}$, i.e., $Fi^d = U(Fi) \supseteq$ $\supseteq W'(Fi)$. Hence $d(i, \mathcal{W}') = d = d(i, \mathcal{W})$. Let $\mathcal{V} = \mathcal{V}_1 \cdot_i \mathcal{V}_2 \cdot_i \ldots \cdot_i \mathcal{V}_n$. Then $\mathcal{V} \supseteq \mathcal{V}_1' \cdot_i \mathcal{V}_2' \cdot_i \ldots \cdot_i \mathcal{V}_n' \supseteq (\ldots((\mathcal{V}_1' \cdot_i \mathcal{V}_2') \cdot_i \mathcal{V}_3') \cdot_i \ldots) \cdot_i \mathcal{V}_n'$, (by Theorem 1). $d(i, \mathcal{V}) = d(i, \mathcal{V}_1) \ldots d(i, \mathcal{V}_n) = d(i, \mathcal{V}_1') \ldots d(i, \mathcal{V}_n')$ and

$$\mathcal{V} \supseteq \mathcal{V}' \supseteq ((\ldots((\mathcal{V}_1' \cdot_i \mathcal{V}_2')' \cdot_i \mathcal{V}_3')' \cdot_i \ldots)' \cdot \mathcal{V}_n')'.$$

Hence

$$\begin{aligned} \mathfrak{c}(i, \mathcal{V}) &= \mathfrak{c}(i, d(i, \mathcal{V}), \mathcal{V}) = \mathfrak{c}(i, d(i, \mathcal{V}'), \mathcal{V}) = \\ &= \mathfrak{c}(i, d, \mathcal{V}) \subseteq \mathfrak{c}(i, d, \mathcal{V}') = \mathfrak{c}(i, \mathcal{V}') \subseteq \quad \text{(by Lemma 11)} \\ &\subseteq \mathfrak{c}(i, ((\ldots((\mathcal{V}_1' \cdot_i \mathcal{V}_2')' \cdot_i \mathcal{V}_3')' \cdot_i \ldots)' \cdot \mathcal{V}_n')') = \\ &= \mathfrak{c}(i, \mathcal{V}_1')\mathfrak{c}(i, \mathcal{V}_2')^{d_1}\mathfrak{c}(i, \mathcal{V}_3')^{d_1 d_2} \ldots \mathfrak{c}(i, \mathcal{V}_n')^{d_1 d_2 \ldots d_{n-1}}, \end{aligned}$$

by Lemma 15 and by induction on n, where $d_j = d(i, \mathcal{V}_j) = d(i, \mathcal{V}_j')$, $1 \leqq j \leqq r$. If $d(i, \mathcal{V}_j) = 1$, $\mathcal{V}_j' \neq \mathcal{E}$, then $\mathfrak{c}(i, \mathcal{V}_j') \neq \mathfrak{k}$, $\mathfrak{c}(i, \mathcal{V}_j') \neq \mathfrak{o}$. Since

$$\mathfrak{c}(i, \mathcal{V}) \subseteq \mathfrak{c}(i, \mathcal{V}_1')\mathfrak{c}(i, \mathcal{V}_2')^{d_1} \ldots \mathfrak{c}(i, \mathcal{V}_n')^{d_1 \ldots d_{n-1}} \subseteq \prod \{\mathfrak{c}(i, \mathcal{V}_j') \colon 1 \leqq j \leqq n\}$$

and $\mathfrak{k}$ is a Dedekind domain, each non-zero proper ideal of $\mathfrak{k}$ is uniquely the product of maximal ideals, possibly non-distinct, of $\mathfrak{k}$. If $\mathfrak{c}(i, \mathcal{V}) = \mathfrak{m}1 \ldots \mathfrak{m}r$, where $\mathfrak{m}1, \ldots, \mathfrak{m}r$ are maximal ideals of $\mathfrak{k}$, possibly equal, each of the ideals $\mathfrak{c}(i, \mathcal{V}_j') \neq \mathfrak{k}$ is a product of some of $\mathfrak{m}1, \cdots, \mathfrak{m}r$. Thus, the number of $\mathcal{V}_j$, such that $\mathcal{V}_j' \neq \mathcal{E}$, $d(i, \mathcal{V}_j) = 1$, is at most r.

We return to the proof of the theorem.

If $\mathcal{V} \in L\mathcal{A}i\mathfrak{k}$, $\mathcal{V} \neq \mathcal{A}i\mathfrak{k}$, $\mathcal{V}' = \mathcal{E}$, then either $\mathcal{V}$ is i-pseudo-indecomposable or $\mathcal{V} = \mathcal{U} \cdot_i \mathcal{W}$ for some $\mathcal{U}, \mathcal{W} \in L\mathcal{A}i\mathfrak{k}$, $\mathcal{U} \neq \mathcal{A}i\mathfrak{k}$, $\mathcal{W} \neq \mathcal{A}i\mathfrak{k}$, $\mathcal{U}' \neq \mathcal{E}$, $\mathcal{W}' \neq \mathcal{E}$. Continuing this procedure, by Lemma 37, after a finite number of steps we get $\mathcal{V} = \mathcal{V}_1 \cdot_i \mathcal{V}_2 \cdot_i \ldots \cdot_i \mathcal{V}_n$ where $\mathcal{V}_1, \ldots, \mathcal{V}_n$ are i-pseudo-indecomposable.

If $\mathcal{U} \in L\mathcal{A}2\mathfrak{k}$, $\mathcal{U}' = \mathcal{E}$, $\mathcal{C} = \mathcal{A}3\mathfrak{k}$, then, by Lemma 23 and Corollary 28, $\mathcal{U} \cdot_2 \mathcal{C} = \mathcal{U} \vee \mathcal{C} = \mathcal{C}$. Thus $\mathcal{C} \cdot_2 \mathcal{C} = \mathcal{C} \cdot_2 (\mathcal{U} \cdot_2 \mathcal{C})$. By Corollary 26, $(\mathcal{C} \cdot_2 \mathcal{U}) \cdot_2 \mathcal{C} = \mathcal{C} \cdot_2 (\mathcal{U} \cdot_2 \mathcal{C}) = \mathcal{C} \cdot_2 \mathcal{C}$. $\mathcal{C} \cdot_2 \mathcal{U} \neq \mathcal{C}$ if $\mathcal{U} \neq \mathcal{E}$. Thus the decomposition of Theorem 3 is not unique. It is an open question as to whether different factorizations are due only to this reason. Also, pseudo-indecomposables cannot be replaced by indecomposables if $\mathfrak{k}$ contains a maximal ideal of finite index. For then, if α is a non-trivial homomorphism of $\mathfrak{k}$ into a finite field G, $\operatorname{var}\mathfrak{G}\alpha\subseteq\mathscr{C}$ and $\mathscr{C}\cdot_2\mathscr{C}=$ $=(\mathscr{C}\cdot_2\operatorname{var}\mathfrak{G}\alpha)\cdot_2\mathscr{C}=(\ldots(\mathscr{C}\cdot_2\operatorname{var}\mathfrak{G}\alpha)\cdot_2\ldots\cdot_2\operatorname{var}\mathfrak{H}\alpha)\cdot_2\mathscr{C}$, where $G\subseteq\ldots\subseteq H$ are any ascending chain of finite fields.

If $\mathfrak{k}$ is a field of characteristic 0, $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}'=\mathscr{E}$, then $x_1+f(x_1)\in V$ for some $f\in F0^2$ (by Lemma 19). By linearization $ax_1\in V$ for some $a\neq 0$, $a\in\mathfrak{k}$. Hence $a^{-1}(ax_1)\in V$, i.e., $V=F0$. Thus, $\mathscr{V}=\mathscr{E}$. Hence, over a field of characteristic 0, i-pseudo-indecomposables are i-indecomposable. This concludes the proof of Theorem 3.

Corollary 38. *Suppose $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $d(i,\mathscr{V})$ is prime and $\mathfrak{c}(i,\mathscr{V})=\mathfrak{k}$. Then $\mathscr{V}$ is i-pseudo-indecomposable, $i=0,1,2,3$.*

This follows from Lemma 37, since $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$ and $\mathscr{V}'\neq\mathscr{E}$.

Corollary 39. *Let $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$. Then either $\mathscr{V}\cdot_i\mathscr{V}=\mathscr{V}$ or $\mathscr{V}$ is a product of a finite number of i-pseudo-indecomposable varieties, $i=2,3$.*

If $\mathscr{V}=\mathscr{A}i\mathfrak{k}$, then $\mathscr{V}\cdot_i\mathscr{V}=\mathscr{V}$. If $\mathscr{V}'=\mathscr{E}$, then $\mathscr{V}\cdot_i\mathscr{V}=\mathscr{V}$ (by Corollary 29). The rest follows from Theorem 3.

Proposition 40. *Suppose $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$ and V contains all words of $G0$ of length n in x_1 for some $n\geq 1$. Then $\mathscr{V}$ is i-pseudo-indecomposable iff $\mathscr{V}$ is i-indecomposable, $i=0,1,2,3$.*

Proof. If $\mathscr{V}$ is i-indecomposable, then $\mathscr{V}$ is i-pseudo-indecomposable. Let $\mathscr{V}$ be i-pseudo-indecomposable. Then $\mathscr{V}\supseteq\mathscr{V}'\neq\mathscr{E}$, $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$. Suppose $\mathscr{V}=\mathscr{U}\cdot_i\mathscr{W}$, $\mathscr{U},\mathscr{W}\in L\mathscr{A}i\mathfrak{k}$. Then $\mathscr{U}'=\mathscr{E}$ or $\mathscr{W}'=\mathscr{E}$, $\mathscr{U},\mathscr{W}\subseteq\mathscr{V}$. By Lemma 19, $x_1+f(x_1)\in U$ or $x_1+f(x_1)\in W$ for some $f\in F0^2$. Thus $x_1=-f(x_1)$ is an identity in $\mathscr{U}$ or in $\mathscr{W}$. By repeated substitutions, $x_1=-f(-f(\ldots(-f(x_1))\ldots))$, we can get a term of order $\geq n$ on the right hand side. Hence $x_1=0$ is an identity in $\mathscr{U}$ or in $\mathscr{W}$; i.e., $\mathscr{U}=\mathscr{E}$ or $\mathscr{W}=\mathscr{E}$.

Corollary 41. *If $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, V contains all words of $G0$ of length n in x_1 for some $n>0$ and $\mathscr{V}\neq\mathscr{E}$, then $\mathscr{V}$ is a product of a finite number of i-indecomposable varieties, $i=0,1,2,3$.*

This follows from Theorem 3 and Proposition 40.

From Corollary 38 $\mathscr{A}j\mathfrak{k}$ is i-pseudo-indecomposable for $0\leq i<j\leq 3$. The variety of all commutative algebras is 0-pseudo-indecomposable. The variety of all Jordan algebras is 0-pseudo-indecomposable. The variety of all Lie algebras is 0-indecomposable and 1-indecomposable. This follows from Corollary 28 and Proposition 40.

8. Changing the domain of operators

We will consider the effect of changing the domain of operators $\mathfrak{k}$ on $\langle L\mathscr{V};\ \cdot_{\mathscr{V}}, \wedge, \vee\rangle$, $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$. Let $\mathfrak{k}'$ be a commutative and associative ring with 1. We assume $\mathfrak{k}'$ is non-trivial. Let α be a ring homomorphism of $\mathfrak{k}$ into $\mathfrak{k}'$ preserving 1. Let $\mathfrak{a}=\ker\alpha$. For every $f\in F0=F\mathscr{A}0\mathfrak{k}$, $f\alpha\in F\mathscr{A}0\mathfrak{k}'$ is defined by replacing all the coefficients of elements of $G0$ in f by their images under α. Let $\mathscr{V}\in\mathscr{A}0\mathfrak{k}$. $\alpha\mathscr{V}$ is the subvariety of $\mathscr{A}0\mathfrak{k}'$ defined by $\{f\alpha\colon f\in V\}=V\alpha$. Θ is the equivalence relation on $L\mathscr{A}0\mathfrak{k}$ such that $\mathscr{U}\Theta\mathscr{V}$ iff $\alpha\mathscr{U}=\alpha\mathscr{V}$. Every $\mathfrak{k}'$-algebra $\mathfrak{R}$ can be considered naturally as a $\mathfrak{k}$-algebra: $ax=(a\alpha)x$, $a\in\mathfrak{k}$, $x\in\mathfrak{R}$. With this understanding $\alpha\mathscr{V}=\mathscr{V}\cap\mathscr{A}0\mathfrak{k}'$. For some special cases, cf. J. M. Osborn [17], p. 187 and M. V. Volkov [25], p. 62.

Lemma 42. *Let $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$. Then $\mathscr{V}\to\alpha\mathscr{V}$ is a homomorphism of $\langle L\mathscr{V};\ \cdot_{\mathscr{V}}, \wedge\rangle$ into $\langle L\alpha\mathscr{V};\ \cdot_{\alpha\mathscr{V}}, \wedge\rangle$ preserving all intersections.*

Proof. Let $\mathscr{V}_t\in L\mathscr{V}$, $t\in I$. $(\sum\{V_t\colon t\in I\})\alpha=\sum\{V_t\alpha\colon t\in I\}$, i.e., $\alpha\cap\{\mathscr{V}_t\colon t\in I\}=\cap\{\alpha\mathscr{V}_t\colon t\in I\}$. Let $\mathscr{U},\mathscr{W}\in L\mathscr{V}$.

$$\alpha(\mathscr{U}\cdot_{\mathscr{V}}\mathscr{W})=(\mathscr{U}\cdot_{\mathscr{V}}\mathscr{W})\cap\alpha\mathscr{V}=(\mathscr{U}\cap\alpha\mathscr{V})\cdot_{\alpha\mathscr{V}}(\mathscr{W}\cap\alpha\mathscr{V})=\alpha\mathscr{U}\cdot_{\alpha\mathscr{V}}\alpha\mathscr{W}.$$

Till the end of the present paper S is a submonoid of the multiplicative monoid of $\mathfrak{k}$ such that $s\alpha$ is not a zero-divisor in $\mathfrak{k}'$ for any $s\in S$ and $\mathfrak{k}'$ is the ring of fractions of $\mathfrak{k}\alpha$ relative to $S\alpha$. Every element in $\mathfrak{k}'$ can be written as x/s where $s\in S$, $x\in\mathfrak{k}$, $x/s=y/t$ iff $tx-sy\in\mathfrak{a}=\ker\alpha$. If $\mathfrak{R}\in\mathscr{A}0\mathfrak{k}$, $T(\mathfrak{R})=\{x\colon x\in\mathfrak{R},\ sx\in\mathfrak{a}\mathfrak{R}$ for some $s\in S\}$ and $\alpha\mathfrak{R}$ is the tensor product of $\mathfrak{k}'$ and $\mathfrak{R}$ as $\mathfrak{k}$-algebras. Clearly, $\alpha\mathfrak{R}\in\mathscr{A}0\mathfrak{k}'$. This construction is a covariant functor from the category $\mathscr{A}0\mathfrak{k}$ into the category $\mathscr{A}0\mathfrak{k}'$. In the case under consideration, which unifies the special case where α is a homomorphism of $\mathfrak{k}$ onto $\mathfrak{k}'$, i.e., $S=\{1\}$, and the one where $\mathfrak{k}'$ is the ring of fractions of $\mathfrak{k}$ relative to S, i.e., $\mathfrak{a}=\ker\alpha=\mathfrak{o}$, respectively, $\alpha\mathfrak{R}$ has a simple construction, cf. P. M. Cohn [6], p. 21. The carrier of $\alpha\mathfrak{R}$ is the set $S\times\mathfrak{R}/\sim$ where $(s,x)\sim(t,y)$ iff $sy-tx\in T(\mathfrak{R})$. The equivalence class of (s,x) will be denoted by $(x/s)^{\sim}$. We have

$$(x/s)^{\sim}+(y/t)^{\sim}=((tx+sy)/st)^{\sim},\quad (x/s)^{\sim}(y/t)^{\sim}=(xy/st)^{\sim},\quad (a/s)(y/t)^{\sim}=(ay/st)^{\sim},$$
$$a\in\mathfrak{k},\ s,t\in S,\ x,y\in\mathfrak{R}.$$

Put $x\alpha'=(x/1)^{\sim}$.

Some of the properties of $\alpha\mathfrak{R}$, α' are summarized in the following:

Lemma 43. *Let $\mathfrak{R}\in\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$ and let $\mathfrak{R}$ be generated by Y. Then*

(i) *$\alpha\mathfrak{R}\in\alpha\mathscr{V}$,*

(ii) *α' is a homomorphism of $\mathfrak{k}$-algebras whose kernel is $T(\mathfrak{R})$ and the $\mathfrak{k}$-subalgebra of $\alpha\mathfrak{R}$ generated by $Y\alpha'$ is isomorphic to $\mathfrak{R}/T(\mathfrak{R})$,*

(iii) *$\alpha\mathfrak{R}\cong\alpha(\mathfrak{R}/T(\mathfrak{R}))$, and*

(iv) *if β is a homomorphism of $\mathfrak{k}$-algebras from $\mathfrak{R}$ into $\mathfrak{R}_1\in\mathscr{A}0\mathfrak{k}'$, then there is a unique homomorphism γ of $\mathfrak{k}'$-algebras from $\alpha\mathfrak{R}$ into $\mathfrak{R}_1$ such that $\beta=\alpha'\gamma$.*

Conversely, let $\mathfrak{R}_1 \in \mathscr{A}0\mathfrak{k}'$ *be generated as a* $\mathfrak{k}'$*-algebra by* Y, *and let* $\mathfrak{R}$ *be the* $\mathfrak{k}$*-subalgebra of* $\mathfrak{R}_1$ *generated by* Y. *Then* $y \to y\alpha'$ *can be extended to an isomorphism of* $\mathfrak{R}_1$ *onto* $\alpha\mathfrak{R}$. *If* $\mathfrak{R}_1$ *satisfies* $g=0$ $(g \in F\mathscr{A}0\mathfrak{k}')$ *and* $h\alpha = g$, $h \in F\mathscr{A}0\mathfrak{k}$, *then* $\mathfrak{R}$ *satisfies* $h=0$.

Proof. That $\alpha\mathfrak{R} \in \mathscr{A}0\mathfrak{k}'$ is standard. By the methods of the proof of L. H. ROWEN's [19] Proposition 1.3, p. 393, $\alpha\mathfrak{R}$ satisfies $f\alpha=0$ if $\mathfrak{R}$ satisfies $f=0$. (ii), (iii), (iv) follow from the construction of $\alpha\mathfrak{R}$. To check the converse, let $x \in T(\mathfrak{R})$. Then $sx = \sum \{a_i y_i : 1 \leqq i \leqq n\}$, $a_1, \ldots, a_n \in \mathfrak{a}$, $y_1, \ldots, y_n \in \mathfrak{R}$. But $\mathfrak{R} \subseteq \mathfrak{R}_1$, $sx = (s\alpha)x = \sum \{(a_i\alpha)y_i : 1 \leqq i \leqq n\} = 0$. Thus $x = (1/s)sx = 0$, i.e., $T(\mathfrak{R}) = 0$. Thus α' is injective from $\mathfrak{R}$ into $\alpha\mathfrak{R}$. If $z \in \mathfrak{R}_1$, then $z = f(y_1, \ldots, y_n)$ where $f \in F\mathscr{A}0\mathfrak{k}'$, $y_1, \ldots, y_n \in Y$. The coefficients in f are of the form $a_1/s_1, \ldots, a_m/s_m$, $a_1, \ldots, a_m \in \mathfrak{k}$, $s_1, \ldots, s_m \in S$; they can be rewritten as $b1/s, \ldots, bm/s$, $b1, \ldots, bm \in \mathfrak{k}$, $s \in S$. Thus $z = (1/s)u$, where $u \in \mathfrak{R}$. The mapping $(1/t)v \to (v/t)^{\sim}$ is well defined from $\mathfrak{R}_1$ onto $\alpha\mathfrak{R}$. $(1/s)u = (1/t)v$ iff $(u/s)^{\sim} = (v/t)^{\sim}$. This mapping is a homomorphism and it is injective, i.e., it is an isomorphism. If $\mathfrak{R}_1$ satisfies $g=0$ and $h\alpha = g$, then $\mathfrak{R}$ satisfies $h=0$ since in $\mathfrak{R}$, $ax = (a\alpha)x$, $a \in \mathfrak{k}$, $x \in \mathfrak{R}$.

Corollary 44. *If* $\mathscr{V} \in L\mathscr{A}0\mathfrak{k}$, *then* $\alpha\mathscr{V}$ *is the class of all isomorphic copies of* $\alpha\mathfrak{R}$, $\mathfrak{R} \in \mathscr{V}$. α *maps* $L\mathscr{A}0\mathfrak{k}$ *onto* $L\mathscr{A}0\mathfrak{k}'$.

Proof. From Lemma 43, $\alpha\mathfrak{R} \in \alpha\mathscr{V}$ if $\mathfrak{R} \in \mathscr{V}$ and $\mathfrak{R}_1 \in \alpha\mathscr{V}$ iff $\mathfrak{R}_1 \cong \alpha\mathfrak{R}$, $\mathfrak{R} \in \mathscr{V}$. If $\mathscr{W} \in L\mathscr{A}0\mathfrak{k}'$, then $F\mathscr{W} \cong \alpha\mathfrak{R}$ where $\mathfrak{R}$ is the $\mathfrak{k}$-subalgebra of $F\mathscr{W}$ generated by X. Let $\mathscr{U} = \operatorname{var} \mathfrak{R}$. Then $\alpha\mathscr{U} = \mathscr{W}$, since by Lemma 43, $\mathfrak{R}$ satisfies $h=0$ for all $h \in F\mathscr{A}0\mathfrak{k}$, $h\alpha \in \mathscr{W}$.

Corollary 45. *For any cardinal number* n, $F(n, \alpha\mathscr{V}) \cong \alpha F(n, \mathscr{V})$.

Proof. Let $\mathscr{V}$ be non-trivial and $\mathfrak{R}_1 = F(n, \alpha\mathscr{V})$. By Lemma 43, $\mathfrak{R}_1 \cong \alpha\mathfrak{R}$ where $\mathfrak{R}$ is the $\mathfrak{k}$-subalgebra of $\mathfrak{R}_1$ generated by $X(n)$, $\mathfrak{R} \in \mathscr{V}$. Hence there is a homomorphism β of $F(n, \mathscr{V})$ onto $\mathfrak{R}$ such that $x\beta = x$ for all $x \in X(n)$. Hence, by Lemma 43 there is a homomorphism γ of $\alpha F(n, \mathscr{V})$ onto $\alpha\mathfrak{R}$, i.e., onto $\mathfrak{R}_1$ such that $x\alpha'\gamma = x$ for all $x \in X(n)$. But, there is a homomorphism δ of $\mathfrak{k}'$-algebras from $\mathfrak{R}_1$ onto $\alpha F(n, \mathscr{V}) \in \alpha\mathscr{V}$ such that $x\delta = x\alpha'$ for all $x \in X(n)$. Hence $x\delta\gamma = x$ for all $x \in X$. Thus δ is injective and so δ is an isomorphism. If $\alpha\mathscr{V}$ is trivial, then $F(n, \alpha\mathscr{V})$ is trivial and $\alpha F(n, \mathscr{V}) \in \alpha\mathscr{V}$.

For any variety $\mathscr{V} \in L\mathscr{A}0\mathfrak{k}$, $T(V) = \{f : f \in FA0\mathfrak{k} = F0, sf \in V + \mathfrak{a}F0 \text{ for some } s \in S\}$. Clearly $T(V)$ is a T-ideal of $F0$ containing V.

Lemma 46. *Let* $\mathscr{U}, \mathscr{V} \in L\mathscr{A}0\mathfrak{k}$. *Then* $\alpha\mathscr{U} = \alpha\mathscr{V}$ *iff* $T(U) = T(V)$.

Proof. Let $f \in T(U)$. Then $sf \in U + \mathfrak{a}F0$ for some $s \in S$. $\alpha\mathscr{U}$ satisfies $g=0$ iff $\alpha\mathscr{U}$ satisfies $sg=0$. But $sf = u + a_1 f_1 + \ldots + a_n f_n$, $a_1, \ldots, a_n \in \mathfrak{a}$, $f_1, \ldots, f_n \in F0$. Thus

$(sf)\alpha = u\alpha + (a_1 f_1 + \ldots + a_n f_n)\alpha = u\alpha + 0$. Thus $\alpha\mathscr{U}$ satisfies $f\alpha$ for all $f \in T(U)$. Hence $\alpha T(\mathscr{U}) = \alpha\mathscr{U}$. If $T(U) = T(V)$, then $\alpha\mathscr{U} = \alpha T(\mathscr{U}) = \alpha T(\mathscr{V}) = \alpha\mathscr{V}$. Conversely, if $\alpha\mathscr{U} = \alpha\mathscr{V}$, then $\alpha T(\mathscr{U}) = \alpha T(\mathscr{V})$. Hence $F\alpha T(\mathscr{U}) = F\alpha T(\mathscr{V})$. The $\mathfrak{k}$-subalgebra of $F\alpha T(\mathscr{U})$ generated by X is isomorphic to $\mathfrak{R}/T(\mathfrak{R})$ (by Lemmas 43 and Corollary 45), where $\mathfrak{R} = FT(\mathscr{U}) \cong F0/T(U)$. Let $x \in T(\mathfrak{R})$. Then $sx \in \mathfrak{a}\mathfrak{R}$ for some $s \in S$, i.e., if $x = g + T(U)$, then $sg + T(U) \subseteq T(U) + aF0 = T(U)$. Thus $T(\mathfrak{R}) = 0$, and the $\mathfrak{k}$-subalgebra of $F\alpha T(\mathscr{U})$ generated by X is isomorphic to $F0/T(U)$. Hence $F0/T(U) \cong$ $\cong F0/T(V)$. Since $T(U)$, $T(V)$ are T-ideals of $F0$, $T(U) = T(V)$.

$T(\mathscr{U})$ is the smallest variety among the varieties $\mathscr{W}$ such that $\alpha\mathscr{W} = \alpha\mathscr{U}$, since if $\alpha\mathscr{W} = \alpha\mathscr{U}$, then $W \subseteq T(W) = T(U)$. In the case $\mathfrak{a} = \mathfrak{o}$, the least variety $\mathscr{W}$ such that $\alpha\mathscr{W} = \alpha\mathscr{U}$ is called by M. V. Volkov [25], p. 66, the S-knotted variety associated to $\mathscr{U}$. Modifying slightly the terminology of M. V. Volkov when $\mathfrak{a} \neq \mathfrak{o}$, define a binary relation λ on $LA0\mathfrak{k}$ by $\mathscr{U}\lambda\mathscr{V}$ iff there is $s \in S$ such that $sU \subseteq V + \mathfrak{a}F0$, $sV \subseteq U + \mathfrak{a}F0$. A variety $\mathscr{V} \in L\mathscr{A}0\mathfrak{k}$ is S-joined if the restrictions of Θ and λ on $L\mathscr{V}$ coincide. M. V. Volkov [25], Lemma 9, p. 67, showed that λ is a congruence on $\langle L\mathscr{A}0\mathfrak{k}; \wedge, \vee\rangle$ if $\mathfrak{a} = \mathfrak{o}$. Thus λ is a lattice congruence on the lattice of varieties satisfying $ax = 0$ for all $a \in \mathfrak{a}$. However, λ is a congruence on the meet semi-lattice $\langle L\mathscr{A}0\mathfrak{k}; \wedge\rangle$. Let $\mathscr{U}, \mathscr{V}, \mathscr{W} \in L\mathscr{A}0\mathfrak{k}$, $s \in S$, $sU \subseteq V + \mathfrak{a}F0$, $sV \subseteq U + \mathfrak{a}F0$. Then $s(U+W) \subseteq V + W + \mathfrak{a}F0$ and $s(V+W) \subseteq U + W + \mathfrak{a}F0$. Also, $\lambda \subseteq \Theta$. The relation between λ and Θ is described by

Proposition 47. *Let $\mathscr{U}, \mathscr{V} \in L\mathscr{A}0\mathfrak{k}$. Then $\alpha\mathscr{U} = \alpha\mathscr{V}$ iff there are $\mathscr{U}_i, \mathscr{V}_i \in L\mathscr{A}0\mathfrak{k}$, $\mathscr{U}_i\lambda\mathscr{V}_i$, $i \in I$, and $\mathscr{U} = \bigcap\{\mathscr{U}_i\colon i \in I\}$, $\mathscr{V} = \bigcap\{\mathscr{V}_i\colon i \in I\}$.*

Proof. Since $\lambda \subseteq \Theta$ and α preserves all intersections, we need to show the only if part. Let $\alpha\mathscr{U} = \alpha\mathscr{V}$. By Lemma 46, $T(U) = T(V)$. Let $I = \{(f, g)\colon f \in U,$ $g \in V,\ sf - tg \in \mathfrak{a}F0$ for some $s, t \in S\}$. If $i \in I$, $i = (f, g)$, then $\mathscr{U}_i$ is the variety of all algebras satisfying $f = 0$ and $\mathscr{V}_i$ is the variety of algebras satisfying $g = 0$. Thus $\mathscr{U} \subseteq \bigcap\{\mathscr{U}_i\colon i \in I\}$ and $\mathscr{V} \subseteq \bigcap\{\mathscr{V}_i\colon i \in I\}$. Let $f \in U \subseteq T(U) = T(V)$. Then there is $s \in S$ such that $sf \in V + \mathfrak{a}F0$, i.e., there is $g \in V$ such that $sf - g \in \mathfrak{a}F0$. Thus, $\mathscr{U} \subseteq \mathscr{U}_i$, $i = (f, g)$. Since $\mathscr{U} = \bigcap\{\mathscr{U}_f\colon f \in U\}$, where $\mathscr{U}_f$ is the variety of all algebras satisfying $f = 0$, $\mathscr{U} = \bigcap\{\mathscr{U}_i\colon i \in I\}$ and, similarly, $\mathscr{V} = \bigcap\{\mathscr{V}_i\colon i \in I\}$. If $i = (f, g)$, then $sf - tg \in \mathfrak{a}F0$. Hence, $sU_i \subseteq tV_i + \mathfrak{a}F0$ and $stU_i \subseteq t^2V_i + \mathfrak{a}F0 \subseteq V_i + \mathfrak{a}F0$. Similarly $stV_i \subseteq U_i + \mathfrak{a}F0$, i.e., $\mathscr{U}_i\lambda\mathscr{V}_i$.

It is implicit in M. V. Volkov [25] that the join of S-knotted varieties is S-knotted (in the case $\mathfrak{a} = \mathfrak{o}$). This also follows once we check that $T(U \cap V) = T(U) \cap T(V)$ if $U, V \supseteq \mathfrak{a}F0$. If $\mathscr{V}$ is a variety satisfying $ax = 0$ for all $a \in \mathfrak{a}$, then $\alpha(\mathscr{U} \vee \mathscr{W}) =$ $= \alpha\mathscr{U} \vee \alpha\mathscr{W}$ for any $\mathscr{U}, \mathscr{W} \in L\mathscr{V}$. This follows from M. V. Volkov [25], p. 63. We give here another proof using T-ideals. The T-ideal αU of the variety $\alpha\mathscr{U}$ is the T-ideal of $F\mathscr{A}0\mathfrak{k}'$ generated by $U\alpha$. Thus $\alpha U = \alpha T(U)$ is generated by $T(U)\alpha$. If $U \supseteq \mathfrak{a}F0$, then $\alpha U = \alpha T(U)$ is the set of all elements of the form $(f/s)^{\sim}$, $f \in T(U)$,

$s \in S$ where $(f/s)^{\sim} = (g/t)^{\sim}$ iff $tf - sg \in \mathfrak{a}F0$. If $(f/s)^{\sim} \in \alpha T(U \cap W)$; then $f \in T(U \cap W) = T(U) \cap T(W)$, i.e., $(f/s)^{\sim} \in \alpha T(U) \cap \alpha T(W)$. If $(f/s)^{\sim} \in \alpha T(U) \cap \cap \alpha T(W)$, then $f/s = g/t$, where $f \in T(U)$, $g \in T(W)$. Since $sg - tf \in \mathfrak{a}F0$, $sg \in T(U) + + \mathfrak{a}F0 = T(U)$. Thus $g \in T(U)$, i.e., $(f/s)^{\sim} \in \alpha(T(U) \cap T(W))$.

We conclude that $\mathscr{U} \to \alpha\mathscr{U}$ is a homomorphism of $\langle L\mathscr{V}; \cdot_{\mathscr{V}}, \wedge, \vee \rangle$ onto $\langle L\alpha\mathscr{V}; \cdot_{\alpha\mathscr{V}}, \wedge, \vee \rangle$ for any variety $\mathscr{V} \in L\mathscr{A}0\mathfrak{k}$ satisfying $ax = 0$ for all $a \in \mathfrak{a}$. This follows from Lemma 41, Corollary 44 and $\alpha(\mathscr{U} \vee \mathscr{W}) = \alpha\mathscr{U} \vee \alpha\mathscr{W}$ if $\mathscr{U}, \mathscr{W} \in L\mathscr{V}$.

A number of characterizations of S-joined varieties, in the case $\mathfrak{a} = \mathfrak{o}$, were given by M. V. Volkov [25]. The same characterizations can be modified to describe the case $\mathfrak{a} \neq \mathfrak{o}$. For instance, $\mathscr{V}$ is S-joined iff for every subvariety $\mathscr{W}$ of $\mathscr{V}$, $T(\mathscr{W})$ is finitely based relative to $\mathscr{W}$; $\mathscr{V}$ is S-joined iff for every subvariety $\mathscr{W}$ of $\mathscr{V}$ there is $s \in S$ such that $sT(W) \subseteq W + \mathfrak{a}F0$. This is true since if $\mathscr{V}$ is S-joined then $\mathscr{W} \lambda T(\mathscr{W})$ since $\lambda = \Theta$ on $L\mathscr{V}$. Thus there is $s \in S$ such that $sT(W) \subseteq W + \mathfrak{a}F0$. Conversely, if for every $\mathscr{W} \in L\mathscr{V}$ there is $s \in S$ such that $sT(W) \subseteq W + \mathfrak{a}F0$, then $\mathscr{W} \lambda T(\mathscr{W})$. If $\mathscr{U}, \mathscr{W} \in L\mathscr{V}$, $\alpha\mathscr{U} = \alpha\mathscr{W}$, then $T(\mathscr{U}) = T(\mathscr{W})$ and $\mathscr{U} \lambda T(\mathscr{U})$, $\mathscr{W} \lambda T(\mathscr{W})$, i.e., $\mathscr{U} \lambda \mathscr{W}$. The following will show the behavior of S-joined varieties under multiplication of varieties:

Proposition 48. *The S-joined subvarieties of $\mathscr{V} \in L\mathscr{A}0\mathfrak{k}$ form a subgroupoid with* 1 *of* $\langle L\mathscr{V}; \cdot_{\mathscr{V}} \rangle$.

Proof. Let $\mathscr{U}$, $\mathscr{W}$ be S-joined varieties, $\mathscr{U}, \mathscr{W} \in L\mathscr{V}$, and let $\mathscr{K} \subseteq \mathscr{U} \cdot_{\mathscr{V}} \mathscr{W}$, $\mathscr{K} \in L\mathscr{V}$. We need to show that there is $s \in S$ such that $sT(K) \subseteq K + \mathfrak{a}F0$. In other words, if $x \in F\mathscr{K}$ and there is $t \in S$ such that $tx \in \mathfrak{a}F\mathscr{K}$, then $sx \in \mathfrak{a}F\mathscr{K}$. Let $\mathfrak{R} = F\mathscr{K}$. Then $\mathfrak{R} \in \mathscr{U} \cdot_{\mathscr{V}} \mathscr{W}$, $\mathfrak{R}/W(\mathfrak{R}) \in \mathscr{W}$ and $\mathfrak{R}/W(\mathfrak{R}) = F\mathscr{M}$ where $\mathscr{M} \subseteq \mathscr{W}$, $W(\mathfrak{R}) \in \mathscr{U}$. $W(\mathfrak{R})$ generates a variety $\mathscr{M}_1 \subseteq \mathscr{U}$. Let $x \in \mathfrak{R}$, $t \in S$ and $tx \in \mathfrak{a}\mathfrak{R}$. Hence $t\bar{x} \in F\mathscr{M}$ where $\bar{x} = x + W(\mathfrak{R}) \in F\mathscr{M}$. Thus there is $s \in S$ not depending on x or t such that $s\bar{x} \in \mathfrak{a}F\mathscr{M}$, i.e., $sx \in W(\mathfrak{R}) + \mathfrak{a}\mathfrak{R}$ for all $x \in \mathfrak{R}$ such that there is $t \in S$ and $tx \in \mathfrak{a}\mathfrak{R}$. Thus $tsx \in \mathfrak{a}\mathfrak{R}$ and $tsx = 0$ in $W(\mathfrak{R}) + \mathfrak{a}R/\mathfrak{a}\mathfrak{R} \cong W(\mathfrak{R})/W(\mathfrak{R}) \cap \mathfrak{a}\mathfrak{R}$. x is a polynomial f from $F0$, and $tsx = tsf(x_1, \ldots, x_n) = 0$ is an identity in $W(\mathfrak{R})/W(\mathfrak{R}) \cap \cap \mathfrak{a}\mathfrak{R}$. Thus $tsf = 0$ is an identity in $F\mathscr{M}_1/\mathfrak{a}F\mathscr{M}_1$. Hence, there is $u \in S$ not depending on x or t such that $usx = 0$ in $F\mathscr{M}_1/\mathfrak{a}F\mathscr{M}_1$; i.e., $usx = 0$ in $W(\mathfrak{R})/W(\mathfrak{R}) \cap \mathfrak{a}\mathfrak{R} \cong \cong W(\mathfrak{R}) + \mathfrak{a}\mathfrak{R}/\mathfrak{a}\mathfrak{R}$. But $sx \in W(\mathfrak{R}) + \mathfrak{a}\mathfrak{R}$. Hence $usx \in \mathfrak{a}\mathfrak{R}$ for any $x \in \mathfrak{R}$ such that $tx \in \mathfrak{a}\mathfrak{R}$ for some $t \in S$, i.e., $usT(K) \subseteq K + \mathfrak{a}F0$. The variety $\mathscr{E}$ is S-joined.

Corollary 49. *The S-joined varieties of $L\mathscr{V}$, $\mathscr{V} \in L\mathscr{A}0\mathfrak{k}$, form a lattice ideal of* $\langle L\mathscr{V}; \wedge, \vee \rangle$.

Since a subvariety of an S-joined variety is S-joined and $\mathscr{U} \vee \mathscr{W} \subseteq \mathscr{U} \cdot_{\mathscr{V}} \mathscr{W}$ if $\mathscr{U}, \mathscr{W} \in L\mathscr{V}$, the corollary follows from Proposition 48.

That the S-joined varieties of $L\mathscr{A}0\mathfrak{k}$ $(\mathfrak{a} = \mathfrak{o})$ form a lattice ideal of $\langle L\mathscr{A}0\mathfrak{k}; \wedge, \vee \rangle$ was shown by M. V. Volkov [25], Proposition 8, p. 72.

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