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Holiday numbers: sequences resembling to the Stirling numbers of second kind

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Dedicated to Professor K. Tandori on his 60th birthday

1. Introduction. The first appearance of the often rediscovered Stirling numbers seems to be Stirling's work Methodus differentialis in 1730, but some mathematicians attribute them to Euler without prima facie evidences. Although their importance was clear in that time, Ch. Jordan had to summarize their meaning in finite difference calculus in 1933 [6]. Combinatorial properties of Stirling numbers were exhibited by E. T. BELL [1], [2], [3], [4], but we must know that Dobinski's formula for the sum of Stirling numbers $\frac{1}{e}\sum_{j=0}^{\infty}\frac{j^n}{j!}$ was found as early as 1877 [5]!

The aim of the present paper is to investigate the analytic and combinatorial properties of two sequences introduced by Z. I. Szabó. Investigating Hilbert's fourth problem, in order to define a transformation on some cylindric functions whose domain is $\mathbf{R}^n$, Z. I. Szabó introduced the following transformation on continous real functions of one variable:

$$\overset{(n)}{f^{\wedge}}(x) = \int_0^{\pi/2} \cos^{n-2}\alpha f(x\sin\alpha)\,d\alpha,$$

and its inverse transformation for odd and even numbers as follows. We use the abbreviations $E=(1/t)(d/dt)$, $D=t(d/dt)$ and E^m, D^m for their powers. Now the inverses are

$$\overset{(2m+1)^{-1}}{\wedge}(f) = \frac{1}{2^{m-1}(m-1)!}\{E^{m-1}(t^{2m-1}\overset{(2m+1)}{f^{\wedge}}(t)\},$$

for $m\geqq 1$ and

$$\overset{(2m)^{-1}}{\wedge}(f) = \frac{1}{\pi(2m-3)!!}\{E^{m-1}(t^{2m-2}\overset{(2m)}{f^{\wedge}}(t))\}^{\overset{(2)}{\wedge}},$$

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for $m \geqq 2$. The inverses can be rewritten with some constants $a_{m,i}$ and $b_{m,i}$ as

$$\overset{(2m+1)^{-1}}{\wedge}(f) = \sum_{i=0}^{m} a_{m,i} t^i f^{\overset{(2m+1)}{\wedge}}(t)^{(i)}$$

and

$$\overset{(2m)^{-1}}{\wedge}(f) = \sum_{i=0}^{m-1} b_{m,i} \left\{ t^{i+1} f^{\overset{(2m)}{\wedge}}(t)^{(i)} \right\}^{\overset{(2)}{\wedge}},$$

if $f^{\overset{(2m+1)}{\wedge}}$ (resp. $f^{\overset{(2m)}{\wedge}}$) is m times differentiable.

It is time to define the holiday numbers. We call $\psi(m, i)$ the *holiday numbers of the first kind* (resp. $\varphi(m, i)$ *of the second kind*) where

$$(0'') \qquad \Psi_m(y) = \{E^{m-1}(t^{2m-1}y)\}' = \sum_{i=0}^{m} \psi(m, i) t^i y^{(i)}$$

and

$$(0''') \qquad \Phi_m(y) = E^m(t^{2m}y) = \sum_{i=0}^{m} \varphi(m, i) t^i y^{(i)}.$$

The background of these names will be given in the fourth section.

Now we have by easy calculation that

$$a_{m,i} = \frac{1}{2^{m-1}(m-1)!} \psi(m, i) \quad \text{and} \quad b_{m,i} = \frac{2}{\pi(2m-3)!!} \varphi(m, i).$$

We note that Z. I. Szabó was interested only in the existence of $a_{m,i}$ and $b_{m,i}$ and not in their behaviour.

Our investigation is based on the substitution of the exponential function into Ψ_m and Φ_m, what is essentially the same as done by Bell [2], Rota [10], Rota, Kahaner, and Odlyzko [11], where the exponential function is substituted into the formula

$$(0') \qquad S_m(y) = D^m(y) = \sum_{i=1}^{m} S(m, i) t^i y^{(i)}.$$

The reason of the applicability of the same method is that in

$$x^{-\alpha_m}(\dots x^{-\alpha_2}(x^{-\alpha_1}(x^{\sum_{i=1}^{m}(\alpha_i+1)} y)')\dots)'$$

the coefficient of $y^{(i)}$ is a monomial of x.

If $\alpha_i = 0$, we have a trivial case.

If $\alpha_i = 1$, we have $(0''')$, if $\alpha_i = -1$, we have $(0')$. We mention the Lah numbers, which have properties similar to the Stirling and holiday numbers [7].

In the following we number the analogous formulae concerning with S, Ψ, Φ by (n'), (n''), (n'''). Even though the present paper does not contain any new result on Stirling numbers, we sketch proofs for them, since these proofs are carried over to the holiday numbers. All these results can be found either in RIORDAN [9] or in LOVÁSZ [8], in analytic and in combinatorial treatment. More references on Stirling numbers can be found in [10] and [11].

We are indebted to Z. I. Szabó and L. Lovász for the encouraging talks on holiday numbers.

2. Generating functions. We complete the definitions with

$$S(0,0)=\psi(0,0)=\varphi(0,0)=1.$$

Applying S_m, Ψ_m and Φ_m to t^k we have

$$S_m(t^k)=k^m t^k, \tag{1'}$$

$$\Psi_m(t^k)=(2m+k-1)(2m+k-3)\dots(k+1)t^k, \tag{1''}$$

$$\Phi_m(t^k)=(2m+k)(2m+k-2)\dots(k+2)t^k, \tag{1'''}$$

thus applying them to e^t we have

$$\sum_{n=0}^{\infty}\frac{n^m t^n}{n!}=S_m(e^t)=e^t\sum_{k=1}^{m}S(m,k)t^k, \tag{2'}$$

$$\sum_{n=0}^{\infty}\frac{(2m+n-1)(2m+n-3)\dots(n+1)}{n!}t^n=\Psi_m(e^t)=e^t\sum_{k=0}^{m}\Psi(m,k)t^k, \tag{2''}$$

$$\sum_{n=0}^{\infty}\frac{(2m+n)(2m+n-2)\dots(n+2)}{n!}t^n=\Phi_m(e^t)=e^t\sum_{k=0}^{m}\varphi(m,k)t^k. \tag{2'''}$$

On the one hand, dividing by e^t it gives explicit formulae

$$S(m,k)=\sum_{j=1}^{k}(-1)^{k+j}\binom{k}{j}\frac{j^m}{k!}, \tag{3'}$$

$$\psi(m,k)=\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}\frac{(2m+j-1)(2m+j-3)\dots(j+1)}{k!}, \tag{3''}$$

$$\varphi(m,k)=\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}\frac{(2m+j)(2m+j-2)\dots(j+2)}{k!}, \tag{3'''}$$

and so, the right sides of (3′), (3″) and (3‴) are zero for $k>m$. On the other hand, substituting $t=1$ into (2′), (2″), (2‴), we have

$$\sum_{k=1}^{m} S(m,k)=\frac{1}{e}\sum_{n=0}^{\infty}\frac{n^m}{n!},$$

$$\sum_{k=0}^{m}\psi(m,k)=\frac{1}{e}\sum_{n=0}^{\infty}\frac{(2m+n-1)(2m+n-3)\dots(n+1)}{n!},$$

$$\sum_{k=0}^{m}\varphi(m,k)=\frac{1}{e}\sum_{n=0}^{\infty}\frac{(2m+n)(2m+n-2)\dots(n+2)}{n!}.$$

Calculating from (2′), (2″), (2‴) the generating functions, we have

$$\text{(4′)}\quad G_S(t,z)=1+\sum_{m=1}^{\infty}\frac{z^m}{m!}\sum_{k=1}^{m}S(m,k)t^k=1+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(zn)^m}{m!}\frac{t^n}{n!}e^{-t}=e^{t(e^z-1)},$$

$$\text{(4″)}\quad \begin{aligned} G_\Psi(t,z)&=1+\sum_{m=1}^{\infty}\frac{z^m}{m!}\sum_{k=0}^{m}\psi(m,k)t^k=\\ &=1+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(2m+n-1)(2m+n-3)\dots(n+1)}{m!}z^m\frac{t}{n!}e^{-t}=\frac{1}{\sqrt{1-2z}}e^{t\left(\frac{1}{\sqrt{1-2z}}-1\right)},\end{aligned}$$

$$\text{(4‴)}\quad \begin{aligned} G_\Phi(t,z)&=1+\sum_{m=1}^{\infty}\frac{z^m}{m!}\sum_{k=0}^{m}\varphi(m,k)t^k=\\ &=1+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(2m+n)(2m+n-2)\dots(n+2)}{m!}z^m\frac{t^n}{n!}e^{-t}=\frac{1}{1-2z}e^{t\left(\frac{1}{\sqrt{1-2z}}-1\right)}.\end{aligned}$$

3. Recurrences. Since $(ty)^{(n)}=\sum_i\binom{n}{i}t^{(i)}y^{(n-i)}=ty^{(n)}+ny^{(n-1)}$, we have from (1′), (1″), (1‴)

$$S_m(y)=S_{m-1}\big((ty)'\big)-S_{m-1}(y)$$

$$\Psi_m(y)=\Psi_{m-1}\big((ty')\big)+(2m-2)\Psi_{m-1}(y)$$

$$\Phi_m(y)=\Phi_{m-1}\big((ty)'\big)+(2m-1)\Phi_{m-1}(y)$$

and the following recurrences:

$$\text{(5′)}\qquad S(m,k)=kS(m-1,k)+S(m-1,k-1)$$

$$\text{(5″)}\qquad \psi(m,k)=(2m+k-1)\psi(m-1,k)+\psi(m-1,k-1)$$

$$\text{(5‴)}\qquad \varphi(m,k)=(2m+k)\varphi(m-1,k)+\varphi(m-1,k-1).$$

Now it is an easy task to tabulate some holiday numbers.

ψ $m \backslash k$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	3	5	1	0	0	0	0
3	15	33	12	1	0	0	0
4	105	279	141	22	1	0	0
5	945	2 895	1 830	405	35	1	0
6	10 395	35 685	26 685	7500	930	51	1

φ $m \backslash k$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	2	1	0	0	0	0	0
2	8	7	1	0	0	0	0
3	48	57	15	1	0	0	0
4	384	561	207	26	1	0	0
5	3 840	6 555	3 045	545	40	1	0
6	46 080	89 055	49 185	11 220	1185	57	1

We gain some more complicated recurrences comparing the genetating functions with their partial derivatives. Applying $\partial/\partial t$ to the generating functions we notice

$$\frac{\partial}{\partial t} G_S(t, z) = (e^z-1)\, G_S(t, z),$$

$$\frac{\partial}{\partial t} G_\Psi(t, z) = \left(\frac{1}{\sqrt{1-2z}}-1\right) G_\Psi(t, z),$$

$$\frac{\partial}{\partial t} G_\Phi(t, z) = \left(\frac{1}{\sqrt{1-2z}}-1\right) G_\Phi(t, z).$$

Comparing the coefficients in the previous identities we have the following recurrences.

(6′) $S(m, k+1) = \dfrac{1}{k+1} \sum_{j=1}^{m} \binom{m}{j} S(m-j, k)$ for $k \geqq 1$ and $S(m, 1) = 1$,

(6″) $\Psi(m, k+1) =$

$$= \frac{1}{k+1} \sum_{j=1}^{m} \binom{m}{j} (2j-1)!!\, \Psi(m-j, k) \quad \text{for} \quad k \geqq 0 \quad \text{and} \quad \Psi(m, 0) = (2m-1)!!$$

(6‴)

$$\varphi(m, k+1) = \frac{1}{k+1} \sum_{j=1}^{m} \binom{m}{j} (2j-1)!!\, \Psi(m-j, k) \quad \text{for} \quad k \geqq 0 \quad \text{and} \quad \varphi(m, 0) = 2^m m!$$

Applying $\partial/\partial z$ to the generating functions we notice

$$\frac{\partial}{\partial z} G_S(t, z) = te^z G_S(t, z),$$

$$\frac{\partial}{\partial z} G_\Psi(t, z) = \{(1-2z)^{-1}+t(1-2z)^{-3/2}\} G_\Psi(t, z),$$

$$\frac{\partial}{\partial z} G_\Phi(t, z) = \frac{2+t(1-2z)^{-1/2}}{1-2z} G_\Phi(t, z).$$

Comparing the coefficients (and using for (7‴) the easy identity

$$2^S s! \left\{1+\frac{(2t-1)!!}{2^t t!}\right\} = (2s+1)!!)),$$

we have

$$S(m+1, k) = \sum_{s=1}^{m} \binom{m}{s} S(m-s, k-1), \tag{7'}$$

$$\psi(m+1, k) = \sum_{s=0}^{m} \binom{m}{s} 2^s s! \psi(m-s, k) + \sum_{s=0}^{m} \binom{m}{s} (2s+1)!! \psi(m-s, k-1), \tag{7''}$$

$$\varphi(m+1, k) = \sum_{s=0}^{m} \binom{m}{s} 2^{s+1} s! \varphi(m-s, k) + \sum_{s=0}^{m} \binom{m}{s} (2s+1)!! \varphi(m-s, k-1). \tag{7'''}$$

From (7′), (7″), (7‴) we have recurrences analogous to the recurrence of Bell numbers (the sum of Stirling numbers of second kind):

$$\sum_{k=1}^{m+1} S(m+1, k) = \sum_{s=1}^{m} \binom{m}{s} \sum_{k=1}^{m-s} S(m-s, k),$$

$$\sum_{k=0}^{m+1} \psi(m+1, k) = \sum_{s=0}^{m} \binom{m}{s} (2^S s! + (2s+1)!!) \sum_{k=0}^{m-s} \psi(m-s, k),$$

$$\sum_{k=0}^{m+1} \varphi(m+1, k) = \sum_{s=0}^{m} \binom{m}{s} (2^{S+1} s! + (2s+1)!!) \sum_{k=0}^{m-s} \varphi(m-s, k).$$

4. The combinatorial meaning of the holiday numbers. The leader of the social department of a company is to make plans for m married couples for their holidays. We say that he is to compile a Ψ-plan, if his tasks were (i), (ii), (iii).

(i) He is to compile k nonempty, pairwise disjoint groups of married couples. It is possible that some couples do not belong to any group.

(ii) He is to make a complete matching in the groups made in (i). Every man or woman of the group can be matched with every man or woman of the same group.

(iii) He is to make a complete matching in the rest that may have been made in (i) on the way written in (ii). We say that the leader of the social department is to compile a Φ-plan, if (iii) were changed for (iv):

(iv) he is to order the married couples of the rest (for the next year holidays) and to write in his notebook the name of either the husband or the wife.

Let us denote by $\tilde{\psi}(m, k)$ (resp. $\tilde{\varphi}(m, k)$) the number of all possible Ψ-plans (resp. Φ-plans) for m married couples into k groups.

Theorem. *$\tilde{\psi}(m, k)=\psi(m, k)$ and $\tilde{\varphi}(m, k)=\varphi(m, k)$.*

Proof. We prove that $\tilde{\psi}$, ψ and $\tilde{\varphi}$, φ have the same initial values and obey the same recurrence. It is easy to see that $\psi(m, 0)=(2m-1)!!=\tilde{\psi}(m, 0)$ and $\varphi(m, 0)=$ $=2^m m!=\tilde{\varphi}(m, 0)$. Now we prove that (6″) and (6‴) hold with $\tilde{\psi}$ and $\tilde{\varphi}$ instead of ψ and φ. In both foımulae the right-hand side means the choice of j couples and a matching on them, and a plan for $m-j$ couples into groups. This way all the plans are enumerated $k+1$ times.

We note that (6′) has a combinatorial proof on a similar way. The reader can give alternative proofs on the theorem using other formulae rather than (6″), (6‴), e.g. using (5″) and (5‴) and distinguishing cases by the mth married couple (see [8] 1.6); using (3″) and (3‴) and sifting (see [8] 2.4); using (7″) and (7‴) and distinguishing cases whether the $(m+1)$th married couple travel or not and by the length of the alternating circle of couples and matched pairs containing the $(m+1)$th married couple in the latter case for Ψ (and by the position of the $(m+1)$th married couple in the notebook for Φ).

As a corollary of the theorem of the present section we give new explicit formulae for the holiday numbers:

$$\psi(m, k) = \sum_{\substack{x_1\ge 1 \\ x_1+\dots+x_k\le m}} \sum_{x_2\ge 1} \cdots \sum_{x_k\ge 1} \frac{1}{k!}\binom{m}{x_1}\binom{m-x_1}{x_2}\cdots\binom{m-x_1 \dots - x_{k-1}}{x_k}\times$$

$$\times\Big(2m-1-2\sum_{i=1}^{k} x_i\Big)!!\prod_{i=1}^{k}(2x_i-1)!!$$

and

$$\varphi(m, k) = \sum_{\substack{x_1\ge 1 \\ x_1+\dots+x_k\ge m}} \sum_{x_2\ge 1} \cdots \sum_{x_k\ge 1} \frac{1}{k!}\binom{m}{x_1}\binom{m-x_1}{x_2}\cdots\binom{m-x_1-\dots-x_{k-1}}{x_k}\times$$

$$\times 2^{m-\sum_{i=1}^{k} x_i}\Big(m-\sum_{i=1}^{k} x_i\Big)!\prod_{i=1}^{k}(2x_i-1)!!.$$

5. Holiday transformation of sequences.

Theorem. *Suppose* $b_m=\sum_k \psi(m,k)a_k$ *and* $d_m=\sum_k \varphi(m,k)c_k$. *Then*

$$a_k=\sum_t s(k,t)(-1)^t\sum_i \binom{t}{i}2^i\sum_j S(i,j)\frac{b_j}{(-2)^j},$$

and

$$c_k=\sum_t s(k,t)(-2)^t\sum_i \binom{t}{i}\sum_j S(i,j)\frac{d_j}{(-2)^j},$$

where $s(k,t)$ *is the Stirling number of the first kind.*

Proof. At first we state new explicit formulae for the holiday numbers in terms of the Stirling numbers of the first and the second kind

$$\psi(m,k)=2^m\sum_{i,t}(-1)^{m+i}\binom{i}{t}2^{-i}s(m,i)S(t,k), \tag{8''}$$

$$\varphi(m,k)=2^m\sum_{i,t}(-1)^{m+i}\binom{i}{t}2^{-t}s(m,i)S(t,k). \tag{8'''}$$

Having applied Ψ_m to t^x, (0″) and (1″) give

$$(2m-1+x)(2m-3+x)\dots(1+x)=\sum_k \psi(m,k)x(x-1)\dots(x-k+1).$$

But we have

$$(2m-1+x)(2m-3+x)\dots(x+1)=(-2)^m m!\binom{-\frac{x}{2}-\frac{1}{2}}{m}$$

$$=(-2)^m\sum_i s(m,i)\left(-\frac{x}{2}-\frac{1}{2}\right)^i=(-2)^m\sum_i s(m,i)\left(-\frac{1}{2}\right)^i\sum_t\binom{i}{t}\times$$

$$\times\sum_k S(t,k)x(x-1)\dots(x-k+1),$$

and comparing the coefficients of the linearly independent polynomials $x(x-1)\dots(x-k+1)$, we get (8″).

Analogously, having applied Φ_m to t^x, (0‴) and (1‴) give

$$(2m+x)(2m-2+x)\dots(2+x)=\sum_k \varphi(m,k)x(x-1)\dots(x-k+1),$$

and we have

$$(2m+x)(2m-2+x)\dots(2+x)=(-2)^m m!\binom{-\frac{x}{2}-1}{m}=$$

$$=(-2)^m\sum_i s(m,i)\left(-1-\frac{x}{2}\right)^i=(-2)^m\sum_i s(m,i)(-1)^i\times$$

$$\times\sum_t\binom{i}{t}2^{-t}\sum_k S(t,k)x(x-1)\dots(x-k+1).$$

Comparing the coefficients we get (8‴). (8″) and (8‴) implies the theorem through a Stirling, a binomial and again a Stirling inversion as follows:

$$b_m = \sum_k \psi(m, k)a_k = \sum_k 2^m \sum_{i,t} (-1)^{m+1}\binom{i}{t} 2^{-i} s(m, i)S(t, k)a_k$$

is equivalent to

$$\sum_t \binom{i}{t} \sum_k S(t, k)a_k = (-2)^i \sum_j S(i, j)\frac{b_j}{(-2)^j}$$

which is equivalent to

$$\sum_k S(t, k)a_k = \sum_i (-1)^{t-i}\binom{t}{i} \sum_j (-2)^i S(i, j)\frac{b_j}{(-2)^j},$$

which is equivalent to the first part of the theorem. Analogously we find that

$$d_m = \sum_k \varphi(m, k)c_k = \sum_k 2^m \sum_{i,t} (-1)^{m+i}\binom{i}{t} 2^{-t} s(m, i)S(t, k)c_k$$

is equivalent to

$$\sum_t \binom{i}{t} 2^{-t} \sum_k S(t, k)c_k = \sum_j (-1)^i S(i, j)\frac{d_j}{(-2)^j},$$

which is equivalent to

$$\sum_k S(t, k)c_k = \sum_i (-1)^{t-i}\binom{t}{i} 2^t \sum_j (-1)^i S(i, j)\frac{d_j}{(-2)^j},$$

which is equivalent to the second statement of the theorem.

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