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Some Fourier multiplier criteria for anisotropic $H^p(\mathbf{R}^n)$-spaces

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Dedicated to Professor K. Tandori on the occassion of his 60th birthday

1. Introduction. In this paper we introduce anisotropic H^p-spaces along the pattern of COIFMAN and WEISS [7] and discuss the question when an operator T, given by its Fourier transform, is bounded on H^p. The multiplier criteria obtained partly improve, partly generalize results of MIYACHI [15], [16] and PERAL and TORCHINSKY [17]. Stress is laid on the practicableness of the multiplier criteria which are in the nature of best possible.

To fix ideas let us give some notations. By $L^p=L^p(\mathbf{R}^n)$, $0<p\leqq\infty$, we denote the standard Lebesgue spaces with (quasi-) norm $\|\cdot\|_p$, by S the set of all $C^\infty(\mathbf{R}^n)$-functions, rapidly decreasing at infinity, and by S' its dual, the set of all tempered distributions. As Fourier transformation F we define

$$F[f](\xi)=f^\wedge(\xi)=\int f(x)e^{-i\xi x}\,dx, \quad f\in S,$$

(when the integration domain is all of $\mathbf{R}^n$ we omit indicating it). By F^{-1} we denote the inverse Fourier transformation.

Let $A_t=t^P$ be a dilation matrix, $P=\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, $\nu=\operatorname{tr} P$, $\lambda_j>0$; we define the dilation operator δ_t by $\delta_t f(x)=f(A_t x)$. Following BESOV, IL'IN and LIZORKIN [2] (see also DAPPA [9]) we call $\varrho\in C(\mathbf{R}^n)$ an A_t-homogeneous distance function if $\varrho(x)>0$ for $x\neq 0$ and $\varrho(A_t x)=t\varrho(x)$ for all $t>0$, $x\in\mathbf{R}^n$; all ϱ's are comparable with the typical distance function $\varrho_\varkappa(x)$ is the sense that

$$(1.1)\qquad C\varrho(x)\leqq\varrho_\varkappa(x):=\Big(\sum_{j=1}^n |x_j|^{\varkappa/\lambda_j}\Big)^{1/\varkappa}\leqq C\varrho(x), \quad \varkappa>0$$

(see [3], [20], [9]).

A p-atom a is a bounded function on $\mathbf{R}^n$ with the following properties:

i) there is a ϱ-ball $B_r(x_0)=\{x\in\mathbf{R}^n\colon \varrho(x-x_0)\leqq r\}$

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with supp $a \subset B_r(x_0)$,

ii) $\|a\|_\infty \leqq r^{-\nu/p}$,

iii) $\int x^\sigma a(x)\,dx = 0$ for $|\sigma| \leqq \left[\frac{\nu}{\lambda_{\min}}\left(\frac{1}{p}-1\right)\right] =: N,\ \lambda_{\min} = \min_j \lambda_j.$

Following COIFMAN and WEISS [7] we define $H^p = H^p(\mathbf{R}^n;\ P)$, $0<p\leqq 1$, as the set of all $f\in S'$ which can be represented in the form

$$f = \sum_{j=0}^{\infty} \mu_j a_j, \quad \sum |\mu_j|^p < \infty,$$

a_j being p-atoms for $j\geqq 0$, and

$$\|f\|_{H^p}^p = \inf\left\{\sum |\mu_j|^p : f = \sum \mu_j a_j\right\}.$$

If $\lambda_j \geqq 1$, $j=1, \dots, n$, then these H^p-spaces coincide with those in CALDERÓN and TORCHINSKY [5] (choose there A_t diagonal; see [13]). A bounded function m is said to be a Fourier multiplier for H^p if T_m, $T_m f = F^{-1}[mf^\wedge]$, maps H^p boundedly into H^p. The set of all multipliers m is normed by the operator (quasi-) norm of T_m:

$$\|m\|_{M(H^p)} = \sup\{\|T_m f\|_{H^p}:\ \|f\|_{H^p} \leqq 1\}.$$

Our aim is to give sufficient, nearly best possible multiplier criteria of Hörmander type for m to belong to $M(H^p)$, $0<p\leqq 1$. For this purpose we introduce function spaces $S(q, \gamma;\ B, D)$ as follows:

Let $\varphi\in C^\infty(\mathbf{R}_+)$ be a bump function with support in $[1/2, 2]$ and satisfy

$$\int_0^\infty \varphi^2\left(\frac{s}{t}\right)\frac{dt}{t} = 1, \quad s>0.$$

Let $B(t)$ and $D(t)$ be positive continuous functions on $[0, \infty)$ with

$$0 < c \leqq \frac{B(st)}{B(t)},\ \frac{D(st)}{D(t)} \leqq C < \infty \tag{1.2}$$

for all s in a compact interval of $(0, \infty)$ and assume additionally that

$$B(t) \geqq c > 0, \quad t>0. \tag{1.3}$$

Then $S(q, \gamma;\ B, D)$ consists of all $m\in L^1_{\text{loc}}(\mathbf{R}^n_0)$ which have finite norm

$$\|m\|_{S(q,\gamma)} = \sup_{t>0} D(t)\{\|\varphi\delta_t m\|_q + B(t)^{-\gamma}\|D^\gamma(\varphi\delta_t m)\|_q\}, \quad 1<q<\infty,$$

where $D^\gamma f = F^{-1}[|\xi|^\gamma f^\wedge]$ is the γ-th, n-dimensional Riesz derivative. Using Stein's Lemma [18; p. 133], an elementary calculation shows that

$$\sup_{t>0} D(t) B(t)^{-n/q}\|(\varphi\delta_t m)(\cdot/B(t))\|_{L^q_\gamma} \tag{1.4}$$

is an equivalent norm on $S(q, \gamma\colon B, D)$; here L^q_γ is the standard Bessel potential space [18; p. 135].

We have

$$\|m\|_\infty \leqq C\|m\|_{S(q,\gamma)}, \quad \gamma > n/q, \tag{1.5}$$

if $B(t)^{n/q}/D(t)$ is uniformly bounded in $t \geqq 0$, since by the imbedding properties of the Bessel potential spaces there holds

$$\|m\|_\infty \leqq C \sup_{t>0} \|\varphi(\xi/B(t))\delta_t m(\xi/B(t))\|_\infty \leqq C(\sup_{t>0} B(t)^{n/q}/D(t))\|m\|_{S(q,\gamma)}.$$

Our results now read as follows.

Theorem 1. *Let* $0<p<1$, $m \in S(2, \gamma;\, B, D)$ *for* $\gamma > n(1/p-1/2)$ *and* $D(t) \geqq B(t)^{n(1/p-1/2)}$. *Then there holds*

$$\|T_m f\|_{H^p} \leqq C\|m\|_{S(2,\gamma)}\|f\|_{H^p}, \quad f \in H^p.$$

This will be proved in Sect. 2. Using Theorem 1 and interpolation of analytic families of operators acting on H^p-spaces we will derive in Sect. 3

Theorem 2. *Let* $1 \leqq p < 2$ *and* $D(t) \geqq B(t)^{n(1/p-1/2)}$. *If* $\gamma > n(1/p-1/2)$, $1/q < 1/p - 1/2$, *then*

$$\|T_m f\|_{H^p} \leqq C\|m\|_{S(q,\gamma)}\|f\|_{H^p}.$$

(Note that $H^p = L^p$ for $p>1$). In particular we deduce in Sect. 4 for quasi-radial multipliers $m(\xi) = m_0 \circ \varrho(\xi)$, m_0 defined on $\mathbf{R}_+$, the following

Corollary. *Let* $0<p<2$, $D(t) \geqq B(t)^{n(1/p-1/2)}$, $\gamma > n(1/p-1/2)$, *and* $\varrho \in C^{[\gamma]+1}(\mathbf{R}^n_0)$. *Then*

$$\|m_0 \circ \varrho\|_{M(H^p)} \leqq C \sum_{j=0}^{[\gamma]+1} \sup_{t>0} D(t)B(t)^{-j} \left(\int_t^{2t} |s^j m_0^{(j)}(s)|^q \frac{ds}{s}\right)^{1/q},$$

where $q=2$ *for* $0<p<1$ *and* $1/q < 1/p - 1/2$ *in the case* $1 \leqq p < 2$. *In particular, if* $B(t) = D(t) = 1$, *then we have also for fractional* $\gamma > n(1/p-1/2)$, $0 < p \leqq 1$, *that*

$$\|m_0 \circ \varrho\|_{M(H^p)} \leqq C \left\{\|m_0\|_\infty + \sup_{t>0}\left(\int_t^{2t} |s^\gamma m_0^{(\gamma)}(s)|^2 \frac{ds}{s}\right)^{1/2}\right\}. \tag{1.6}$$

Here the notion of a fractional derivative is that of Gasper and Trebels [12] (see also [6]).

Remarks. **1.** Theorem 1 is due to Miyachi [15] in the isotropic case (for $B=D=1$ see also [21]); Theorem 2 for $q=\infty$ is proved in Miyachi [16] (for the isotropic case).

2. It is not hard to generalize Theorem 1 in the sense that $F^{-1}[(1+|\xi|^2)^{\gamma/2}]$, $\gamma>n((1/p)-(1/2))$, $D(t)\geqq B(t)^{n(1/p-1/2)}$ is replaced by $F^{-1}[(1+\tilde\varrho(\xi))^{\beta}]$, $\beta>\tilde\nu((1/p)-(1/2))$, $D(t)\geqq B(t)^{\tilde\nu(1/p-1/2)}$, where $\tilde\varrho$ is a $C^\infty(\mathbf{R}_0^n)$-distance function homogeneous with respect to another dilation matrix $\tilde A_t=t^{\tilde P}$, the eigenvalues of $\tilde P$ have positive real parts, $\tilde\nu=\mathrm{tr}\,\tilde P$; thus we could partly regain a result of CALDERÓN and TORCHINSKY [5; II Theorem 4.6] in the case $\tilde A_t=A_t$.

3. Our results for $0<p\leqq 1$ are nearly optimal. As test multipliers consider the well discussed examples:

$$e^{i|\xi|^a}(1+|\xi|)^{-b}\in M(H^p),\quad 0<p\leqq 2,\quad a\neq 1, \tag{1.7}$$

if and only if $b\geqq an((1/p)-(1/2))$ (cf. [16]) and

$$(1-|\xi|)_+^a\in M(H^p),\quad 0<p\leqq\frac{2(n+1)}{n+3}, \tag{1.8}$$

if and only if $a>n((1/p)-(1/2))-1/2$ (see [19], [11], [9]).

It is not hard to verify the conditions of Corollary for the functions $e^{it^a}(1+t)^{-b}$ and $(1-t)_+^a$ so that Corollary gives the correct positive results for $0<p\leqq 1$, if we choose $A_t=\mathrm{diag}\,(t,\dots,t)$, $\varrho(\xi)=|\xi|$.

4. The multiplier condition (1.6) is an essential improvement of a result of PERAL and TORCHINSKY [17; Theorem 1.4] in the case of diagonal dilation matrices with eigenvalues $\lambda\geqq 1$ since $\gamma>\nu((1/p)-(1/2))+1/2$, $\nu=\mathrm{tr}\,P\geqq n$ is assumed in [17] in comparison to our $\gamma>n((1/p)-(1/2))$.

5. The results of MADYCH [14] (see also DAPPA and LUERS [10] in the quasiradial case) suggest that Theorem 1 remains valid if the diagonal matrix A_t is replaced by t^{P^*}, P^* being a real $n\times n$ matrix whose eigenvalues have positive real parts.

We now give some applications of Corollary.

i) $$(1-\varrho(\xi))_+^a\in M(H^p),\quad a>n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2},\quad 0<p\leqq 1.$$

ii) Let $\Phi\in C^\infty(\mathbf{R}_+)$ be 1 for $t\geqq 2$ and 0 for $t\leqq 1$; choose

$$B_1(t)=\begin{cases} t^a\log^c(1+t), & t\geqq 1\\ \log^c 2, & t\leqq 1,\end{cases}\qquad D_1(t)=\begin{cases} t^b\log^d(1+t), & t\geqq 1\\ \log^d 2, & t\leqq 1,\end{cases}$$

where $a, b, c, d\geqq 0$, then

$$\Phi\circ\varrho(\xi)e^{iB_1\circ\varrho(\xi)}/D_1\circ\varrho(\xi)\in M(H^p),\quad 0<p<2,$$

if $d/c, b/a\geqq n((1/p)-(1/2))$ or $b/a>n((1/p)-(1/2))$ and $c, d\geqq 0$.

(iii) Let $B_1, D_1, a, b, c, d, \Phi$ be as in ii).

$$\Phi\circ\varrho(\xi)\left(\cos_+ B_1\circ\varrho(\xi)\right)^{\alpha+i\beta}/D_1\circ\varrho(\xi)\in M(H^p),\quad 0<p\leqq 1$$

for $\alpha>n((1/p)-(1/2))-1/2>0$; it is easy to verify the first condition in the Corollary for integer $\gamma>n((1/p)-(1/2))$; complex interpolation then gives the rest of the assertion.

2. Proof of Theorem 1. This is a modification of the corresponding proof of [15; Theorem 1] so that we will be quite concise at some part of the proof. We have only to prove

$$\|T_m a\|_p \leqq C\|m\|_{S(2,\gamma)}$$

for p-atoms a with C independent of a; for it is proved in [16; Theorem 3.4.] that this implies T_m to be bounded from H^p into H^p in the isotropic case; this argument can be generalized to the anisotropic case by a result of TRIEBEL [22]. Since T_m is translation invariant we may assume that supp $a\subset\{x:\ \varrho(x)\leqq r\}$; further we choose $M>0$ so big that $\varrho(x)>Mr$ and $\varrho(y)\leqq r$, $0<s<1$, imply $\varrho(x)>2\varrho(sy)$. Then, by Hölder's inequality, the Parseval formula and (1.5),

$$\int_{\varrho(x)\leqq Mr} |T_m a(x)|^p\,dx \leqq C\|m\|^p_{S(2,\gamma)}. \tag{2.1}$$

If we set

$$\hat{K}_j(\xi) = \int_{2^j}^{2^{j+1}} \varphi^2(\varrho(\xi)/t)\,m(\xi)\frac{dt}{t}, \tag{2.2}$$

there remains to estimate

$$\int_{\varrho(x)\geqq Mr} |T_m a(x)|^p\,dx \leqq \sum_{j=-\infty}^{\infty}\ \int_{\varrho(x)\geqq Mr} |K_j * a(x)|^p\,dx. \tag{2.3}$$

Now observe that, by the properties of the p-atoms and by Taylor's formula

$$|K_j * a(x)| \leqq r^{-\nu/p}\int_{\varrho(y)\leqq r} |K_j(x-y)|\,dy, \tag{2.4}$$

$$|K_j * a(x)| \leqq Cr^{-\nu/p}\sum_{|\sigma|=N+1}\iint_{\Omega} |D^\sigma K_j(x-sy)|\,|y^\sigma|\,dy\,ds, \tag{2.5}$$

where $\Omega=\{(s,y):\ 0<s<1,\ \varrho(y)<r\}$. In order to estimate the latter integral we use a covering argument for Ω. First observe that, by the triangle inequality and the boundedness of A_t, there is a $\delta=\delta(j,r)>0$ such that

$$\big||A_{2^j}(x-sy)|-|A_{2^j}(x-s'y')|\big| \leqq |A_{2^j}(sy-s'y')| \leqq \frac{1}{2}|A_{2^j}(x-sy)| \tag{2.6}$$

for $|sy-s'y'|<\delta$, $\varrho(x)\geqq Mr$. Now define a family of balls in $\mathbf{R}^{n+1}$ by

$$B_\varepsilon(s,y) = \{(s',y'):\ |s-s'|+|y-y'|\leqq\varepsilon\}\quad (s,y)\in\Omega;$$

choose $\delta'>0$ such that $|sy-s'y'|<\delta$ for $(s', y')\in B_{5\delta'}(s, y)$ and such that $|y-z|\leqq 5\delta'$, $\varrho(y)<r$ imply $\varrho(z)<2r$. Then (cf. [18; p. 9]) select a disjoint sequence of balls $B_{\delta'}(s_i, y_i)=B_i$ such that the expanded balls B_i^* (same center as B_i but with diameter five times as large) cover Ω. An elementary homogeneity consideration shows that at most K balls B_i^* overlap; here K does not depend on $\delta=\delta(j, r)$ (but only on the ratio $|B_i^*|/|B_i|$, $|B_i|$ the Lebesgue measure of B_i). We now have by (2.6)

$$\iint_\Omega |D^\sigma K_j(x-sy)||y^\sigma|\,dy\,ds \leqq C\sum_i (1+|A_{2^j}(x-s_iy_i)/B(2^j)|)^{-\gamma p}\iint_{B_i^*}|D^\sigma K_j(x-sy)|\times$$
$$\times(1+|A_{2^j}(x-sy)/B(2^j)|)^{\gamma p}|y^\sigma|\,dy\,ds$$

and therefore, by the Hölder and the integral Minkowski inequality,

$$\Big(\int_{\varrho(x)\geqq Mr}|r^{-\nu/p}\iint_\Omega |D^\sigma K_j(x-sy)|\,|y^\sigma|\,ds\,dy|^p\,dx\Big)^{1/p}\leqq$$

$$\leqq Cr^{-\nu/p}\sum_i\|(1+|A_{2^j}(x-s_iy_i)/B(2^j)|)^{-\gamma p}\|_{2/(2-p)}^{(2-p)/2p}\times$$

$$(2.7)\qquad \times\iint_{B_i^*}\|(1+|A_{2^j}(x-sy)/B(2^j)|)^\gamma D^\sigma K_j(x-sy)\|_2|y^\sigma|\,ds\,dy\leqq$$

$$\leqq Cr^{-\nu/p}(2^j)^{-\nu(1/p-1/2)}B(2^j)^{n(1/p-1/2)}\int_{\varrho(y)\leqq 2r}\|\ldots\|_2|y^\sigma|\,dy\leqq$$

$$\leqq Cr^{\nu(1-1/p)+\lambda\sigma}(2^j)^{-\nu(1/p-1/2)}B(2^j)^{n(1/p-1/2)}\|\ldots\|_2,$$

where $\lambda\sigma=\sum_{j=1}^n\lambda_j\sigma_j$. Here the second inequality follows by the translation invariance of the L^2-norm and the fact that at most K of the B_i^* overlap, and the last inequality, on account of (1.1) with $\varkappa=1$, by

$$\int_{\varrho(y)\leqq 2r}|y^\sigma|\,dy\leqq C\prod_{j=1}^n\Big(\int_{|y_j|\leqq cr^{\lambda_j}}|y_j|^{\sigma_j}\,dy_j\Big).$$

Analogously,

$$(2.8)\qquad \Big(\int_{\varrho(x)\geqq Mr}\Big|r^{-\nu/p}\int_{\varrho(y)\leqq r}|K_j(x-y)|\,dy\Big|^p\,dx\Big)^{1/p}\leqq$$
$$\leqq Cr^{\nu(1-1/p)}(2^j)^{-\nu(1/p-1/2)}B(2^j)^{n(1/p-1/2)}\|(1+|A_{2^j}x/B(2^j)|)^\gamma K_j(x)\|_2.$$

Thus there remains to estimate $\|\ldots\|_2$ for $|\sigma|=0$ and for $|\sigma|=N+1$ (recall $N=\Big[\frac{\nu}{\lambda_{\min}}\Big(\frac{1}{p}-1\Big)\Big]\Big)$. Using the definition (2.2) of K_j, the fact that

$$|A_{2^j}x/B(2^j)|\approx|A_tx/B(t)|,\quad 2^j\leqq t\leqq 2^{j+1},$$

the integral Minkowski inequality and repeated substitutions, we see that $\|\ldots\|_2$ can be majorized by

$$C2^{j\nu/2}B(2^j)^{-n/2}\int_{2^j}^{2^{j+1}}\left\|F^{-1}[\varphi^2\circ\varrho(\xi/B(t))(A_t i\xi/B(t))^\sigma m(A_t\xi/B(t))](x)(1+|x|)^\gamma\right\|_2\frac{dt}{t}\leqq$$

$$\leqq C2^{j\nu/2}B(2^j)^{-n/2}\int_{2^j}^{2^{j+1}}\left\|\varphi^2\circ\varrho(\xi/B(t))(A_t\xi/B(t))^\sigma m(A_t\xi/B(t))\right\|_{L^2_\gamma}\frac{dt}{t}$$

by the Parseval formula. Now observe that

$$\|gf\|_{L^2_\gamma}\leqq C\sup_{|\tau|\leqq[\gamma]+1}\|D^\tau g\|_\infty\|f\|_{L^2_\gamma}=C\|g\|_{W^\infty_{[\gamma]+1}}\|f\|_{L^2_\gamma}$$

which is obvious for integer γ and hence, by interpolation, also for fractional γ. Thus we have for $2^j\leqq t\leqq 2^{j+1}$, on account of (1.2),

(2.9) $$\|\ldots\|_2\leqq C\left\|\{\varphi\circ\varrho(\xi/B(t))(A_t\xi/B(t))^\sigma\}\right\|_{W^\infty_{[\gamma]+1}}(2^{j\nu/2}/D(2^j))\|m\|_{S(2,\gamma)}.$$

Now, by (1.3),

(2.10) $$\left\|\varphi\circ\varrho(\xi/B(t))(A_t\xi/B(t))^\sigma\right\|_{W^\infty_{[\gamma]+1}}\leqq Ct^{\lambda\sigma},$$

for $|\sigma|=0$ and $|\sigma|=N+1$. Combining (2.10), (2.9) with (2.7) and (2.8) we arrive at

$$\int_{\varrho(x)\geqq Mr}|K_j*a(x)|^p\,dx\leqq C(2^jr)^{\nu(p-1)}\min\{1,(2^jr)^{\lambda\sigma p}\}\|m\|^p_{S(2,\gamma)}$$

which clearly implies the convergence of the series in (2.3). Thus Theorem 1 is established.

3. Proof of Theorem 2. We essentially interpolate between $S(2,\gamma_0;B,D_0)\subset\subset M(H^{p_0})$, $p_0<1$ near 1, and $L^\infty\subset M(H^2)$. We use the following imbedding and interpolation properties of the Besov and Bessel potential spaces (see [18; p. 155], [1; p. 153]):

(3.1) $$L^2_\gamma=B^{22}_\gamma;\ L^q_\gamma\subset B^{qq}_\gamma\quad\text{for}\quad q\geqq 2;$$

$$B^{\infty\infty}_\gamma\subset L^\infty\quad\text{for}\quad\gamma>0;\ [B^{22}_{\gamma_0},B^{\infty\infty}_{\gamma_1}]_\Theta=B^{qq}_\gamma,\quad\gamma_0\neq\gamma_1,$$

when

$$(\gamma,1/q)=(1-\Theta)\left(\gamma_0,\frac{1}{2}\right)+\Theta(\gamma_1,0),\quad 0<\Theta<1.$$

Now choose

$$\gamma_0>n\left(\frac{1}{p}-\frac{1}{2}\right),\quad\gamma_1>0(\text{small}),\quad D_0(t)=D(t)^{1/(1-\Theta)}$$

then $D_0(t)\geqq B(t)^{n(1/p_0-1/2)}$. With $V=L^q_\gamma$ or B^{qq}_γ we define the auxiliary function space $X_0(V;B,D)$ as consisting of those functions, locally integrable away from the origin, which satisfy

$$\sup_{k\in\mathbf{Z}} \|m\|_{X_{0,k}} < \infty, \quad \lim_{k\to\pm\infty} \|m\|_{X_{0,k}} = 0,$$

where

$$\|m\|_{X_{0,k}} = D(2^k)B(2^k)^{-n/q}\|\varphi\circ\varrho\delta_{2^k}m\|_V.$$

Observe that we have for $m\in X_0(L^q_\gamma;\ B, D)$

$$\sup_k \|m\|_{X_{0,k}} \approx \|m\|_{S(q,\gamma;\,B,\,D)}.$$

By the methods in [6] (see also [8]) one can show that

$$(V_i = L^{q_i}_{\gamma_i} \quad \text{or} \quad B^{q_i q_i}_{\gamma_i}, \quad q_0 = 2,\ q_1 = \infty)$$

$$[X_0(V_0;\ B, D_0),\ X_0(V_1;\ B, D_1)]_\Theta = X_0([V_0, V_1]_\Theta;\ B, D)$$

where $[,]_\Theta$ is Calderón's lower interpolation method [4]. By (3.1) we obtain

$$X_0(L^q_\gamma;\ B, D) \subset X_0(B^{qq}_\gamma;\ B, D) = [X_0(B^{22}_{\gamma_0};\ B, D_0), X_0(B^{\infty\infty}_{\gamma_1};\ B, 1)]_\Theta \subset$$
$$\subset [S(2, \gamma_0;\ B, D_0), L^\infty]_\Theta.$$

Now consider the dense subspace H^{p_0} of H^p-functions whose Fourier transforms have compact support away from the origin; for $f\in H^{p_0}$ let

$$\psi(\xi) = \int_\varepsilon^N \varphi^2(\varrho(\xi)/t)\,\frac{dt}{t}$$

such that $\psi = 1$ on $\operatorname{supp} f^\wedge$ for suitable ε and N. Then, by the interpolation of analytic families of operators on H^p-spaces [5], [7], it finally follows that

$$\|F^{-1}[mf^\wedge]\|_{H^p} = \|F^{-1}[m\psi f^\wedge]\|_{H^p} \leqq C\|m\psi\|_{S(q,\gamma)}\|f\|_{H^p} \leqq C\|m\|_{\dot S(q,\gamma)}\|f\|_{H^p},$$

$1/p = (1-\Theta)(1/p_0) + (\Theta/2)$, thus the assertion.

4. Proof of Corollary. We observe that a further equivalent norm on $S(q, \gamma;\ B, D)$ for integer γ is given by

$$\sup_{t>0} D(t)\,B(t)^{-n/q} \sum_{0\leqq|\sigma|\leqq\gamma} \|D^\sigma((\varphi\circ\varrho\delta_t m)(\xi/B(t)))\|_q,$$

which follows from (1.4) and the identification of the Bessel potential space with the Sobolev space [18; p. 135]. If we now consider quasi-radial functions $m = m_0\circ\varrho$, m_0 being defined on $\mathbf{R}_+$, then the first part of Corollary is an immediate consequence of

$$\text{(4.1)} \qquad \left\|D^\sigma\{\varphi\circ\varrho(\cdot/B(t))\,m_0(t\varrho(\cdot/B(t)))\}\right\|_q \leqq$$
$$\leqq C \sum_{j=0}^{|\sigma|} t^{j-\nu/q} B(t)^{n/q-|\sigma|} \Big(\int_{t\leqq\varrho(\xi)\leqq 2t} |m_0^{(j)}(\varrho(\xi))|^q\, d\xi\Big)^{1/q};$$

for the introduction of polar coordinates $\varrho(\xi) \doteq s$, $d\xi \doteq s^{\nu-1}\,ds\,d\omega$ ($d\omega$ being a finite measure on $\{\xi:\ \varrho(\xi)=1\}$ leads at once to the assertion. To establish (4.1) we need the Leibniz rule

(4.2)
$$D^{\sigma}\{\varphi\circ\varrho(\xi/B(t))m_0(t\varrho(\xi/B(t)))\} = \sum_{\sigma=\sigma'+\sigma''} D^{\sigma'}(\varphi\circ\varrho(\xi/B(t)))D^{\sigma''}(m_0(t\varrho(\xi/B(t)))$$

and the following consequence of the chain rule:

$$D^{\sigma}(g\circ\varrho) = \sum_{j=1}^{|\sigma|} g^{(j)}(\varrho(\xi)) \sum \prod_{i=1}^{j} D^{\tau^{(i)}}\varrho(\xi), \tag{4.3}$$

where the sum is taken over all possible representations of $\sigma=\sum_{i=1}^{j}\tau^{(i)}$. Now

$$\|D^{\sigma'}(\varphi\circ\varrho(\xi/B(t)))\|_\infty \leqq \sum_{j=1}^{|\sigma'|}\|\varphi^{(j)}\circ\varrho(\xi/B(t))\sum\prod_{i=1}^{j}D^{\tau^{(i)}}(\varrho(\xi/B(t)))\|_\infty$$

which is clearly bounded for $|\sigma'|\geqq 0$ on account of (1.3), since $\varphi\in C^\infty(\mathbf{R}_+)$ and, by definition of φ, we have only to consider

$$1/2\leqq \varrho(\xi/B(t))\leqq 2.$$

If $\sigma''=(0,\dots,0)$, then

$$\Big(\int_{1/2\leqq\varrho(\xi/B(t))\leqq 2}|m_0(t\varrho(\xi/B(t)))|^q d\xi\Big)^{1/q} \leqq CB(t)^{n/q}t^{-\nu/q}\Big(\int_{t/2\leqq\varrho(\xi)\leqq 2t}|m_0\circ\varrho(\xi)|^q d\xi\Big)^{1/q}$$

which is of the desired type. Let $|\sigma''|\neq 0$; then

$$\Big(\int_{1/2\leqq\varrho(\xi/B(t))\leqq 2}|D^{\sigma''}\{m_0(t\varrho(\xi/B(t)))\}|^q d\xi\Big)^{1/q}\leqq$$
$$\leqq C\sum_{j=1}^{|\sigma''|}\Big(\int_{1/2\leqq\varrho(\xi/B(t))\leqq 2}|m_0^{(j)}(t\varrho(\xi/B(t)))t^jB(t)^{-|\sigma''|}|^q d\xi\Big)^{1/q}.$$

(4.1) in combination with the above estimates gives the assertion.

In the particular case $B(t)=D(t)=1$, $0<p<1$ the first condition in Corollary reduces to

$$\sup_{t>0}\sum_{j=0}^{\gamma}\Big(\int_t^{2t}|s^j m_0^{(j)}(s)|^2\frac{ds}{s}\Big)^{1/2},\quad \gamma \text{ integer},$$

which is an equivalent norm on

$$S(2,\gamma;\ \mathbf{R}_+)=\{m\in L^2_{loc}(\mathbf{R}_+):\ \|m\|_{S(2,\gamma;\mathbf{R}_+)}<\infty\},$$
$$\|m\|_{S(2,\gamma;\mathbf{R}_+)}=\sup_{t>0}\Big(\int_{\mathbf{R}}|F^{-1}[(1+\zeta^2)^{\gamma/2}[\varphi(\cdot)m_0(t\cdot)]^\wedge(\zeta)](s)|^2ds\Big)^{1/2}$$

(here $^\wedge$ and F^{-1} denote the one-dimensional Fourier transformation and its inverse, resp.).

Now define the operator

$$T_\varrho\colon S(2,\gamma;\ \mathbf{R}_+)\to S(2,\gamma;\ 1,1;\ \mathbf{R}^n),\quad T_\varrho m_0 = m_0\circ\varrho.$$

Then the above estimates show that T_ϱ is bounded if $\gamma\in\mathbf{N}_0$. Therefore, for $\gamma_0, \gamma_1\in\mathbf{N}_0$, $\gamma_0\neq\gamma_1$,

$$T_\varrho\colon [S(2,\gamma_0;\ \mathbf{R}_+),\ S(2,\gamma_1;\ \mathbf{R}_+]^\Theta\to[S(2,\gamma_0;\ \mathbf{R}^n),\quad S(2,\gamma_1;\ \mathbf{R}^n)]^\Theta$$

is bounded, where $[,]^\Theta$ is Calderón's [4] upper interpolation method. But it is shown in [6] that

$$[S(2,\gamma_0;\ \mathbf{R}_+),\ S(2,\gamma_1;\ \mathbf{R}_+)]^\Theta = S(2,\gamma;\ \mathbf{R}_+),$$

where $\gamma=(1-\Theta)\gamma_0+\Theta\gamma_1$, $0<\Theta<1$; the same argument applies for the n-dimensional situation so that

$$T_\varrho\colon\ S(2,\gamma;\mathbf{R}_+)\to S(2,\gamma;\mathbf{R}^n)$$

is bounded, i.e.,

$$\|m_0\|_{S(2,\gamma;\,R_+)}\leqq C\|m_0\circ\varrho\|_{S(2,\gamma;\,1,1;\,\mathbf{R}^n)},\quad \gamma>0.$$

Further, in [6] it is shown that $\|m_0\|_{S(2,\gamma;\mathbf{R}_+)}$ for $\gamma>1/2$ is equivalent to the condition in Corollary, which hence is proved up to the case $p=1$. By the same reasoning we obtain for $p=1$

$$S(q,\gamma;\ \mathbf{R}_+)\subset M(H^1),\quad \gamma>n/2,$$

and $q>2$ arbitrarily close to 2. A slight increase of γ allows to take $q=2$ by the imbedding properties of the L^q_γ-spaces, thus the assertion holds.

Concluding let us observe that we have estimated the n-dimensional potential norm of the quasi-radial function $m=m_0\circ\varrho$ by a one-dimensional potential norm of m_0; i.e., loosely speaking, on the space of quasi-radial functions we have majorized an n-dimensional fractional differential operator by a tractable, one-dimensional fractional operator.

Added in proof: The authors realized that all the results of the paper remain valid if, in the definition of $S(q,\gamma,B,D)$, the diagonal matrix P is replaced by a real $n\times n$ matrix whose eigenvalues have positive real parts; then we have multiplier theorems on $H^p(\mathbf{R}^n, P^*)$. The key for this generalization is to be seen in a right application of Taylor's formula: replace (2.5) by

$$|K_j*a(x)|\leqq Cr^{-\nu/p}\iint_\Omega|(y\cdot\nabla)^{N+1}K_j(x-sy)|\,dyds$$

and modify appropriately the following estimates.

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer-Verlag (Berlin, 1976).

[2] O. V. Besov, V. P. Il'in and P. I. Lizorkin, L_p-estimates of certain class of nonisotropic singular integrals, *Soviet Math. Dokl.*, 7 (1966), 1065—1069.

[3] O. V. Besov and P. I. Lizorkin, Singular integral operators and sequences of convolutions in L_p spaces, *Math. USSR-Sb.*, **2** (1967), 57—76.

[4] A. P. Calderón, Intermediate spaces and interpolation, *Studia Math.*, **24** (1964), 113—190.

[5] A. P. Calderón and A. Torschinsky, Parabolic maximal functions associated with a distribution. I, II, *Adv. in Math.*, **16** (1975), 1—64; **24** (1977), 101—171.

[6] A. Carbery, G. Gasper and W. Trebels, On localized potential spaces, *J. Approx. Theory*, to appear.

[7] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.*, **83** (1977), 569—645.

[8] W. C. Connett and A. L. Schwartz, The theory of ultraspherical multipliers, *Mem. Amer. Math. Soc.*, 9 (1977), no. 183.

[9] H. Dappa, Quasiradiale Fouriermultiplikatoren, Dissertation (Darmstadt, 1982).

[10] H. Dappa and H. Luers, A Hörmander type criterion for quasi-radial Fourier multipliers, *Proc. Amer. Math. Soc.*, to appear.

[11] C. Fefferman, A note on spherical summation multipliers, *Israel J. Math.*, **15** (1973), 44—52.

[12] G. Gasper and W. Trebels, A characterization of localized Bessel potential spaces and applications to Jacobi and Hankel multipliers, *Studia Math.*, **65** (1979), 243—278.

[13] R. H. Latter and A. Uchiyama, The atomic decomposition for parabolic H^p-spaces, *Trans. Amer. Math. Soc.*, **253** (1979), 391—396.

[14] W. Madych, On Littlewood-Paley functions, *Studia Math.*, **50** (1974), 43—63.

[15] A. Miyachi, On some Fourier multipliers, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **27** (1980), 157—179.

[16] A. Miyachi, On some singular Fourier multipliers, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **28** (1981), 267—315.

[17] J. Peral and A. Torchinsky, Multipliers in $H^p(\mathbf{R}^n)$, $0<p<\infty$, *Ark. Mat.*, **17** (1979), 224—235.

[18] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press (Princeton, 1970).

[19] E. M. Stein, M. Taibleson and G. Weiss, Weak type estimates for maximal operators on certain H^p classes, *Rend. Circ. Mat. Palermo* (2), Suppl., (1981), 81—97.

[20] E. M. Stein, and S. Wainger, Problems in harmonic analysis related to curvature, *Bull. Amer. Math. Soc.*, **84** (1978), 1239—1295.

[21] M. H. Taibleson and G. Weiss, The molecular characterization of certain Hardy spaces, *Astérisque*, 77 (1980), 67—149.

[22] H. Triebel, Theorems of Littlewood-Paley type for BMO and for anisotropic Hardy spaces, in *Proc. Intern. Conf. "Constructive Function Theory"* (Sofia, 1980), 525—532.

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