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Embedding theorems and strong approximation

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Dedicated to Professor Károly Tandori on his 60th birthay

1. Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. We denote by $s_n = s_n(x) = s_n(f; x)$ the n-th partial sum of (1), and the usual supremum norm by $\|\cdot\|$. We define a class of functions in connection with the strong approximation:

$$S_p(\lambda) := \left\{f\colon \left\| \sum_{n=0}^{\infty} \lambda_n |s_n - f|^p \right\| < \infty\right\},$$

where $\lambda = \{\lambda_n\}$ is a monotonic sequence of positive numbers and $0 < p < \infty$.

It is well known that the classical de la Vallée Poussin means

$$\tau_n = \tau_n(f;\ x) := \frac{1}{n} \sum_{k=n+1}^{2n} s_k(x), \quad n = 1, 2, \ldots$$

usually approximate the function f, in the supremum norm, better than the partial sums do. Thus it was reasonable that L. LEINDLER and A. MEIR [2] introduced, in analogy to $S_p(\lambda)$, the following class of functions:

$$V_p(\lambda) := \left\{f\colon \left\| \sum_{n=0}^{\infty} \lambda_n |\tau_n - f|^p \right\| < \infty\right\}.$$

They proved the following result concerning the relation between $S_p(\lambda)$ and $V_p(\lambda)$.

Theorem ([2, Theorem 1]). *If $p \geqq 1$ and $\{\lambda_n\}$ is a monotonic sequence of positive numbers satisfying the restriction*

$$\lambda_n / \lambda_{2n} \leqq K \quad n = 1, 2, \ldots \tag{2}$$

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with fixed positive K, then

(3) $$S_p(\lambda) \subset V_p(\lambda)$$

holds.

Furthermore it is well known that in many cases the classical Fejér means

$$\sigma_n = \sigma_n(f; x) := \frac{1}{n+1} \sum_{k=0}^{n} s_k(x)$$

approximate the function better than the partial sums do, but worse than the τ_n means. Therefore we introduce a new class of functions in connection with the approximation by $\sigma_n(x)$:

$$F_p(\lambda) := \{f\colon \|\sum_{n=0}^{\infty} \lambda_n |\sigma_n - f|^p\| < \infty\}.$$

The aim of the present paper is to investigate some relations among this new class and the previous ones.

2. We shall establish the following results.

Theorem 1. *If $p>1$ and $\{\lambda_n\}$ is a monotonic nonincerasing sequence of positive numbers, then*

(4) $$S_p(\lambda) \subset F_p(\lambda)$$

holds.

Our Remarks show that the assumptions $p>1$ and $\lambda_n \downarrow$, in certain sense, are necessary. If we omit one or the other (4) does not hold any more.

Remark 1. *For every $p>1$ there exist nondecreasing sequences $\{\lambda_n^*\}$ and $\{\lambda_n^{**}\}$ such that*

$$S_p(\lambda^*) \subset F_p(\lambda^*)$$

and

(*) $$S_p(\lambda^{**}) \not\subset F_p(\lambda^{**}).$$

Remark 2. *For any $0<p\leqq 1$ there exist a nonincreasing sequence $\{\lambda_n^*\}$ and a nondecreasing sequence $\{\lambda_n^{**}\}$ such that*

$$S_p(\lambda^*) \not\subset F_p(\lambda^*) \quad \textit{and} \quad S_p(\lambda^{**}) \not\subset F_p(\lambda^{**}).$$

The proof of these remarks is very elementary and simple, therefore we only present the suitable sequences $\{\lambda_n^*\}$ and $\{\lambda_n^{**}\}$, and the function f, furthermore we detail the proof of (*).

Theorem 2. *If* $0<p<\infty$ *and* $\{\lambda_n\}$ *is a monotonic sequence of positive numbers satisfying* (2) *then*

$$F_p(\lambda)\subset V_p(\lambda) \tag{5}$$

holds.

Proof of Theorem 1. Since

$$\sum_{n=1}^{\infty}\lambda_n|\sigma_n-f|^p = \sum_{n=1}^{\infty}\lambda_n\left|\frac{1}{n+1}\sum_{k=0}^{n}(s_k-f)\right|^p \leqq \sum_{n=1}^{\infty}\lambda_n\frac{1}{n^p}\Big(\sum_{k=0}^{n}|s_k-f|\Big)^p = I_1, \tag{6}$$

we have to estimate I_1 from above. Using an inequality of LEINDLER (see inequality (8) of [1])

$$\sum_{n=1}^{\infty}\lambda_n\Big(\sum_{k=1}^{n}a_k\Big)^p \leqq K_1\sum_{n=1}^{\infty}\lambda_n\left(\frac{a_n}{\lambda_n}\sum_{k=n}^{\infty}\lambda_k\right)^p \tag{7}$$

which holds for any $\lambda_n>0$, $a_n\geqq 0$ and $p\geqq 1$, we have

$$I_1\leqq K_2\sum_{n=1}^{\infty}\frac{\lambda_n}{n^p}\left(\frac{\sum_{k=1}^{\infty}\frac{\lambda_k}{k^p}}{\frac{\lambda_n}{n^p}}\right)^p|s_n-f|\leqq K_3\sum_{n=1}^{\infty}\lambda_n|s_n-f|^p, \tag{8}$$

where the last inequality follows from the fact that $\{\lambda_n\}$ is monotonic nonincreasing sequence and that $p>1$. Inequalities (6) and (8) clearly imply (4), which proves Theorem 1.

The first part of Remark 1 instantly follows taking $\lambda_n^*=n^{(p-1)/2}$ (using inequality (7)). Now we prove the second part of Remark 1.

Let $\lambda_n^{**}=\dfrac{n^{p-1}}{\log^p(n+1)}$ and $f(x)=\sum_{n=1}^{\infty}\dfrac{\cos nx}{n^2}$. We show that $f\in S_p(\lambda^{**})$ and $f\notin F_p(\lambda^{**})$. Since

$$\sum_{n=1}^{\infty}\lambda_n|s_n-f|^p = \sum_{n=1}^{\infty}\frac{n^{p-1}}{\log^p(n+1)}\left|\sum_{k=n+1}^{\infty}\frac{\cos kx}{k^2}\right|^p\leqq$$

$$\leqq\sum_{n=1}^{\infty}\frac{n^{p-1}}{\log^p(n+1)}\left(\sum_{k=n+1}^{\infty}\frac{1}{k^2}\right)^p\leqq K\sum_{n=1}^{\infty}\frac{n^{p-1}}{\log^p(n+1)}\frac{1}{n^p}=\sum_{n=1}^{\infty}\frac{1}{n\log(n+1)}<\infty,$$

that is, $f\in S_p(\lambda^{**})$. Let $x=0$ and N be large enough. Then we get

$$\sum_{n=1}^{N}\lambda_n|\sigma_n(0)-f(0)|^p = \sum_{n=1}^{N}\frac{n^{p-1}}{\log^p(n+1)}\cdot\frac{1}{n^p}\left(\sum_{k=1}^{n}\sum_{l=k+1}^{\infty}\frac{1}{l^2}\right)^p\geqq$$

$$\geqq K\sum_{n=1}^{N}\frac{n^{p-1}}{\log^n(n+1)}\cdot\frac{1}{n^p}\log^p(n+1)=K\sum_{n=1}^{N}\frac{1}{n},$$

which gives that $f\notin F_p(\lambda^{**})$. Remark 2 can be obtained taking $\lambda_n^*=\frac{n^{p-1}}{(\log n)^{1+p}}$ and $f(x)=\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$; or $\lambda_n^{**}=\log n$ and $f(x)=\sum_{n=2}^{\infty}\frac{\cos nx}{n^{1+1/p}(\log n)^{3/p}}$.

Proof of Theorem 2. If $0<p\leqq 1$, then using the trivial inequality

$$|a|^p \leqq |b|^p+|a+b|^p, \tag{9}$$

we get

$$\sum_{n=1}^{\infty}\lambda_n|\tau_n-f|^p = \sum_{n=1}^{\infty}\lambda_n\left|\frac{\sum_{k=n+1}^{2n}(s_k-f)}{n}\right|^p \leqq \tag{10}$$

$$\leqq \sum_{n=1}^{\infty}\lambda_n\left|\frac{\sum_{k=0}^{n}(s_k-f)}{n}\right|^p+\sum_{n=1}^{\infty}\lambda_n\left|\frac{\sum_{k=0}^{2n}(s_k-f)}{n}\right|^p = I_1.$$

Using condition (2) we obtain

$$I_1 \leqq \sum_{n=1}^{\infty}\lambda_n\left|\frac{\sum_{k=0}^{n}(s_k-f)}{n}\right|^p+K_1\sum_{n=1}^{\infty}\lambda_{2n}\left|\frac{\sum_{k=0}^{2n}(s_k-f)}{2n}\right|^p \leqq \tag{11}$$

$$\leqq K_2\sum_{n=1}^{\infty}\lambda_n\left|\frac{\sum_{k=0}^{n}(s_k-f)}{n}\right|^p \leqq K_3\sum_{n=1}^{\infty}\lambda_n|\sigma_n-f|^p.$$

Estimates (10) and (11) give (5) in the case $0<p\leqq 1$.

In the case $p>1$ we use the inequality

$$|a|^p \leqq 2^{p-1}(|b|^p+|a+b|^p)$$

instead of (9) and we get

$$\sum_{n=1}^{\infty}\lambda_n|\tau_n-f|^p \leqq K\sum_{n=1}^{\infty}\lambda_n|\sigma_n-f|^p \tag{12}$$

similarly as before. Inequality (12) immediately gives assertion (5).

References

[1] L. Leindler, Generalization of inequality of Hardy and Littlewood, *Acta Sci. Math.*, **31** (1970) 279—285.

[2] L. Leindler and A. Meir, Embedding theorems and strong approximation, *Acta Sci. Math.*, **47** (1984), 371—375.

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