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A local spectral theorem for closed operators

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday

1. Introduction

Operators with spectral singularities were first studied in the particular case of ordinary differential operators by NAIMARK [10], SCHWARTZ [14], LJANCE [7], and PAVLOV [11], [12]. The first attempts at constructing a general theory for the case of a bounded operator have been made by BACALU [1] and NAGY [8]. This theory has been extended in the more general case of a closed operator by NAGY [9]. The aim of this paper is to present sufficient conditions for a closed operator (in a reflexive Banach space), having in its spectrum what may be loosely called an "exposed arc", to be K-scalar with a certain subset K of the spectrum (see the definitions below). These conditions are clearly of local character, and are satisfied e.g. by wide classes of ordinary differential operators. However, for reasons of space, the application of these results to the above-mentioned classes will be published elsewhere.

The main result (Theorem 1) may be regarded as an extension of a result of DUNFORD and SCHWARTZ [2; XVIII. 2.34]. Since the proof there does not seem to be adaptable to the local case, we follow completely different lines, and make essential use of a remarkable result of SUSSMANN [15]. We note that related results have been obtained by JONAS [5], [6] under certain assumptions on the global structure of the spectrum of the operator (which will not be made here).

Now we fix some notations and recall some definitions and results. In what follows X will denote a complex Banach space, and $C(X)$ and $B(X)$ will denote the set of closed and bounded linear operators in X, respectively. $\mathbf{C}$ and $\bar{\mathbf{C}}$ denote the complex plane and its one-point compactification, respectively. If $T\in C(X)$, then $\sigma(T)$ denotes its extended spectrum, i.e. its usual spectrum $s(T)$ if $T\in B(X)$, and

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$s(T)\cup\{\infty\}$ otherwise. If Y is a T-invariant linear manifold, then $T|Y=T_Y$ denotes the restriction of T to $Y\cap D(T)$. By a locally holomorphic (in general X-valued) function we mean a function that is holomorphic in each component of its domain.

If $H\subset\overline{\mathbf{C}}$, then H^c will mean $\overline{\mathbf{C}}\setminus H$, and $\overline{H}$ will denote the closure of H in the topology of $\overline{\mathbf{C}}$. Let $T\in C(X)$, and let $K=\overline{K}\subset(T)$. The σ-algebra B_K consists of those Borel sets in $\overline{\mathbf{C}}$ that either contain K or are contained in K^c. A K-resolution of the identity E for T is a Boolean algebra homomorphism of B_K into a Boolean algebra of projections in $B(X)$ with $E(\overline{\mathbf{C}})=I$ which is countably additive on B_K in the strong operator topology of $B(X)$ (i.e. a B_K-spectral measure), and satisfies

$$E(b)T\subset TE(b),\ \sigma(T|E(b)X)\subset\overline{b}\quad (b\in B_K).$$

An operator $T\in C(X)$ having a K-resolution of the identity E will be called a K-spectral operator. It can be shown, as in [8; Corollary 1 to Theorem 1] for the case of a bounded T, that the K-resolution of the identity for T is then unique. If E is the K-resolution of the identity for T, and $K_1=K\cap\sigma(T)$, then the definition

$$E_1(b)=\begin{cases}E(b\cup K) & \text{if } b\supset K_1\\ E(b\cap K^c) & \phantom{\text{if }} b\subset K_1^c\end{cases}\quad (b\in B_{K_1})$$

extends E to the K_1-resolution of the identity E_1 for T (cf. [8; p. 38]). So T is K-spectral if and only if T is K_1-spectral.

If E is the K-resolution of the identity for T, and the restriction $T|E(K^c)X$ is spectral of scalar type in the sense of Bade (cf. [2; XVIII.2.12]), then T will be called a K-scalar operator.

Note that an operator $T\in C(X)$ is spectral in the classical sense due to Bade (cf. [2; XVIII.2.1]) if and only if T is $\emptyset$-spectral with the $\emptyset$-resolution of the identity E for which $E(\{\infty\})=0$. By definition (cf. [9]), if an operator is K-spectral, then the spectral singularities of the operator are contained in the set K. So any statement establishing that an operator is K-spectral is an estimation of the set of the spectral singularities as well as a structure theorem for the operator.

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2. The results

Lemma 1. *Let X be a reflexive Banach space, and $T\in B(X)$. Let $a<b$, $c>0$ be real numbers, and*

$$J=\{z\in\mathbf{C}:\ a<\operatorname{Re} z<b,\ |\operatorname{Im} z|<c\}.$$

Let p be a one-to-one holomorphic **C**-*valued mapping on a region containing* $\bar{J}$, $p(J)=H$, *and*

$$G = H \cap \sigma(T) \subset p((a, b)) = G_1. \tag{1}$$

Assume that there are dense linear manifolds $X_0 = X_0(H)$ *in* X *and* $X'_0 = X'_0(H)$ *in* X^* *(the dual of X), respectively, such that with the notation* $R(z, T) = (z-T)^{-1}$

1° *for every* $(x_0, x_0^*) \in X_0 \times X'_0$ *there exist in almost every point* $s \in (a, b)$

$$R^{\pm}(p(s), x_0^*, x_0) = \lim_{t \to 0\pm} x_0^* R(p(s-it), T) x_0,$$

2° *there is a positive number* $M = M(T, H)$ *such that for every* (x_0, x_0^*) *in* $X_0 \times X'_0$

$$\int_G |R^+(z, x_0^*, x_0) - R^-(z, x_0^*, x_0)|\, |dz| \leqq M |x_0^*|\, |x_0|.$$

Assume further that there is a positive number $M_1 = M_1(T, H)$ *such that for every* $z \in H \setminus G_1$ *(here d is the distance in* **C**)

$$|R(z, T)| \leqq M_1 d(z, G_1)^{-1}. \tag{3°}$$

With the notation

$$K = H^c \cap \sigma(T) \tag{2}$$

the operator T is then K-scalar with the K-resolution of the identity E, for which

$$x_0^* E(b) x_0 = (2\pi i)^{-1} \int_b (R^+(z, x_0^*, x_0) - R^-(z, x_0^*, x_0))\, dz \quad (b \text{ Borel}, \quad b \subset G).$$

Remark. If X is an arbitrary Banach space, the spectrum of $T \in B(X)$ satisfies (1), with the notation (2) T is K-scalar, and H_1 is a compact set in H, then there is an $M_1 = M_1(T, H_1) > 0$ such that for every z in $H_1 \setminus G_1$ the relation 3° holds. Indeed, if E denotes the K-resolution of the identity for T, then

$$R(z, T) = R(z, T|E(G_1)X) E(G_1) + R(z, T|E(G_1^c)X) E(G_1^c). \tag{3}$$

Since $E(G_1^c) = E(G_1^c \cap \sigma(T)) = E(K)$, hence

$$\sigma(T|E(G_1^c)X) \subset K,$$

the second term on the right-hand side of (3) is bounded on the set H_1. The operator $T|E(G_1)X$ is spectral of scalar type, therefore there is an $L > 0$ such that

$$|R(z, T|E(G_1)X)| = \left| \int_{G_1} (z-v)^{-1} E(dv) | E(G_1) X \right| \leqq L d(z, G_1)^{-1}.$$

In view of (3) we obtain 3°.

The proof of Lemma 1. Let $C^r(\mathbf{C}, K)$ denote the algebra of those functions $f\colon \mathbf{C} \to \mathbf{C}$ which are of class C^r on $\mathbf{C} = \mathbf{R}^2$ and are locally holomorphic on K (i.e. on some neighbourhood $U=U(f)\subset \mathbf{C}$ of K; we may assume that U has a finite number of components.) Sussmann (see [15; Theorem 10 and Remark, p. 188]) has proved that 3° implies the existence of an algebra homomorphism

$$A\colon C^3(\mathbf{C}, K) \to B(X)$$

such that $A(p_0)=I$ and $A(p_1)=T$ hold, where

$$p_k\colon z \mapsto z^k \quad (z\in \mathbf{C}).$$

Though Sussmann has stated this result for the case of a densely defined operator in Hilbert space, it is easy to check that it is valid (with the proof unchanged) under our conditions (cf. Vasilescu [16; V.5]).

Let $B_1=B_1(G, K)$ denote the algebra of those functions $f\colon \mathbf{C}\to\mathbf{C}$, for which the restriction $f|G$ is bounded Borel, and $f|K=0$. Let $f\in B_1$, $x_0\in X_0$, $x_0^*\in X_0'$, and define the bilinear form b_f on the set $X_0'\times X_0$ as follows:

$$b_f(x_0^*, x_0) = (2\pi i)^{-1} \int_G f(z)\big(R^+(z, x_0^*, x_0)-R^-(z, x_0^*, x_0)\big)\,dz. \tag{4}$$

By 2°, this form is continuous. Since it is densely defined, it can be uniquely extended to a continuous bilinear form $\overline{b_f}$ on $X^*\times X$. Since X is reflexive, there is a unique $Q(f)\in B(X)$ such that

$$x^*Q(f)x = \overline{b_f}(x^*, x) \quad (x^*\in X^*, x\in X).$$

The mapping $Q\colon B_1\to B(X)$ will be called a B_1 functional calculus. If $k(G)$ denotes the characteristic function of the set G, then clearly

$$Q(f) = Q\big(fk(G)\big) \quad (f\in B_1).$$

Let $\{f_n\}\subset B_1$ be a sequence, and $f_0\in B_1$. We shall write

$$\operatorname{Lim} f_n = f_0,$$

if $\lim_n f_n(z)=f_0(z)$ $(z\in G)$ pointwise, and

$$\sup\{|f_n(z)|\colon z\in G, \quad n = 1, 2, \dots\} = L < \infty.$$

By 2°, we have then

$$|x^*Q(f_n)x| \leqq LM|x^*||x| \quad (x^*\in X^*,\ x\in X,\ n = 0, 1, \dots).$$

Therefore, for every $x_0^*\in X_0'$, $x_0\in X_0$, $n=0, 1, \dots$

$$|x^*Q(f_n)x-x_0^*Q(f_n)x_0| \leqq LM(|x^*-x_0^*|\,|x|+|x_0^*|\,|x-x_0|).$$

Further we have

$$|x^*Q(f_n)x-x^*Q(f_0)x| \leqq |x^*Q(f_n)x-x_0^*Q(f_n)x_0|+|x_0^*Q(f_n)x_0-x_0^*Q(f_0)x_0|+$$
$$+|x_0^*Q(f_0)x_0-x^*Q(f_0)x|.$$

The first and third terms of the right-hand side are uniformly small in n, if $|x^*-x_0^*|$ and $|x-x_0|$ are small. By 2°, (4) and the Lebesgue convergence theorem, the second term is small, if n is large. Therefore, $\operatorname{Lim} f_n = f_0$ implies

(5) $$\lim_n x^* Q(f_n)x = x^* Q(f_0)x \quad (x^* \in X^*,\ x \in X).$$

Let $B_0 = B_0(G, K)$ denote the subalgebra (in general without unit) of B_1 consisting of those f in B_1, the suppoits of which satisfy $\operatorname{supp} f \cap K = \emptyset$. We show that

(6) $$A(f) = Q(f) \quad (f \in C^3(\mathbf{C}, K) \cap B_0).$$

Since $f \in B_0$, therefore

$$\operatorname{supp} f \cap \sigma(T) = \operatorname{supp} f \cap G$$

is a compact (in $\mathbf{C}$) subset of the analytic arc G_1. Sussmann [15; Lemma 6] shows that $A(g) = 0$ for every $g \in C^3(\mathbf{C}, K)$ that vanishes in a neighbourhood of $\sigma(T)$. Hence for the function f there is $f_0 \in C_0^3(H) \subset C^3(\mathbf{C}, K) \cap B_0$ such that

(7) $$A(f) = A(f_0), \quad Q(f) = Q(f_0).$$

(To obtain f_0 use a suitable partition of unity.) Further, the quoted proofs ([15; Theorem 10] and [16; V.5.2]) show that $f_0 \in C_0^3(H)$ together with 1° and 2° imply

$$x_0^* A(f_0) x_0 = (2\pi i)^{-1} \int_G f_0(z)\big(R^+(z, x_0^*, x_0) - R^-(z, x_0^*, x_0)\big)\,dz = x_0^* Q(f_0) x_0$$

for every $x_0^* \in X_0'$, $x_0 \in X_0$. From (7) we obtain that (6) is valid.

Now we show that the (clearly linear) mapping Q is an algebra homomorphism, i.e. for every $f, g \in B_1$

(8) $$Q(f)Q(g) = Q(fg).$$

At first let $f, g \in C^3(\mathbf{C}, K) \cap B_0$. Since A is an algebra homomorphism, (6) yields

$$Q(f)Q(g) = A(f)A(g) = A(fg) = Q(fg).$$

Now let $f \in C^0(\mathbf{C}) \cap B_0$, g as above, then there is a sequence $\{f_n\} \subset C^3(\mathbf{C}, K) \cap B_0$ such that $\operatorname{Lim} f_n = f$. For every pair (f_n, g) the equality (8) holds, therefore (5) implies that (8) holds for the pair (f, g). Let g be as before, and let

$$D = \{f \in B_1 \colon Q(f)Q(g) = Q(fg)\}.$$

Since $\{f_n\} \subset D$, $\operatorname{Lim} f_n = f$ imply, by (5), $f \in D$, we obtain that $D = B_1$. Fix now $f \in B_1$, and repeat this argument for the function g, then we obtain that (8) holds for every pair $f, g \in B_1$.

Let $B = B(G, K)$ denote the algebra of those functions $f \colon \mathbf{C} \to \mathbf{C}$ which are locally holomorphic on a neighbourhood $U = U(f) \subset \mathbf{C}$ of K, and for which $f|G$ is a bounded Borel function. Let $f \in B$, and let the open set U_1 contain K and be such that $f|U_1$ is locally holomorphic. Let U_0 have the same properties and let $\overline{U}_0 \subset U_1$.

With the notation $U_2=C\setminus\overline{U_0}$, let (g_1, g_2) be a partition of unity subordinate to the open covering (U_1, U_2) of $\mathbf{C}$, and define the operator $F(f)\in B(X)$ as follows:

$$F(f)=A(fg_1)+Q(fg_2).$$

This operator is well-defined: clearly, $fg_1\in C^\infty(\mathbf{C}, K)$ and $fg_2\in B_0$; further, if (g_1^*, g_2^*) is a partition of unity subordinate to the open covering (U_1^*, U_2^*) of $\mathbf{C}$ (with the same properties), then $g_1-g_1^*=g_2^*-g_2$ implies

$$A(fg_1)-A(fg_1^*)=A\big(f(g_1-g_1^*)\big)=Q\big(f(g_2^*-g_2)\big)=Q(fg_2^*)-Q(fg_2),$$

since $f(g_1-g_1^*)\in C^\infty(\mathbf{C}, K)\cap B_0$ and, as (6) shows, the calculi A and Q coincide on this set. The mapping $F\colon B\to B(X)$ is clearly linear. We show that it is also multiplicative, i.e. it is an algebra homomorphism.

The mapping F is an extension of the calculi A and $Q_0=Q|B_0$. Indeed, let e.g., $f\in C^3(\mathbf{C}, K)$. Since $\overline{G_1}$ is compact in $\mathbf{C}$, we have $f\in B$. Let (U_1, U_2) and (g_1, g_2) be as above. Then $fg_2\in C^3(\mathbf{C}, K)\cap B_0$ and, by (6),

$$F(f)=A(fg_1)+Q(fg_2)=A(fg_1)+A(fg_2)=A(f).$$

The proof for the calculus Q_0 is similar.

If $f_1\in C^3(\mathbf{C}, K)$ and $f_2\in B_0$, then $f_1f_2\in B_0$. Further, we have

$$A(f_1)Q(f_2)=Q(f_2)A(f_1)=Q(f_1f_2). \tag{9}$$

Indeed, if, in addition, $f_2\in B_0\cap C^3(\mathbf{C}, K)$, then $f_1f_2\in C^3(\mathbf{C}, K)\cap B_0$. By (6) and by the multiplicativity of the calculus A, we obtain (9) for this case. Let

$$D_0=\{f_2\in B_0\colon \text{(9) holds with every } f_1\in C^3(\mathbf{C}, K)\}.$$

A reasoning similar to that in the proof of the multiplicativity of the calculus Q shows that $D_0=B_0$.

Now let $f_1, f_2\in B$, let $\overline{U_1}$ be contained in the intersection of the domains of local holomorphy of f_1 and f_2, otherwise let (U_1, U_2) and (g_1, g_2) be as above. By (9), we obtain

$$F(f_1)F(f_2)=\big(A(f_1g_1)+Q(f_1g_2)\big)\big(A(f_2g_1)+Q(f_2g_2)\big)=$$
$$=A(f_1f_2g_1^2)+2Q(f_1f_2g_1g_2)+Q(f_1f_2g_2^2).$$

Since F is an extension of the calculi A and Q_0, respectively, further $g_1+g_2\equiv 1$, we have

$$F(f_1)F(f_2)=F\big(f_1f_2(g_1+g_2)^2\big)=F(f_1f_2),$$

so F is multiplicative.

Applying our earlier notations, $p_i\in C^3(\mathbf{C}, K)$ $(i=0, 1)$ imply that $F(p_0)=A(p_0)=I$, and $F(p_1)=T$.

Let $k(b)=k(b;z)$ denote the characteristic function of the Borel set b in $\mathbf{C}$. If $k(b)\in B$, then define the operator $E(b)\in B(X)$ by

$$E(b)=F(k(b)). \tag{10}$$

Let S be a closed neighbourhood (in $\mathbf{C}$) of the set K. If the set b belongs to the σ-algebra B_S, then $k(b)\in B$. Since F is an algebra homomorphism, E is a homomorphism of the Boolean algebra B_S onto a Boolean algebra of projections in $B(X)$. If $\{b_n\}\subset B_S\cap S^c$ is a nondecreasing sequence of sets with the union b_0, then $E(b_n)=Q(k(b_n))$ and $\operatorname{Lim} k(b_n)=k(b_0)$. Hence, by (5),

$$\lim_n x^*E(b_n)x=x^*E(b_0)x \quad (x\in X, x^*\in X^*).$$

Therefore $E|B_S$ is countably additive in the weak and, by [2; IV. 10.1], in the strong operator topology of $B(X)$, i.e. it is a B_S-spectral measure. The multiplicativity of F implies that every $E(b)$ commutes with $T=F(p_1)$. Further, we show that

$$\sigma(T|E(b)X)\subset \bar{b} \quad (b\in B_S). \tag{11}$$

Let $z_0\notin\bar{b}$, and let $r\colon z\mapsto(z_0-z)^{-1}k(b;z)$ $(z\in\mathbf{C})$. Then $r\in B$, and $r(z)(z_0-z)=k(b;z)$. The multiplicativity of the calculus F yields that

$$F(r)(z_0-T)=(z_0-T)F(r)=E(b).$$

From this we see that (11) holds. Hence $E|B_S$ is the S-resolution of the identity of the operator T.

Let $b\in B_S\cap S^c$, $x_0^*\in X_0'$, $x_0\in X_0$. Since $E(b)=E(b\cap\sigma(T))$, by (4) we obtain

$$x_0^*E(b)x_0=x_0^*Q(k(b\cap\sigma(T)))x_0=(2\pi i)^{-1}\int_{b\cap G}(R^+(z,x_0^*,x_0)-R^-(z,x_0^*,x_0))\,dz.$$

Applying the notation $S_0=S^c\cap\sigma(T)=S^c\cap G$, the set S_0 is bounded, therefore

$$\int_{S_0} zE(dz)\in B(X).$$

Further, by (4),

$$\begin{aligned} x_0^*\int_{S_0} zE(dz)x_0 &= \int_{S_0} z x_0^*E(dz)x_0 = \\ &= (2\pi i)^{-1}\int_{S_0} z(R^+(z,x_0^*,x_0)-R^-(z,x_0^*,x_0))dz = \\ &= x_0^*Q(p_1k(S_0))x_0 = x_0^*F(p_1k(S_0))x_0. \end{aligned}$$

By the multiplicativity of F, we have

$$\int_{S^c} zE(dz)=\int_{S_0} zE(dz)=F(p_1)F(k(S_0))=TE(S^c).$$

Hence for every $x_0=E(S^c)x_0$

$$Tx_0 = \int_{S^c} z(E|E(S^c)X)(dz)x_0,$$

i.e. the operator T is S-scalar for every closed neighbourhood S of the set K.

If $b\in B_K\cap K^c$, hence $k(b)\in B_1$, then let

$$E(b) = Q(k(b)), \quad E(\mathbf{C}\setminus b) = I-E(b).$$

This definition is clearly an extension of the definition of the mapping E in (10) to the σ-algebra B_K. If $\{b_n\}$ is a nondecreasing sequence in $B_K\cap K^c$, converging to b, further $x\in X$, $x^*\in X^*$, then $\operatorname{Lim} k(b_n)=k(b)$ implies

$$x^*E(b)x = x^*Q(k(b))x = \lim_n x^*Q(k(b_n))x = \lim_n x^*E(b_n)x.$$

Hence, as above, it follows that E is countably additive on B_K in the strong operator topology of $B(X)$. In particular, if $\{S_n\}$ is a sequence of closed neighbourhoods of K, converging nonincreasingly to K, and b is as above, then

$$E(b)x = \lim_n E(b\cap S_n^c)x \quad (x\in X)$$

in the norm topology of X. The technique of [8; Theorem 3] shows that the operator T is K-scalar.

Remark. It can be seen from the proof above that instead of 3° it is sufficient to have an estimation

3′ $$|R(z, T)| \leqq M_1 d(z, G_1)^{-r}$$

with some positive integer r. In this case the only necessary modification in the proof is that A will be an algebra homomorphism of $C^{r+2}(\mathbf{C}, K)$ into $B(X)$. The other parts of the proof remain unchanged.

In the following lemma we apply the notation $p_k\colon \overline{\mathbf{C}}\to\overline{\mathbf{C}}$, $p_k(z)=z^k$ ($z\in\overline{\mathbf{C}}$, k integer), further for $h\subset\overline{\mathbf{C}}$ we set $h^{-1}=p_{-1}(h)$.

Lemma 2. *If $T\in C(X)$ is K-scalar $(K=\overline{K}\subset\sigma(T))$, and there exists $T^{-1}\in C(X)$, then T^{-1} is K^{-1}-scalar.*

Proof. Let E denote the K-resolution of the identity for T. It is not hard to see that the projection-valued mapping E_1 defined by

$$E_1(b^{-1}) = E(b) \quad (b^{-1}\in B_{K^{-1}})$$

is the K^{-1}-resolution of the identity for T^{-1}. We shall show that the restriction $T^{-1}|E_1((K^{-1})^c)X$ is spectral of scalar type in the sense of Bade (cf. [2; XVIII.2.12]). Let

$$Y = E_1((K^{-1})^c)X = E(K^c)X.$$

Since T and E commute, and T is injective, $(T|Y)^{-1}=T^{-1}|Y$. Since $T|Y$ is spectral of scalar type, from [2; XVIII.2.11(h)] (and with the notations there) we obtain that

$$T^{-1}|Y=(T|Y)^{-1}=T_Y(p_1)^{-1}=T_Y(p_{-1}).$$

Thus for y in Y we have

$$T^{-1}y=\lim_{n\to\infty}\int_{|p_{-1}(z)|\leqq n} z^{-1}E(dz)y$$

in the sense that $y\in D(T^{-1}|Y)$ if and only if the right-hand side limit exists in the norm topology of Y. Hence, by [2; III.10.8. (f)],

$$T^{-1}y=\lim_{n\to\infty}\int_{|z|\leqq n} zE_1(dz)y,$$

again in the above sense. Therefore the operator T^{-1} is K^{-1}-scalar.

Theorem 1. *Let X be a reflexive Banach space, and $U\in C(X)$. Let $a<b$, $c>0$ be real numbers, and*

$$J=\{z\in\mathbf{C}:\ a<\operatorname{Re} z<b,\ |\operatorname{Im} z|<c\}.$$

Let p be a one-to-one conformal mapping of a region containing $\bar{J}$ into $\overline{\mathbf{C}}$ (hence p is meromorphic with at most one pole in this region), $p(J)=H\subset\overline{\mathbf{C}}$, and

$$G=H\cap\sigma(U)\subset p((a,b))=G_1. \tag{1}$$

Assume that there are dense linear manifolds $X_0=X_0(H)$ in X and $X_0'=X_0'(H)$ in X^* such that

1° *for every (x_0, x_0^*) in $X_0\times X_0'$ there exists in almost every $s\in(a,b)$*

$$R^\pm(p(s), x_0^*, x_0)=\lim_{t\to 0\pm} x_0^*R(p(s-it), U)x_0,$$

2° *there is a positive number $M=M(T,H)$ such that for every (x_0, x_0^*) in $X_0\times X_0'$*

$$\int_G |R^+(z,x_0^*,x_0)-R^-(z,x_0^*,x_0)|\,|dz|\leqq M|x_0^*|\,|x_0|.$$

Assume further that there are a positive number $M_1=M_1(U,H)$ and a positive integer r such that for every $z\in H\setminus G_1$ (here $\bar{d}$ denotes the chordal distance in $\overline{\mathbf{C}}$)

3° $$|R(z,U)|\leqq M_1\bar{d}(z,G_1)^{-r}.$$

With the notation

$$K=H^c\cap\sigma(U)$$

the operator U is then K-scalar with the K-resolution of the identity E, for which

$$x_0^*E(b)x_0\doteq(2\pi i)^{-1}\int_b \big(R^+(z,x_0^*,x_0)-R^-(z,x_0^*,x_0)\big)dz \quad (b \text{ Borel},\ b\subset G).$$

Proof. Due to (1), the resolvent set $\varrho(U)$ of the operator U is nonvoid, and there is a point $z\in\varrho(U)\cap G_1^c$. There is a positive number $c_1\leqq c$ such that (with understandable notations) the corresponding image set $H(c_1)=p\big(J(c_1)\big)$ is a subset of $H=H(c)$, and $z\notin\overline{H(c_1)}$. With these notations then

$$K(c_1)=H(c_1)^c\cap\sigma(U)=H(c)^c\cap\sigma(U)=K(c)=K.$$

So we may assume that there is a point $z\in\varrho(U)\cap\overline{H}^c$. Hence the operator $T=(U-z)^{-1}$ belongs to $B(X)$. Without restricting the generality we may and will assume that $z=0$: i.e. that $0\in\varrho(U)\cap\overline{H}^c$ and $T=U^{-1}\in B(X)$.

With the notation p_k of the preceding lemma we have $\sigma(T)=p_{-1}\big(\sigma(U)\big)$, and the function $p_{-1}\circ p$ is a one-to-one holomorphic mapping of a region containing $\bar{J}$ into $\mathbf{C}$. Further, $p_{-1}\circ p(J)=p_{-1}(H)$, and

$$p_{-1}(G)=p_{-1}(H)\cap\sigma(T)\subset p_{-1}\circ p\big((a,b)\big)=p_{-1}(G_1).$$

So with the function $\bar{p}=p_{-1}\circ p$ replacing p and with the "reciprocals" of the sets occurring in condition (1), the bounded operator T satisfies condition (1) of Lemma 1. Now we show that it satisfies conditions 1° and 2° there. Since

$$R(z,T)=z^{-1}-z^{-2}R(z^{-1},U)\quad\big(z^{-1}\in\varrho(U)\big),$$

for every (x_0,x_0^*) in $X_0\times X_0'$ there exists in almost every point $s\in(a,b)$ the limit

$$R_1^{\pm}\big(\bar{p}(s),x_0^*,x_0\big)=\lim_{t\to 0\pm}x_0^*R\big(\bar{p}(s-it),T\big)x_0.$$

Further, for almost every point $z=\bar{p}(s)$ on $p_{-1}(G_1)$ $\big(s\in(a,b)\big)$

$$R_1^+(z,x_0^*,x_0)-R_1^-(z,x_0^*,x_0)=-z^{-2}\big(R^+(z^{-1},x_0^*,x_0)-R^-(z^{-1},x_0^*,x_0)\big).$$

If the integral of a function f on G_1 exists, i.e.

$$\int_{G_1}|f(z)|\,|dz|=\int_a^b|f(p(t))p'(t)|\,dt<\infty,$$

then (cf. [2; III.10.8])

$$\int_{G_1}|f(z)|\,|dz|=\int_{p_{-1}(G_1)}|f(z^{-1})z^{-2}|\,|dz|.$$

Hence

$$\int_{p_{-1}(G_1)}|R_1^+(z,x_0^*,x_0)-R_1^-(z,x_0^*,x_0)|\,|dz|=\int_{G_1}|R^+(z,x_0^*,x_0)-$$
$$-R^-(z,x_0^*,x_0)|\,|dz|\leqq M|x_0^*|\,|x_0|\quad(x_0\in X_0,\ x_0^*\in X_0'),$$

so condition 2° of Lemma 1 is also satisfied.

Eschmeier [3; III.1.7. Korollar, p. 58] has shown that the growth condition 3°

on the set $H\setminus G_1$ implies

$$|R(z,T)| \leqq M_2 d(z, p_{-1}(G_1))^{-r-2} \quad (z\in p_{-1}(H\setminus G_1)).$$

(The exponent on the right-hand side can be $-r$ if $\infty\notin G_1$.) Thus Lemma 1 and the Remark after it yield that the bounded operator T is K^{-1}-scalar. Applying Lemma 2, we obtain that U is K-scalar.

For the K^{-1}-resolution of the identity E_1 of the operator T Lemma 1 gives that

$$x_0^* E_1(b^{-1})x_0 = (2\pi i)^{-1} \int_{b^{-1}} \left(R_1^+(z, x_0^*, x_0) - R_1^-(z, x_0^*, x_0)\right) dz$$

$$(b^{-1} \quad \text{Borel,} \quad b^{-1}\subset G^{-1}).$$

If E denotes the K-resolution of the identity for U then, by Lemma 2, $E(b)=E_1(b^{-1})$ $(b\in B_K)$, and an integral transformation yields again

$$x_0^* E(b)x_0 = (2\pi i)^{-1} \int_b \left(R^+(z, x_0^*, x_0) - R^-(z, x_0^*, x_0)\right) dz \quad (b \quad \text{Borel,} \quad b\subset G).$$

Remark. It is similarly seen as in the bounded case that if X is an arbitrary (not necessarily reflexive) Banach space, the spectrum of the operator $U\in C(X)$ satisfies (1), and with the notation above U is K-scalar, then for every $z_0\in G_1\cap\mathbf{C}$ there are a neighborhood $N=N(z_0)$ and a positive number $M_3=M_3(U,N)$ such that

$$|R(z,U)| \leqq M_3 d(z, G_1)^{-1} \quad (z\in N\setminus G_1).$$

In particular cases of a spectrum of similar local structure several authors (cf. Pavlov [13], Gasymov and Maksudov [4]) have considered the spectral singularities as those points of the curve, in a neighbourhood N of which the resolvent operator satisfies a growth condition of order larger than one, i.e. the set $\{d(z,G_1)R(z,U)\colon z\in N\setminus G_1\}$ is unbounded.

Corollary. *Under the conditions of Theorem 1 and with the notations there the set $G\cap\mathbf{C}$ is contained in the continuous spectrum of the operator U.*

Proof. Let $z\in G\cap\mathbf{C}$, and let E denote the K-resolution of the identity of the K-scalar operator U. By Theorem 1, we have $E(\{z\})=0$. Let $e\in B_K$. Since U is K-spectral, with the notation $U_e=U|E(e)X$ we have $\sigma(U_e)\subset\bar{e}$.

Let d be an open neighbourhood (in $\mathbf{C}$) of z such that $d\in B_K$. Then the set $e=d^c$ belongs to B_K, and $z\in\varrho(U_e)$. Hence

$$E(e)X = (z-U_e)E(e)X\subset(z-U)X,$$

where VY means $V(Y\cap D(V))$ for any operator V and any set Y. Since E is countably additive, we obtain that $E(\{z\}^c)X\subset\overline{(z-U)X}$. Since $E(\{z\})=0$, we have

$$X = \overline{(z-U)X}. \tag{4}$$

Assume now that $(z-U)x=0$ for some x in X. Then, for every e as above,

$$(z-U_e)E(e)x = E(e)(z-U)x = 0.$$

Since $z\in\varrho(U_e)$, we obtain that $E(e)x=0$. By the countable additivity of E, we have $E(\{z\}^c)x=0$, hence $x=0$. So z is no eigenvalue, and (4) shows that it belongs to the continuous spectrum of U.

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