Acta Sci. Math., 48 (1985), 261—267

A Bohr type inequality on abstract normed linear spaces and its applications for special spaces

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Dedicated to Professor Károly Tandori on his 60th birthday

1. Introduction. BOHR [1] proved (in another form) that if a 2π-periodic integrable function g is orthogonal to every trigonometric polynomial of order at most n then the following inequality is true

$$\left|\int_0^x g(t)\,dt\right| \leqq \frac{c_1}{n}|g(x)| \quad (-\infty < x < \infty,\ n = 1, 2, \ldots) \tag{1}$$

where c_1 (and later c_k, $k=2, 3, \ldots$) denotes an absolute constant. Later an inequality of type (1) was discussed by many authors (see e.g. [2], [3], [4], [6], [9]).

Let $L^p_{2\pi}$ $(1 \leqq p \leqq \infty)$ be the Banach space of all 2π-periodic functions with the usual norm

$$\|f\|_p = \left\{\int_0^{2\pi} |f(x)|^p\,dx\right\}^{1/p} \quad (1 \leqq p < \infty),$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{-\infty<x<\infty} |f(x)|.$$

We denote by T_n the set of all trigonometric polynomials of order at most n $(n=0, 1, \ldots)$. For $f \in L^p_{2\pi}$ let

$$E^p_n(f) = \inf_{t_n \in T_n} \|f - t_n\|_p \quad (n = 0, 1, 2, \ldots). \tag{2}$$

Let $D^p_{2\pi}$ be the set of all 2π-periodic functions f which are absolutely continuous on $(-\infty, \infty)$ and for which $f' \in L^p_{2\pi}$. It is vell known that

$$E^p_n(f) \leqq \frac{c_2}{n}\|f'\|_p \quad (1 \leqq p \leqq \infty,\ f \in D^p_{2\pi},\ n = 1, 2, \ldots). \tag{3}$$

Using the inequality (3) (case $p=1$) we can prove the inequality (1) and conversely.

Received May 21, 1984.

In this paper we prove this statement in abstract normed linear spaces and we give applications for special spaces.

2. A Bohr type inequality in abstract spaces. Let X be an arbitrary normed linear space. The norm in X is denoted by $\|\cdot\|$. Let furthermore X^* be the dual space of X (the space of all continuous linear functionals defined on X). The norm in X^* is denoted by $\|\cdot\|^*$. Let L be a subspace of X and

$$L^{\perp} = L^{\perp}(X) := \{g \in X^*: g(x) = 0 \quad \forall x \in L\}.$$

We can prove that $L^{\perp}$ is a subspace of X^*. We define the best approximation of an element $x \in X$ by elements of L:

$$E_L(x) = \inf_{y \in L} \|x-y\|.$$

Let T be the following operator:

$$T: D(T) \to X \quad \text{linear and} \quad T(D) = X \tag{4}$$

where $D = D(T) (\subseteqq X)$ denotes the domain of T.

Suppose that there exists an operator I which has domain $D(I) \subseteqq X^*$,

$$I: D(I) \to X^* \quad \text{is linear,} \tag{5}$$

I and T satisfy the following relation

$$g(x) = I_g(Tx) \quad (\forall x \in D(T), \ \forall g \in D(I)). \tag{6}$$

Then the following statement is true:

Theorem 1. *Let T and I be two operators satisfying* (4), (5), (6). a) *If $D(I) = L^{\perp}$ then the following statements are equivalent for $\lambda > 0$:*

$$E_L(x) \leqq \lambda \|Tx\| \quad (\forall x \in D(T)), \tag{7}$$

$$\|Ig\|^* \leqq \lambda \|g\|^* \quad (\forall g \in D(I)). \tag{8}$$

b) *In the case $D(I) \subset L^{\perp}$ the inequality* (7) *implies* (8).

Proof. a) (7)→(8): We have by the duality principle of Nikolskii (see e.g. SINGER [8, p. 22])

$$\sup_{\substack{g \in L^{\perp} \\ \|g\|^* \leqq 1}} |Ig(Tx)| = \sup_{\substack{g \in L^{\perp} \\ \|g\|^* \leqq 1}} |g(x)| = E_L(x) \leqq \lambda \|Tx\|.$$

So for any fixed $g \in D(I) \subseteqq L^{\perp} (\|g\|^* \leqq 1)$ we have

$$|Ig(Tx)| \leqq \lambda \|Tx\| \quad (\forall x \in D(T)).$$

Hence by (4) we obtain

$$|Ig(y)| \leqq \lambda \|y\| \quad (\forall y \in X)$$

therefore we get (8) from the definition of norm in X^*.

b) (8)→(7): We have by duality principle and by (8)

$$E_L(x) = \sup_{\substack{g\in L^\perp \\ \|g\|^*\leq 1}} |g(x)| = \sup_{\substack{g\in L^\perp \\ \|g^*\|\leq 1}} |I_g(Tx)| \leq \sup_{\substack{g\in L^\perp \\ \|g\|^*\leq 1}} \|Ig\|^*\|Tx\| \leq \lambda\|Tx\|.$$

3. Applications. a) Let $X=L^p_{2\pi}$ $(1\leq p<\infty)$ and let $L=T_n$ $(n=1,2,\ldots)$. Then we have $X^*=L^q_{2\pi}$ $(1/p+1/q=1,\ 1\leq p<\infty)$ and $(L^\infty_{2\pi})^*\supset L^1_{2\pi}$. Let

$$T_n^\perp(L^p_{2\pi}) = \Big\{g\in L^q_{2\pi}\colon \int_0^{2\pi} gt_n\,dx = 0,\ \forall t_n\in T_n\Big\} \quad (1\leq p<\infty).$$

$$T_n^\perp(L^\infty_{2\pi}) \supset \Big\{g\in L^1_{2\pi};\ \int_0^{2\pi} gt_n\,dx = 0,\ \forall t_n\in T_n\Big\} := \Omega(L^1_{2\pi}).$$

Let $Tf:=f'$ $\big(f\in D(T):=D^p_{2\pi}\big)$ and

$$Ig(x) := \int_0^x g(t)\,dt \quad \left[g\in D_{q,n}(I) := \begin{cases} T_n^\perp(L^p_{2\pi}) & (1\leq p<\infty), \\ \Omega(L^1_{2\pi}) & (p=\infty) \end{cases}\right].$$

It is easy to see that T and I satisfy the conditions (4), (5), (6) (with $D(T)=L^\perp$ in the case $1\leq p<\infty$, $D(T)\subset L^\perp$ in the case $p=\infty$). So by Theorem 1 we have

Theorem 2. *Let* $1\leq q\leq\infty$, $n=1,2,\ldots$. *For every* $g\in D_{q,n}(I)$ *we have*

$$\Big\|\int_0^x g(t)\,dt\Big\|_{L^q_{2\pi}} \leq \frac{c_3}{n}\|g\|_{L^q_{2\pi}}.$$

b) Let $X=L^p(w)$ $1\leq p\leq\infty$ be the Banach space of all measurable functions defined on $[-1,1]$ with norm

$$\|f\|_{p,w} = \Big\{\int_{-1}^1 |f|^p w\,dx\Big\}^{1/p} \quad (1\leq p<\infty),$$

$$\|f\|_{\infty,w} = \|f\|_\infty = \operatorname*{ess\,sup}_{x\in[-1,1]} |f(x)|$$

where

$$w(x) = (1-x)^\alpha(1+x)^\beta \quad (\alpha,\beta>-1,\ x\in[-1,1]).$$

We have $X^*=[L^p(w)]^*=L^q(w)$ $(1\leq p<\infty,\ 1/p+1/q=1)$ and $[L^\infty(w)]^*\supset L^1(w)$.

Let Π_n be the set of all algebraic polynomials of degree at most n $(n=0,1,2,\ldots)$ and let $L=\Pi_n$. Then we have

$$L^\perp = \Pi_n^\perp(L^p(w)) = \Big\{g\in L^q(w)\colon \int_{-1}^1 gp_n w\,dx = 0,\quad \forall p_n\in\Pi_n\Big\}$$

(9) $$(1\leq p<\infty,\quad 1/p+1/q=1),$$

$$\Pi_n^\perp[L^\infty(w)] \supset \Big\{g\in L^1(w)\colon \int_{-1}^1 gp_n w\,dx = 0,\quad \forall p_n\in\Pi_n\Big\} := \Omega_n(w).$$

For any $f\in L^p(w)$ $(1\leqq p\leqq\infty)$ we define

$$E_n^p(w,f)=\inf_{p_n\in\Pi_n}\|f-p_n\|_{p,w}\quad (n=0,1,2,\ldots).$$

The following class of functions was defined in [7]:

$$M_p(w):=\{f\in L^p(w)\colon f \text{ is absolutely continuous in } (-1,1), \sqrt{1-x^2}\,f'(x)\in L^p(w)\}.$$

In [7] we proved that

$$(10)\quad E(w,f)\leqq\frac{c_4}{n}\|\sqrt{1-x^2}\,f'(x)\|_{p,w}\quad (1\leqq p\leqq\infty,\ f\in M_p(w),\quad n=1,2,\ldots).$$

Now, let us define the operators T and I as follows:

$$Tf(x)=T_pf(x):=\sqrt{1-x^2}\,f'(x)\quad (f\in D(T_p):=M_p(w)),$$

$$Ig(x)=I_{q,n}g(x):=\frac{1}{\sqrt{1-x^2}\,w(x)}\int_{-1}^{x}w(t)g(t)\,dt\quad (g\in D(I_{q,n}))$$

where $D(I_{q,n})$ denotes the domain of $I=I_{q,n}$ which is defined by

$$(11)\qquad D(I_{q,n}):=\Pi_n^{\perp}[L^p(w)]\quad (2<q\leqq\infty,\ 1/p+1/q=1),$$

$$(12)\qquad \begin{aligned}&D(I_{q,n}):=\{g\in\Pi_n^{\perp}[L^p(w)]\colon g \text{ satisfies condition (13)}\}\quad (1<q\leqq 2)\\ &D(I_{q,n}):=\{g\in\Omega_n(w)\colon g \text{ satisfies condition (13)}\}\quad (q=1),\end{aligned}$$

where

$$(13)\qquad \int_{-1}^{x}w(t)g(t)\,dt=o\left[w^{1-1/q}(x)\sqrt{1-x^2}\right]\quad (|x|\to 1).$$

We prove that the operators T and I satisfy the conditions in Theorem 1.

Let $f\in D(T_p)$, $g\in D(I_{q,n})$ and let $G(x)=\int_{-1}^{x}w(t)g(t)\,dt$.

In the case $(1\leqq p<2$, so $2<q\leqq\infty)$ we have for $-1<x<0$

$$|G(x)|=\left|\int_{-1}^{x}w(t)g(t)\,dt\right|\leqq\left(\int_{-1}^{x}|g(t)|^q w(t)\,dt\right)^{1/q}\left(\int_{-1}^{x}w(t)\,dt\right)^{1/p}\leqq$$

$$\leqq\|g\|_{q,w}O[(x+1)^{1/p}w^{1/p}(x)]=$$

$$=(x+1)^{1/p-1/p}O\left[w^{1/2}(x)\sqrt{1-x^2}\right]=o\left[w^{1/p}(x)\sqrt{1-x^2}\right]\quad (x\to -1).$$

For $0<x<1$, using the relation

$$G(x)=\int_{-1}^{x}w(t)g(t)\,dt=-\int_{x}^{1}w(t)g(t)\,dt$$

(which follows from the fact that $\int_{-1}^{1} wg\,dt=0$ since $g\in D(I_{q,n})$). By a similar method we obtain

$$G(x) = o\left[w^{1/p}(x)\sqrt{1-x^2}\right] \quad (x \to 1).$$

So relation (13) is true for every $g\in D(I_{q,n})$ $(1\leqq q\leqq\infty,\ n=1,2,\ldots)$. Therefore by integration by part we have

$$\int_{-1}^{1} f(x)g(x)w(x)\,dx = \int_{-1}^{1} f'(x)G(x)\,dx =$$

$$= \int_{-1}^{1} \sqrt{1-x^2} f'(x)\frac{1}{\sqrt{1-x^2}\,w(x)}G(x)\,w(x)\,dx = \int_{-1}^{1} Tf(x)\,Ig(x)w(x)\,dx.$$

Since this integral exists for every $Tf\in L^p(w)$ and $T[D(T)]=L^p(w)$, we have by a well known theorem of functional analysis that $Ig\in L^q(w)$ and the last formula proves (6).

By Theorem 1, using (10) we have

Theorem 3. *Let* $1\leqq q\leqq\infty$, $n=1,2,\ldots$. *For every* $g\in D(I_{q,n})$ *we have*

$$\left\|\frac{1}{\sqrt{1-x^2}\,w(x)}\int_{-1}^{x} w(t)g(t)\right\|_{q,w} \leqq \frac{c_5}{n}\|g\|_{q,w}.$$

c) Let $X=L^p=L^p(-\infty,\infty)$ $(1\leqq p\leqq\infty)$ be the Banach space of functions defined on $(-\infty,\infty)$. Let

$$\varrho(x) = \varrho_{\gamma,\delta}(x) = (1+|x|^\gamma)^{\delta/2\gamma}e^{-|x|^\gamma/2} \quad (\gamma\geqq 2,\ \delta\geqq 0,\ -\infty<x<\infty).$$

We consider the following subspace of L^p:

$$L := H_n := \{\varrho(x)p_n(x):\ p_n\in\Pi_n\} \quad (n=1,2,\ldots).$$

We have

$$L^\perp = H_n^\perp(L^p) = \left\{g\in L^q:\ \int_{-\infty}^{\infty} gp_n\varrho\,dx = 0,\quad \forall p_n\in\Pi_n\right\}$$

$$(1\leqq p<\infty,\ 1/p+1/p=1,\ n=1,2,\ldots)$$

and

$$H_n^\perp(L^\infty) \supset \left\{g\in L^1:\ \int_{-\infty}^{\infty} gp_n\varrho\,dx = 0\quad \forall p_n\in\Pi_n\right\} := \Omega.$$

For any $\varrho f\in L^p$ we define

$$E_n^p(\varrho,f) = \inf_{p_n\in\Pi_n}\|\varrho(f-p_n)\|_p \quad (n=0,1,2,\ldots).$$

FREUD [3] proved the following inequality:

(14) $$E_n^p(\varrho, f) \leqq \frac{c_6}{n^{1-1/\gamma}} \|\varrho f'\|_p \quad (1 \leqq p \leqq \infty,\ \varrho f \in M_p(\varrho),\quad n = 1, 2, \dots)$$

where

(15) $$M_p(\varrho) := \{\varrho f \in L^p : f \text{ is absolutely continuous on } (-\infty, \infty),\ \varrho f' \in L^p\}.$$

We define $T = T_p$ and $I = I_{q,n}$ as follows:

$$T(\varrho f) := \varrho f' \quad (\varrho f \in M_p(\varrho) := D(T_p)),$$

$$Ig(x) := \frac{1}{\varrho(x)} \int_{-\infty}^{x} \varrho(t) g(t)\, dt \quad (f \in D(I)),$$

where

$$D[I_{q,n}) := \left\{ \begin{matrix} g \in H_n^{\perp}(L^p) & (1 \leqq p < \infty) \\ g \in \Omega & (p = \infty) \end{matrix} \quad \left(\begin{matrix} \text{and } g \\ \text{satisfies condition (16)} \end{matrix} \right) \right\},$$

where

(16) $$\int_{-\infty}^{x} \varrho(t) g(t)\, dt = O[|x|^{1/q} \varrho(x)] \quad (|x| \to \infty).$$

First we prove that T and I satisfy the conditions of Theorem 1. Let $f \in D(T_p)$ $(1 \leqq p \leqq \infty)$ and let $g \in D(I_{q,n})$ $(1/p + 1/q = 1)$,

$$G(x) := \int_{-\infty}^{x} g(t)\, \varrho(t)\, dt.$$

Using (16) we obtain

$$|f(x) G(x)| = |G(x)| \left| \int_0^x f'(t)\, dt + f(0) \right| = o[|x^{1/q}| \varrho(x)] + o\left[|x|^{1/q} \varrho(x) \left| \int_0^x |f'(t)|\, dt \right| \right] =$$

$$= o(1) + o\left[\left| \int_0^x |x|^{1/q} \varrho(x) |f'(t)\, dt \right| \right] = o(1) + o\left[|x|^{1/q} \left| \int_0^x \varrho(t) |f'(t)| dt \right| \right] =$$

$$= o(1) + o\left[|x|^{1/q} \|\varrho f'\|_p \left| \left(\int_0^x dt \right)^{1/q} \right] = o(1) + o(1) = o(1) \quad (|x| \to \infty).$$

So we have by integration by part

(17) $$\int_{-\infty}^{\infty} f(x) \varrho(x) g(x)\, dx = \int_{-\infty}^{\infty} f'(x) G(x)\, dx =$$

$$= \int_{-\infty}^{\infty} \varrho(x) f'(x) \frac{1}{\varrho(x)} G(x)\, dx = \int_{-\infty}^{\infty} T(\varrho f) Ig\, dx.$$

Since the integral (17) exists for every $T(\varrho f)\in L^p$ and $T[D(T)]=L^p$ we have $Ig\in L^q$ and (17) proves condition (6). Other properties of T and I follow from the definition.

We have by Theorem 1 and (14)

Theorem 4. *Let* $1\leqq q\leqq\infty$, $n=1,2,\ldots$. *For every* $g\in D(I_{q,n})$ *we have*

$$\left\|\frac{1}{\varrho(x)}\int_{-\infty}^{x}\varrho(t)g(t)\,dt\right\|_q\leqq\frac{c_7}{n^{1-1/\gamma}}\|g\|_q.$$

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