On non-modular n-distributive lattices: The decision problem for identities in finite n-distributive lattices

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To Professor K. Tandori on his sixtieth birthday

1. Introduction. It was proved in [1] that the lattice $\mathfrak{C}(R^{n-1})$ of all convex sets of the $n-1$ dimensional Euclidean space R^{n-1} is a member of the lattice variety $\boldsymbol{D}_n^f$ generated by the finite n-distributive lattices. It is an open question whether this variety equals $\boldsymbol{D}_n$, the class of all n-distributive lattices. An answer might be based on a solution to the word problem for free lattices in $\boldsymbol{D}_n$. In this paper we accomplish a slightly different task and solve the word problem for free lattices in $\boldsymbol{D}_n^f$. Besides, we give a new example of a lattice in this variety, namely we show that the dual of $\mathfrak{C}(R^{n-1})$ is a member of $\boldsymbol{D}_n^f$, too.

We need some notions of universal algebra and lattice theory. By an n-distributive lattice we mean a lattice satisfying the identity

$$x\wedge \bigvee_{i=0}^{n} y_i = \bigvee_{j=0}^{n} \Big(x\wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} y_i\Big).$$

A lattice variety is a class of lattices that can be characterized by a set of identities. The variety generated by a class $\boldsymbol{K}$ of lattices is the smallest lattice variety containing $\boldsymbol{K}$. The decision problem for identities in a class $\boldsymbol{K}$ of lattices is the problem of finding an algorithm which, given any identity $p=q$, decides whether $p=q$ holds in every member of $\boldsymbol{K}$ or not. It is equivalent to the word problem for free lattices in the variety generated by $\boldsymbol{K}$.

We are going to use the following concepts concerning convex sets. Let $\boldsymbol{a}, \boldsymbol{r}_0\in R^{n-1}$. Then the set of all $\boldsymbol{r}\in R^{n-1}$ such that the scalar product $(\boldsymbol{a}, \boldsymbol{r}-\boldsymbol{r}_0)$ equals 0, is called a hyperplane. The set of all $\boldsymbol{r}$ with $(\boldsymbol{a}, \boldsymbol{r}-\boldsymbol{r}_0)\geqq 0$ is called a (closed) halfspace. A finite intersection of halfspaces is a convex polyhedron. The convex closure of a finite number of points is a convex polytope. It is well-known that convex polytopes,

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convex polyhedra and convex sets of R^{n-1} all form lattices, and that in all these three lattices the operations are the intersection and the convex closure of two convex sets. (See [2].) Convex polytopes are exactly the bounded convex polyhedra, thus, in the above list, the former lattice is always a proper sublattice of the latter one.

2. On the dual of $\mathfrak{C}(R^{n-1})$. We prove the following theorem.

Theorem 2.1. *The dual of* $\mathfrak{C}(R^{n-1})$ *is a member of the variety* D_n^f.

Proof. In [1], Lemma 3.1, it was shown that $\mathfrak{C}(R^{n-1})$ is a member of the variety generated by the lattice $\mathfrak{C}_{\text{fin}}(R^{n-1})$ of all $n-1$ dimensional convex polytopes, therefore, it is also a member of the variety generated by $\mathfrak{C}_{\text{fin}}^-(R^{n-1})$ of all $n-1$ dimensional convex polyhedra. Thus it is sufficient to show that the latter lattice is a member of the variety generated by all finite dually n-distributive lattices. By Theorem 1.1 of [1], $\mathfrak{C}_{\text{fin}}^-(R^{n-1})$ is dually n-distributive and its meet-irreducible elements are exactly the halfspaces of R^{n-1}. Let K be a finite set of halfspaces and let $\mathfrak{C}^-(K)$ consist of all those convex polyhedra that are intersections of elements of K. $\mathfrak{C}^-(K)$ is a lattice ordered by the inclusion relation, in fact, it is a meet-sublattice of $\mathfrak{C}_{\text{fin}}^-(R^{n-1})$. Let $\mathscr{K}$ denote the set of all finite subsets of the set of all halfspaces of R^{n-1}. The following two facts obviously include Theorem 2.1.

Lemma 2.2. *For any* $K\in\mathscr{K}$, $\mathfrak{C}^-(K)$ *is dually n-distributive.*

Lemma 2.3. $\mathfrak{C}_{\text{fin}}^-(R^{n-1})$ *is a member of the variety generated by all* $\mathfrak{C}^-(K)$; $K\in\mathscr{K}$.

Proof of Lemma 2.2. The dual n-distributivity of $\mathfrak{C}_{\text{fin}}^-(R^{n-1})$ and the meet-irreducibility of halfspaces in it imply that whenever a halfspace contains the intersection of a finite number of other halfspaces, then it contains the intersection of n of these halfspaces. In fact, let $h, h_1, \ldots, h_m$, $m>n$, be halfspaces and assume that h contains the intersection of the h_i, $i=1, 2, \ldots, m$. Then (denoting by $\vee$ the convex closure)

$$h = h\vee\bigcap_{i=1}^{m} h_i = \bigcap_{\substack{L\subseteq\{1,\ldots,m\}\\ |L|=n}} (h\vee \bigcap_{i\in L} h_i),$$

and, by the irreducibility of h, there is an L, $|L|=n$ with

$$h = h\vee \bigcap_{i\in K} h_i, \quad \text{i.e.,} \quad h \supseteq \bigcap_{i\in K} h_i.$$

Clearly, the lattices $\mathfrak{C}^-(K)$ also satisfy this property, as it refers only to inclusion and intersection, which coincide in $\mathfrak{C}_{\text{fin}}^-(R^{n-1})$ with those in $\mathfrak{C}^-(K)$. This, in turn, implies that the lattices $\mathfrak{C}^-(K)$ are also dually n-distributive. To prove this, let $a, b_0, \ldots$

$\dots, b_n \in \mathfrak{C}^-(K)$. Let $h \in K$, and assume that

$$h \supseteqq a \vee_H \bigcap_{i=0}^{n} b_i.$$

Then h contains a and h also contains n of the halfspaces occurring in the meet-representations of the b_i's. Thus h contains n of the b_i's, too, that is,

$$h \supseteqq \bigcap_{j=0}^{n} (a \vee_H \bigcap_{\substack{i=0 \\ i \neq j}}^{n} b_i).$$

Thus the meet-representations of $a \vee_H \bigcap_{i=0}^{n} b_i$ and of $\bigcap_{j=0}^{n} (a \vee_H \bigcap_{\substack{i=0 \\ i \neq j}} b_i)$ coincide.

Proof of Lemma 2.3. Let $p \geqq q$ be an m-ary lattice inequality holding in all the lattices $\mathfrak{C}^-(K)$, $K \in \mathscr{K}$. Let $a_1, \dots, a_m \in \mathfrak{C}^-_{\text{fin}}(R^{n-1})$. Let A be the set of subpolynomials of p, that is, (i) let $p \in A$, (ii) for $p_1 \wedge p_2 \in A$ or $p_1 \vee p_2 \in A$ let $p_1, p_2 \in A$, and (iii) let A be minimal relative to (i) and (ii). Let B be the set of subpolynomials of q. Finally let C be the set of all polyhedra $r(a_1, \dots, a_m)$, $r \in A \cup B$, and let K be the set of all halfspaces occurring in the irredundant meet-representation of one of the elements of C. (A polyhedron can be represented as an intersection of halfspaces in different ways, however, the irredundant meet-representation is unique.) Let the realization of a polynomial r in the lattice $\mathfrak{C}^-_{\text{fin}}(R^{n-1})$ be also denoted by r and let its realization in $\mathfrak{C}^-(K)$ be denoted by r^K. Then

$$p(a_1, \dots, a_m) = p^K(a_1, \dots, a_m) \geqq q^K(a_1, \dots, a_m) = q(a_1, \dots, a_m),$$

as K was chosen exactly to satisfy the two equalities in the above calculation.

3. On the variety $\boldsymbol{D}_n^f$. Here we deal with the word problem for free lattices of $\boldsymbol{D}_n^f$, in other words with the decision problem for identities in $\boldsymbol{D}_n^f$.

Theorem 3.1. *The word problem for free lattices in $\boldsymbol{D}_n^f$ is solvable.*

Before the proof we introduce some notations. Clearly, every lattice polynomial p can be written in the form

$$p = \bigvee_{i_1 \in I} \bigwedge_{i_2 \in I_{i_1}} \bigvee_{i_3 \in I_{i_1 i_2}} \cdots \bigwedge_{i_{2k-2} \in I_{i_1 \dots i_{2k-3}}} \bigvee_{i_{2k-1} \in I_{i_1 \dots i_{2k-2}}} x_{i_1 \dots i_{2k-1}} \tag{1}$$

if we allow I and the $I_{i_1 \dots i_r}$'s to consist of one element. We define the depth $d(p)$ of p by $d(p) := k$. $m(p)$ denotes the length (that is, the number of components) in the longest meet:

$$m(p) = \max \{\max_{i_1 \in I} |I_{i_1}|, \max_{\substack{i_1 \in I \\ i_2 \in I_{i_1} \\ i_3 \in I_{i_1 i_2}}} |I_{i_1 i_2 i_3}|, \dots\}.$$

Now define

$$c_n(p) := 1+n+n^2\cdot m(p)+n^3\cdot(m(p))^2+\ldots+n^{d(p)}\cdot(m(p))^{d(p)-1}.$$

We are ready to formulate the following lemma.

Lemma 3.2. *Let $p\leqq q$ be a lattice inequality holding in all finite n-distributive lattices containing at most $c_n(p)$ join-irreducible elements. Then $p\leqq q$ holds in every finite n-distributive lattice.*

To decide whether $p\leqq q$ holds in D_n^f requires now to check those finite n-distributive lattices having at most $c_n(p)$ join-irreducibles. This can be carried out in finite time, hence Lemma 3.2 implies Theorem 3.1.

Proof of the lemma. Let L be a finite lattice, let p and q be lattice polynomials in m variables and let $a_1, \ldots, a_m \in L$. Let K denote the set of join-irreducible elements of L. For a lattice polynomial r, let r^L denote the realization of r on L. Let $b\in K$ and let $b\leqq p^L(a_1, \ldots, a_m)$. Under the hypotheses of the lemma, we shall prove that $b\leqq q^L(a_1, \ldots, a_m)$. Let us introduce the following notations for subpolynomials of p. (p is defined by (1).)

$$p_{i_1} = \bigwedge_{i_2\in I_{i_1}} \bigvee_{i_3\in I_{i_1 i_2}} \cdots \bigwedge_{i_{2k-2}\in I_{i_1\ldots i_{2k-3}}} \bigvee_{i_{2k-1}\in I_{i_1\ldots i_{2k-2}}} x_{i_1\ldots i_{2k-1}}, \quad i_1\in I,$$

$$p_{i_1 i_2} = \bigvee_{i_3\in I_{i_1 i_2}} \cdots \bigwedge_{i_{2k-2}\in I_{i_1\ldots i_{2k-3}}} \bigvee_{i_{2k-1}\in I_{i_1\ldots i_{2k-2}}} x_{i_1\ldots i_{2k-1}}, \quad i_1\in I, \quad i_2\in I_{i_1},$$

etc. Now, by the assumption on b, we have

$$b \leqq \bigvee_{i_1\in I} p_{i_1}^L(a_1, \ldots, a_m).$$

Each $p_{i_1}^L(a_1, \ldots, a_m)$ is a join of join-irreducibles. By the n-distributivity of L, we may choose n of these join-irreducibles, say $b_1, \ldots, b_n$ such that

$$b \leqq \bigvee_{j=1}^{n} b_j.$$

(A detailed proof of this fact can be given by dualizing and generalizing the first part of the proof of Lemma 2.2.) Now assign to each b_j one (and only one) $p_{i_1}^L(a_1, \ldots, a_m)$ such that

$$b_j \leqq p_{i_1}^L(a_1, \ldots, a_m).$$

Then, for every b_j, if $p_{i_1}^L(a_1, \ldots, a_m)$ is assigned to b_j, we have

$$b_j \leqq p_{i_1 i_2}^L(a_1, \ldots, a_m), \quad i_2\in I_{i_1},$$

that is,

$$b_j \leqq \bigvee_{i_3\in I_{i_1 i_2}} p_{i_1 i_2 i_3}^L(a_1, \ldots, a_m), \quad i_2\in I_{i_1}.$$

Now we carry out the same construction in these $|I_{i_1}|$ different cases on b_j and on $\bigvee_{i_3\in I_{i_1i_2}} p^L_{i_1i_2i_3}(a_1, a_2, \ldots, a_m)$, with which we started on b and on $\bigvee_{i_1\in I} p^L_{i_1}(a_1, \ldots, a_m)$: For arbitrary fixed $i_2\in I_{i_1}$ choose join-irreducibles $b_{ji_21}, \ldots, b_{ji_2n}$ of L such that

$$b_j \leqq \bigvee_{l=1}^{n} b_{ji_2l}.$$

Again, each b_{ji_2l} is less than or equal to one of the $p^L_{i_1i_2i_3}(a_1, \ldots, a_m)$'s. Assign a $p^L_{i_1i_2i_3}(a_1, \ldots, a_m)$ to b_{ji_2l} such that

$$b_{ji_2l} \leqq p^L_{i_1i_2i_3}(a_1, \ldots, a_m),$$

etc. Let K_0 be the set of join-irreducibles defined during this procedure, that is,

$$K_0 = \{b\}\cup\{b_1, \ldots, b_n\}\cup \bigcup_{j=1}^{n} \bigcup_{\substack{i_2\in I_{i_1} \\ i_1 \text{ is} \\ \text{assigned} \\ \text{to } j}} \{b_{ji_21}, \ldots, b_{ji_2n}\}\cup \ldots.$$

Clearly, $|K_0|\leqq c_n(p)$. Let $\tilde{a}_i = \bigvee_{\substack{c\in K_0 \\ c\leqq a_i}} c$.

Let, furthermore, L_0 consist of all joins of elements of K_0. Then, by the definitions of K_0, L_0 and of $\tilde{a}_i$, $b\leqq p^{L_0}(\tilde{a}_1, \ldots, \tilde{a}_m)$. By the hypotheses, $p^{L_0}(\tilde{a}_1, \ldots, \tilde{a}_m)\leqq q^{L_0}(\tilde{a}_1, \ldots, \tilde{a}_m)$. (Here we need the n-distributivity of L_0, which is a consequence of the fact that whenever a join-irreducible element in L_0 is less than or equal to a join of elements of L_0, then it is less than or equal to an n-element subjoin of that join.) We obviously have $q^{L_0}(\tilde{a}_1, \ldots, \tilde{a}_m)\leqq q^L(\tilde{a}_1, \ldots, \tilde{a}_m)\leqq q^L(a_1, \ldots, a_m)$. Hence $b\leqq q^L(a_1, \ldots, a_m)$, as claimed.

References

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