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The mixture of probability distribution functions by absolutely continuous weight functions

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Dedicated to Professor Károly Tandori on his 60th birthday

1. Introduction

A function $F(x)$ of the real variable X is called a probability distribution function, if it satisfies the following conditions:

a) $F(x)$ is non-decreasing.

b) $F(x)$ is right continuous.

c) $F(\infty)=1$, $F(-\infty)=0$.

If a probability distribution function $F(x)$ is absolutely continuous with respect to the Lebesgue measure then $f(x)=F'(x)$ is called its probability density function.

We say $G(z, x)$ to be a family of probability distribution functions with parameter x, if the following properties are satisfied:

a) For each value of x the function $G(z, x)$ is a probability distribution function in z.

b) $G(z, x)$ is a measurable function of x.

Let $q(x)$ be an arbitrary probability distribution function. We form the expression

$$F(z)=\int_{-\infty}^{\infty} G(z, x)\, dq(x). \tag{1.1}$$

It is not difficult to show ([4], p. 199) that $F(z)$ is a probability distribution function, which is called the mixture of the family of probability distribution functions $G(z, x)$ with the weight function $q(x)$. An important question of the mixture theory of probability distribution functions is the following: Let the probability distribution function $F(z)$, and a family of probability distribution functions $G(z, x)$ with parameter x be given. What is the necessary and sufficient condition of having a probability dis-

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tribution function $q(x)$, which satiesfies equation (1.1)? Or in other words, what is the necessary and sufficient condition for the Fredholm's integral equation of first kind (1.1) to have such a solution, which is a probability distribution function?

This problem was solved by the author [2] in whole generality under the assumption that $q(x)$ is a discrete probability distribution function.

In this paper we give an answer to the raised question in case $q(x)$ is an absolutely continuous probability distribution function. The method in force makes necessary to introduce additional assumptions.

Let a and b, $a<b$, be given real numbers, where $a=-\infty$, $b=\infty$ are permitted, too. Denote by $E(a, b)$ the set of continuous probability distribution functions, which are strictly monotone increasing in $[a, b]$ and have values 0 and 1 at the points a and b, respectively. The inverse of $F\in E(a, b)$ is denoted by F^{-1}.

Without loss of generality we can assume that $G(z, x)$ is a family of probability distribution functions with parameter $x\in[0, 1]$.

The problem, which will be solved in this paper, is the following:

Let $F(z)\in E(a, b)$ and the family of probability distribution functions $G(z, x)\in E(a, b)$ with parameter $x\in[0, 1]$ be given. Assume that $q(x)$ is an absolutely continuous probability distribution function with probability density function $f(x)$, $x\in[0, 1]$. What is the necessary and sufficient condition for the Fredholm's integral equation of first kind

$$\int_0^1 G(z, x) f(x)\, dx = F(z) \tag{1.2}$$

to have a solution with square integrable probability density function $f(x)$?

To solve this integral equation, it will be traced back to the solution of the minimum problem of a symmetric positive definite Hilbert—Schmidt's kernel, using a distance concept between two probability distribution functions ([2], Chapter II).

Besides the Introduction the paper consists of three chapters. In the second one the problem will be traced back to the minimum problem of the above mentioned Hilbert—Schmidt's kernel. We deal with the eigenvalues and with the eigenfunctions of this kernel too. In the third chapter the answer will be given to the above raised question in two theorems. In the fourth one we give a family of probability distribution functions, by which the given probability distribution function is not representable as their mixture.

2. Preliminary

2.1. Let $F\in E(a, b)$, and let

$$F(x_{Nk}) = \frac{k}{N} \quad (k = 0, 1, \ldots, N),$$

where $a=x_{N0}<x_{N1}<\ldots<x_{NN-1}<x_{NN}=b$ are the Nth quantiles of F. Let $G\in E(a, b)$, and let us form the expression

$$D_N(G|F) = N\sum_{k=1}^{N}[G(x_{Nk})-G(x_{Nk-1})]^2.$$

We say

$$D(G|F) = \sup_{N\to\infty} D_N(G|F)$$

to be the discrepancy of $G\in E(a, b)$ with respect to $F\in E(a, b)$.

This discrepancy idea was investigated more generally by the author ([2], Chapter II.) and the following statement shows that this concept is a measure of the distance of two probability distribution functions:

$$D(G|F) \geqq 1$$

with equality if and only if $G=F$ ([2], Theorem 2.1.).

Denote by $H(F)\subset E(a, b)$ the set of the probability distribution functions with finite discrepancy with respect to $F\in E(a, b)$.

It can be shown ([2], Theorems 2.4. and 2.5.) that if $F\in E(a, b)$, $G\in H(F)$, then the probability distribution function $G(F^{-1}(z))$, $z\in[0, 1]$ is absolutely continuous with respect to the Lebesgue measure, and

$$D(G|F) = \int_0^1 \left[\frac{d}{dz}G(F^{-1}(z))\right]^2 dz.$$

Let $G_j\in H(F)$ $(j=1, 2)$ be given. The quantity

$$(G_1, G_2)_F = \int_0^1 \left[\frac{d}{dz}G_1\, F^{-1}(z)\right]\left[\frac{d}{dz}G_2((F^{-1}(z))\right] dz > 0$$

is said to be the common discrepancy of the probability distribution functions G_j $(j=1, 2)$ with respect to F. It is obvious that

$$(G, G)_F = D(G|F)$$

with $G_j=G$, $j=1, 2$. It follows from the Schwartz's inequality that

$$(G_1, G_2)_F \leqq [(G_1, G_1)_F(G_2, G_2)_F]^{1/2}.$$

Let $F\in E(a, b)$, and let

$$G(z, x)\in H(F), \quad x\in[0, 1]. \tag{2.1}$$

Let us introduce the quantity

$$K(x, y) = (G(z, x), \quad G(z, y))_F =$$

$$= \int_0^1 \left[\frac{d}{dz}G(F^{-1}(z), x)\right]\left[\frac{d}{dz}G(F^{-1}(z), y)\right] dz > 0, \quad x, y\in[0, 1], \tag{2.2}$$

and suppose that $K(x, y)$ is continuous in x and y. Moreover, let

$$\int_0^1 \int_0^1 K^2(x, y)\, dx\, dy < \infty, \tag{2.3}$$

i.e. $K(x, y)$ is a continuous Hilbert—Schmidt's kernel ([6], p. 135.).

Let the functions $f, g \in L_2(0, 1)$ be probability density functions. Then the probability distribution functions

$$G_f(z) = \int_0^1 G(z, x) f(x)\, dx,$$

$$G_g(z) = \int_0^1 G(z, x) g(x)\, dx$$

are mixtures of probability distribution functions (2.1) with respect to the weights f and g, respectively. Using Fubini's theorem we obtain

$$(G_f, G_g)_F = \int_0^1 \int_0^1 K(x, y) f(x) g(y)\, dx\, dy > 0.$$

Based on the foregoing we obtain

$$\int_0^1 \int_0^1 K(x, y) f(x) f(y)\, dx\, dy \geqq 1 \tag{2.4}$$

with equality if and only if

$$F(z) = \int_0^1 G(z, x) f(x)\, dx. \tag{2.5}$$

By the help of the kernel (2.2) our problem formulated in the Introduction can be expressed in the following way, too. Let $F \in E(a, b)$ and (2.1) be given. What is the necessary and sufficient condition of having such a probability density function $f \in L_2(0, 1)$ by which equation (2.5) is satisfied, or of having equality in the inequality (2.4).

2.2. Let the kernel (2.2) be given. It is well-known that if $h \in L_2(0, 1)$ is arbitrary, then the function

$$f(x) = \int_0^1 K(x, y) h(y)\, dy \in L_2(0, 1). \tag{2.6}$$

It is obvious that (2.2) is a symmetric kernel. It is well-known too that the integral equation

$$\varphi(x) - \lambda \int_0^1 K(x, y) \varphi(y)\, dy = 0, \quad x \in [0, 1] \tag{2.7}$$

has a solution different from $\varphi(x)=0$ if and only if λ satisfies the equation

$$f(-\lambda)=0, \tag{2.8}$$

where the entire function

$$f(z)=1+\sum_{n=1}^{\infty} a_n z^n \tag{2.9}$$

is the Fredholm's determinant of the function $zK(x, y)$, i.e. the coefficients of (2.9) are given by the following way ([6], p. 159). Let

$$x_k, y_k \in [0, 1] \quad (k=1, \dots, n), \qquad x_k \neq x_l, \quad y_k \neq y_l, \quad k \neq l. \tag{2.10}$$

Moreover, let

$$K\begin{pmatrix} x_1 \dots x_n \\ y_1 \dots y_n \end{pmatrix} = \begin{pmatrix} K(x_1, y_1) \dots K(x_1, y_n) \\ \dots\dots\dots\dots\dots\dots \\ K(x_n, y_1) \dots K(x_n, y_n) \end{pmatrix}.$$

It can be shown ([2], Theorem 3.1.) that the matrix

$$K\begin{pmatrix} x_1 \dots x_m \\ x_1 \dots x_n \end{pmatrix} - M$$

is positive definite or semidefinite, where M is the $n \times n$ matrix with entries 1. Thus

$$\operatorname{Det} K\begin{pmatrix} x_1 \dots x_n \\ x_1 \dots x_n \end{pmatrix} > 0$$

on the n-dimensional unit-cube, except a subset of this cube with zero measure, where inequalities (2.10) are not satisfied. From here we obtain

$$a_k = \frac{1}{k!} \int_0^1 \dots \int_0^1 \operatorname{Det} K\begin{pmatrix} x_1 \dots x_k \\ x_1 \dots x_k \end{pmatrix} dx_1 \dots dx_k > 0 \quad (k=1, 2, 3, \dots).$$

The zeros of equation (2.8) are the eigenvalues of kernel (2.2). Since all zeros of equation (2.8) are different from zero, and since kernel (2.2) is symmetric and positive definite, the eingevalues of kernel (2.2) are positive numbers.

By the help of the inequality concerning to the determinants of positive definite matrices ([1], Chapter 2, Theorem 7), we have

$$a_k \leqq \frac{k_1! \dots k_s!}{k!} a_{k_1} \dots a_{k_s}, \quad k_1 + \dots + k_s = k \quad (s=2, 3, \dots;\ k=2, 3, \dots).$$

In particular, we obtain

$$a_k \leqq \frac{1}{k} a_{k-1} a_1 \quad (k=1, 2, \dots),$$

where a_1 is the trace of the kernel (2.2) satisfying the inequality

$$a_1 = \int_0^1 K(x, x)\,dx \geqq \int_0^1 \int_0^1 K(x, y)\,dx\,dy \geqq 1.$$

Since the inequality

$$\frac{a_{k-1}}{a_k} \geqq \frac{k}{a_1} \quad (k = 1, 2, \ldots) \tag{2.11}$$

holds, based on the theorem of Kakeya ([5], p. 25) the moduli of zeros of the polinomial

$$f_n(z) = \sum_{k=0}^{n} a_k z^k \quad (n = 1, 2, \ldots) \tag{2.12}$$

lie in the closed interval the endpoints of which are the minimum and the maximum of the numbers

$$\frac{a_{k-1}}{a_k} \quad (k = 1, \ldots, n),$$

respectively. Thus the moduli of the zeros of polynomials (2.12) are in the interval $[1/a, \infty)$ based on (2.11). According to the theorem of Hurwitz ([3], p. 78) the zeros of (2.8) are equal to the limit points of the zeros of polynomials (2.12). Therefore, and since the zeros of equation (2.8) are positive numbers, we get that the eigenvalues of the kernel (2.2) lie on the interval $(1/a, \infty)$.

The eigenvectors of kernel (2.2) are the solutions of the integral equation (2.7) if λ runs over the eigenvalues of kernel (2.2).

In what follows let ω be the number of the eigenvalues of $K(x, y)$, i.e. ω is a positive integer, or equal to infinity accordingly to $K(x, y)$ is degenerate, or non-degenerate, respectively.

Let $1/a_1 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots$ be the eigenvalues of $K(x, y)$, and let $\varphi_1(x), \varphi_2(x), \ldots$ be the sequence of the corresponding orthonormal eigenfunctions.

Denote by $E_2(0, 1)$ the set of the functions defined on $[0, 1]$, which can be represented by the formula (2.6) with square integrable functions. Assume that the integrals of the functions of $E_2(0, 1)$ are equal to one. Let $E_2^+(0, 1)$ be the subset of $E_2(0, 1)$ with non-negative elements. It is obvious that these sets are convex.

Let $g, h \in L_2(0, 1)$. In what follows we apply the usual notation $(g, h) = \int_0^1 g(x)h(x)\,dx$.

Lemma 2.1. *Let the kernel* (2.2) *be given. Then the function* (2.4) *is concave on* $E_2(0, 1)$ *and on* $E_2^+(0, 1)$, *respectively.*

Proof. Let

$$f_j \in E_2(0, 1)(E_2^+(0, 1)) \quad (j = 1, \ldots, s), \quad q_j \geqq 0, \quad \sum_{j=1}^{s} q_j = 1.$$

In this case

$$\sum_{j=1}^{s} q_j f_j \in E_2(0, 1)(E_2^+(0, 1)).$$

Using the theorem of Mercer ([6], p. 230) we get partly that

$$(2.13) \quad \int_0^1 \int_0^1 K(x, y) \Big(\sum_{j=1}^{s} q_j f_j(x)\Big)\Big(\sum_{j=1}^{s} q_j f_j(y)\Big)\, dx\, dy = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \Big[\sum_{j=1}^{s} q_j (f_j, \varphi_k)\Big]^2,$$

and partly that

$$(2.14) \quad \sum_{j=1}^{s} q_j \int_0^1 \int_0^1 K(x, y) f_j(x) f_j(y)\, dx\, dy = \sum_{j=1}^{s} q_j \sum_{k=1}^{\infty} \frac{1}{\lambda_k} (f_j;\ \varphi_k)^2.$$

Based on the Cauchy's inequality we obtain

$$\Big(\sum_{j=1}^{s} q_j (f_j, \varphi_k)\Big)^2 \leqq \sum_{j=1}^{s} q_j \sum_{j=1}^{s} q_j (f_j, \varphi_k)^2.$$

This inequality and expressions (2.13) and (2.14) give us the statement of the Lemma.

Lemma 2.2. *Let* $f \in E_2(0, 1)$. *Then*

$$\int_0^1 \int_0^1 K(x, y) f(x) f(y)\, dx\, dy \geqq 1.$$

Proof. First of all we mention that

$$\varphi(z) = \frac{d}{dz} \int_0^1 G\big(F^{-1}(z), x\big) f(x)\, dx \in E_2(0, 1).$$

Since $\varphi(z) \in E_2(0, 1)$, it is sufficient to show that the integral of $\varphi(z)$ is equal to one. But this is obvious. Namely, using Fubini's theorem, we get

$$\int_0^1 \varphi(z)\, dz = \int_0^1 f(x)[G(b) - G(a)]\, dx = 1.$$

Accordingly

$$(2.15) \quad \int_0^1 \int_0^1 K(x, y) f(x) f(y)\, dx\, dy = \int_0^1 \varphi^2(z)\, dz \geqq \int_0^1 \varphi(z)\, dz = 1,$$

and this is the statement of our Lemma.

We have equality in (2.15) if and only if $\varphi(z) = 1$, $z \in [0, 1]$.

3. The solution of the problem

In this chapter we give a solution in connection with the problem mentioned in the Introduction in the following sense. Let $F\in E(a, b)$ and the family of probability distribution functions (2.1) be given. We look for a necessary and sufficient condition of having a solution of (2.5) or having equality in (2.4) with a function $f\in E_2^+(0, 1)$.

Let us define the following quantites:

$$\mu_0 = \inf_{f\in E_2^+(0,1)} \int_0^1 \int_0^1 K(x, y) f(x) f(y)\, dx\, dy,$$

(3.1)

$$\mu_1 = \inf_{f\in E_2(0,1)} \int_0^1 \int_0^1 K(x, y) f(x) f(y)\, dx\, dy.$$

Based on Lemma 2.2 we have

$$\mu_0 \geqq \mu_1 \geqq 1.$$

It follows from here that $\mu_1=1$ is the necessary condition for the mixibility of $F\in E(a, b)$ by the family of probability distribution functions (2.1). This condition is sufficient too if $f\in E_2(0, 1)$ satisfying (3.1) is an element of the set $E_2^+(0, 1)$. Accordingly we can proceed on the following way. We calculate μ_1 and a function $f\in E_2(0, 1)$ satisfying equation (3.1). If $\mu_1=1$ and $f\in E_2^+(0, 1)$ then $F\in E(a, b)$ can be mixed by the family of probability distribution functions (2.1) with weight f. If $\mu_1>1$, or if $\mu_1=1$ but $f\notin E_2^+(0, 1)$, the F cannot be mixed by this family of probability distribution functions.

The number μ_1 can regard as the measure of the mixibility of $F\in E(a, b)$ by the family of probability distribution functions (2.1).

Theorem 3.1. *Let $\{\lambda_k\}$ be the set of eigenvalues of the kernel* (2.2) *enumerated in an increasing way, and let the suitable orthonormal eigenfunctions be the elements of the sequence $\{\varphi_k(x)\}$. Let*

(3.2) $$\alpha_k = (1, \varphi_k) \quad (k = 1, 2, \dots).$$

Then

(3.3) $$\mu_1 = \frac{1}{\sum_{k=1}^{\omega} \alpha_k^2 \lambda_k},$$

where

(3.4) $$0 < \sum_{k=1}^{\omega} \alpha_k^2 \leqq 1.$$

Moreover

(3.5)
$$f(x) = \mu_1 \sum_{k=1}^{\infty} \alpha_k \lambda_k \varphi_k(x) \in E_2(0, 1)$$

is the only solution, which satisfies equality (3.1).

Proof. Let

(3.6)
$$f(x) = \int_0^1 K(x, y)h(y)\, dy = \sum_{k=1}^{\infty} \frac{x_k}{\lambda_k} \varphi_k(x) \in E_2(0, 1),$$

where $h \in L_2(0, 1)$ and

(3.7)
$$\chi_k = (h, \varphi_k) \quad (k = 1, 2, \ldots).$$

Considering the Hilbert—Schmidt's theorem ([6], p. 227) we have

(3.8)
$$\int_0^1 f(x)\, dx = \sum_{k=1}^{\infty} \frac{x_k}{\lambda_k} \alpha_k = 1$$

where the numbers α_k are defined by (3.2). In this case

$$\int_0^1 \int_0^1 K(x, y) f(x) f(y)\, dx dy = \int_0^1 \int_0^1 K_3(x, y) h(x) h(y)\, dx dy,$$

where $K_3(x, y)$ is the third iterated of kernel (2.2) ([6], p. 144). On the basis of Mercer's theorem ([6], p. 230) we get that the series

(3.9)
$$K_3(x, y) = \sum_{k=1}^{\infty} \frac{\varphi_k(x)\varphi_k(y)}{\lambda_k^3} \quad (x, y \in [0, 1])$$

converges uniformly. Thus

(3.10)
$$\int_0^1 \int_0^1 K_3(x, y) h(x) h(y)\, dx dy = \sum_{k=1}^{\infty} \frac{x_k^2}{\lambda_k^3}.$$

Our next duty is to calculate the minimum of (3.10) under condition (3.8)·
Since (3.9) is a concave function on the set $E_2(0, 1)$, using the method of Lagrange's multipliers, (3.10) has an absolute minimum inside of $E_2(0, 1)$ if and only if

$$\frac{\partial \Phi}{\partial x_k} = 0 \quad (k = 1, 2, \ldots)$$

with

$$\Phi(x_1, x_2, \ldots) = \sum_{k=1}^{\infty} \frac{x_k^2}{\lambda_k^3} - 2\lambda \sum_{k=1}^{\infty} \frac{x_k}{\lambda_k} \alpha_k,$$

i.e. if the conditions

(3.11)
$$x_k = \lambda \alpha_k \lambda_k^2 \quad (k = 1, 2, \ldots)$$

are satisfied. On the basis of (3.8) we have

$$\lambda \sum_{k=1}^{\omega} \alpha_k^2 \lambda_k = 1. \tag{3.12}$$

Considering partly that the elements of the sequence $\{\alpha_k\}$ are the Fourier's coefficients of $f(x)=1$, $x\in[0, 1]$ related to the orthonormal system $\{\varphi_k(x)\}$ and partly that

$$\int_0^1 \int_0^1 K(x, y)\, dx dy = \sum_{k=1}^{\omega} \frac{\alpha_k^2}{\lambda_k} \leqq a_1 \sum_{k=1}^{\omega} \alpha_k^2,$$

we obtain

$$\frac{\int_0^1 \int_0^1 K(x, y)\, dx dy}{\int_0^1 K(x, x)\, dx} \leqq \sum_{k=1}^{\omega} \alpha_k^2 \leqq 1,$$

i.e. the inequality (3.4) holds.

Substituting (3.11) into (3.10), we get that

$$\mu_1 = \lambda^2 \sum_{k=1}^{\omega} \alpha_k^2 \lambda_k = \lambda,$$

thus we obtain formula (3.3) on the basis of (3.12).

However substituting (3.11) into (3.6), we get that (3.10) touches his minimum by the function (3.5) of the set $E_2(0, 1)$.

We must mention separately the case if each element of the orthonormal system $\varphi_k(x)$, say $\varphi_s(x)$, is equal to one. In this case the minimum of (3.10) does not fall into the inside of $E_2(0, 1)$, since $\alpha_s=1$ and $\alpha_k=0$, if $k\neq s$. On the basis of (3.8) $x_s=\lambda_s$. Evidently (3.10) has minimum, if $x_k=0$, $k\neq s$, and

$$\mu_1 = \frac{1}{\lambda_s}; \quad f(x) = 1, \ x\in[0, 1].$$

Accordingly, (3.6) gives us the minimum of (3.10) in case the function (3.7) falls to the boundary of $E_2(0, 1)$, too. Thus the proof of Theorem 3.1 is finished.

Corollary 3.1. *Under the assumptions and notations of Theorem 3.1 the representation*

$$G(z) = \int_0^1 G(z, x) f(x)\, dx = \mu_1 \sum_{k=1}^{\omega} \alpha_k \lambda_k \int_0^1 G(x, z) \varphi_k(x)\, dx$$

holds uniformly in $z\in[a, b]$, *where* $(G, G)_F=1$.

Proof. Namely

$$G(z) = \int_0^1 G(z, x)(\mu_1 \sum_{k=1}^{\omega} \alpha_k \lambda_k \varphi_k(x))\, dx.$$

Using the Schwartz's inequality

$$\left|G(z) - \mu_1 \sum_{k=1}^{n} \alpha_k \lambda_k \int_0^1 G(z, x)\varphi_k(x)\, dx\right| = \left|\int_0^1 G(z, x)(\mu_1 \sum_{k=n+1}^{\omega} \alpha_k \lambda_k \varphi_k(x))\, dx\right| \leqq$$

$$\leqq \left(\int_0^1 G^2(z, x)\, dx\right)^{1/2} (\mu_1^2 \sum_{k=n+1}^{\omega} \alpha_k^2 \lambda_k^2)^{1/2} \leqq (\mu_1^2 \sum_{k=n+1}^{\omega} \alpha_k^2 \lambda_k^2)^{1/2}.$$

Since

$$\mu_1^2 \sum_{k=1}^{\omega} \alpha_k^2 \lambda_k^2 = \int_0^1 f^2(x)\, dx < \infty,$$

the sequence

$$\{\mu_1^2 \sum_{k=n+1}^{\omega} \alpha_k^2 \lambda_k^2\}_{n=0}^{\infty}$$

converges to zero.

The chief result of this paper is the following theorem, which arises directly from Theorem 3.1 and from Corollary 3.1.

Theorem 3.2. *Let* $\{\lambda_k\}$ *be the set of eigenvalues of the kernel* (2.2.) *enumerated in an increasing way, and let the suitable orthonormal eigenfunctions be the elements of the sequence* $\{\varphi_k(x)\}$. *Let*

$$\alpha_k = (1, \varphi_k) \quad (k = 1, 2, \ldots).$$

The probability distribution function $F \in E(a, b)$ *can be mixed by the family of probability distribution functions* (2.1) *with weight functions from the set* $E_2^+(0, 1)$ *if and only if the conditions*

$$\sum_{k=1}^{\omega} \alpha_k^2 \lambda_k = 1,$$

and

$$f(x) = \sum_{k=1}^{\omega} \alpha_k \lambda_k \varphi_k(x) \geqq 0, \quad x \in [0, 1]$$

are satisfied. In this case the represantion

$$F(z) = \sum_{k=1}^{\omega} \alpha_k \lambda_k \int_0^1 G(z, x)\varphi_k(x)\, dx$$

holds uniformly in $z \in [a, b]$.

4. Construction such a family of probability distribution functions by which a given probability distribution function cannot be represented as their mixture

The aim of this chapter is to prove the following theorem.

Theorem 4.1. *Let* $\varphi(x)\geqq 1$, $x\in[0, 1]$ *be a continuous function different from a constant. Then* $F(z)\in E(a, b)$ *cannot be represented as the mixture of the family of probability distribution functions*

$$G(z, x) = \big(F(z)\big)^{\varphi(x)}, \quad x\in[0, 1]$$

with a weight function of set $L_2(0, 1)$.

Proof. In this case

$$K(x, y) = \frac{\varphi(x)\varphi(y)}{\varphi(x)+\varphi(y)-1}\,; \quad x, y\in[0, 1]$$

on the basis of (2.2). Taking into account that the identity

$$K(x, y) = \frac{\varphi(x)\varphi(y)}{\varphi(x)\varphi(y)-[\varphi(x)-1][\varphi(y)-1]}$$

holds, we get that

$$\text{(4.1)} \qquad K(x, y) = \sum_{k=0}^{\infty}\left(\frac{\varphi(x)-1}{\varphi(x)}\,\frac{\varphi(y)-1}{\varphi(y)}\right)^k, \quad x, y\in[0, 1].$$

Let

$$\text{(4.2)} \qquad \max_{x\in[0, 1]}\frac{\varphi(x)-1}{\varphi(x)} = \Theta.$$

Using (4.2) and

$$\text{(4.3)} \qquad 0 < \Theta < 1,$$

we obtain that the series (4.1) converges uniformly and

$$1 \leqq \int_0^1\int_0^1 K^2(x, y)\,dx\,dy < \frac{1}{1-\Theta^2}.$$

Moreover, if $h\in L_2(0, 1)$ then representation

$$\text{(4.4)} \qquad \int_0^1\int_0^1 K(x, y)h(x)h(y)\,dx\,dy = \sum_{k=1}^{\infty}\left[\int_0^1\left(\frac{\varphi(x)-1}{\varphi(x)}\right)^k h(x)\,dx\right]^2$$

holds.

Namely let m, n be arbitrary positive integers. Let

$$\Delta(n, m) = \int_0^1 \int_0^1 \sum_{k=n}^{n+m} \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^k \left(\frac{\varphi(y)-1}{\varphi(y)}\right)^k h(x)h(y)\,dx\,dy =$$

$$= \sum_{k=n}^{n+m} \left[\int_0^1 \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^k h(x)\,dx\right]^2.$$

Using the Schwartz inequality, moreover the relations (4.2) and (4.3),

$$\Delta(n, m) \leqq \sum_{k=n}^{n+m} \int_0^1 \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^{2k} dx \int_0^1 h^2(x)\,dx < \frac{\Theta^{2(n+1)}}{1-\Theta^2} \int_0^1 h^2(x)\,dx \quad (m = 1, 2, \ldots).$$

Thus

$$\left|\int_0^1 K(x, y)h(x)h(y)\,dx\,dy - \sum_{k=0}^{n-1} \left[\int_0^1 \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^k h(x)\,dx\right]^2\right| <$$

$$< \frac{\Theta^{2(n+1)}}{1-\Theta^2} \int_0^1 h^2(x)\,dx,$$

which gives us the representation (4.4).

Let now $f \in L_2(0, 1)$ be a probability density function. Using representation (4.4)

$$\int_0^1 \int_0^1 K(x, y)f(x)f(y)\,dx\,dy = 1 + \sum_{k=1}^{\infty} \left[\int_0^1 \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^k f(x)\,dx\right]^2.$$

From here we get that the identity

$$\int_0^1 \int_0^1 K(x, y)f(x)f(y)\,dx\,dy = 1$$

holds if and only if the conditions

$$\int_0^1 \left(\frac{\varphi(x)-1}{\varphi(x)}\right)^k f(x)\,dx = 0 \quad (k = 1, 2, \ldots) \tag{4.5}$$

are satisfied. Since f is a probability density function, neither of (4.5) can be satisfied. This completes the proof.

In particular, let $\varphi(x) = 1+x$, $x \in [0, 1]$. Then

$$K(x, y) = \frac{(1+x)(1+y)}{x+y+1} = \sum_{k=0}^{\infty} \left(\frac{x}{1+x}\,\frac{y}{1+y}\right)^k,$$

and $\Theta = 1/2$. Using Theorem 4.1 we obtain the following result.

Corollary 4.1. *Under the assumption of Theorem* 4.1 $F\in E(a,b)$ *cannot be represented as the mixture of family of probability distribution functions*

$$G(z,x)=\big(F(x)\big)^{1+x},\quad x\in[0,1]$$

with a weight function from the set $L_2[0,1]$.

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