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Representation of functions in the space φ(L) by Vilenkin series

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To Professor K. Tandori on his 60th birthday

1. Let Φ be the set of all even real functions, which are nondecreasing on $[0, +\infty)$ and have the following properties:

(i) $$\varphi(0) = \varphi(+0) = 0$$

(ii) $$\varphi(x) > 0 \quad (x > 0)$$

(iii) $$\varphi(2x) = O(\varphi(x)) \quad (x \to +\infty) \quad (\varphi\in\Phi).$$

(The last property is called "Δ_2-condition".) For every $\varphi\in\Phi$ let us define the space $\varphi(L)$ as the set of measurable and almost everywhere finite functions f defined on $[0, 1]$, for which

$$\|f\|_\varphi := \int_0^1 \varphi(f(x))\,dx < +\infty$$

holds. If the functions f, g belong to $\varphi(L)$, then let their φ-distance be defined as $\|f-g\|_\varphi$, which determines the φ-convergence in the usual way. It is well-known [1] that $\varphi(L)$ is a linear space if and only if the Δ_2-condition holds. Furthermore, as special cases we get the L_p spaces for $0<p<+\infty$ $\big(\varphi(x):=|x|^p\ (x\in\mathbf{R})\big)$, the Orlicz spaces (if φ is convex), the space of a.e. finite functions with the convergence in measure $\left(\varphi(x):=\dfrac{|x|}{1+|x|}\ (x\in\mathbf{R})\right)$.

The system of functions $g_n\in\varphi(L)$ $(n\in\mathrm{N}:=\{0, 1, \dots\})$ is called a system of representation in $\varphi(L)$, if for every $f\in\varphi(L)$ there exists a series $\sum a_k g_k$ with coefficients a_n $(n\in\mathrm{N})$ such that $\lim_{n\to\infty} \|f-\sum_{k=0}^n a_k g_k\|_\varphi=0$. We remark that the uniqueness of such series for all f is not assumed. If this holds too, then the system is a Schauder basis. The following problem is due to P. L. ULJANOV [1]: by what means can be

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characterized the spaces $\varphi(L)$, in which the classical systems of functions are systems of representation? He himself gave in [2] a necessary and sufficient condition for this with respect to the Faber—Schauder system. The analogous question was answered by P. Oswald [3], [4] for $\varphi(L)\subset L_1:=L_1[0,1]$ and for the trigonometric, resp. the Haar system. In [5] we formulated whithout proof the next statement.

Theorem 1. *If* $\varphi\in\Phi$, $\varphi(L)\not\subset L_1$ $\left(i.e.\ \liminf_{x\to+\infty}\frac{\varphi(x)}{x}=0\right)$, $p\geqq 1$ *and* $\limsup_{x\to+\infty}\frac{\varphi(x)}{x^p}<+\infty$, *then every orthogonal basis in* L_p *is a system of representation in* $\varphi(L)$, *whereas the representation is not unique.*

The aim of this work is to solve the above mentioned Uljanov's problem with respect to the Vilenkin systems [6]. To the definition of these systems we fix a sequence of natural numbers $m=(m_0, m_1, \ldots)$ for which $m_k\geqq 2$ $(k\in\mathbf{N})$ holds. Define the group G_m as the set of all sequences $x=(x_0, x_1, \ldots)$ $(0\leqq x_k<m_k,\ x_k\in\mathbf{N},\ k\in\mathbf{N})$ with the group-operation $x+y:=((x_0+y_0)(\mathrm{mod}\ m_0),\ (x_1+y_1)(\mathrm{mod}\ m_1), \ldots)$ $(x, y\in G_m)$. The topology of G_m is given by the neighborhoods $I_n(x):=\{y\in G_m: y_0=x_0, \ldots, y_{n-1}=x_{n-1}\}$ $(x\in G_m, n\in\mathbf{N})$, thus G_m forms a compact Abelian group. Let us introduce in G_m the normalized Haar measure. If $M_0:=1$, $M_{k+1}:=m_kM_k$ $(k\in\mathbf{N})$, then the group G_m can be transformed in the interval $[0, 1]$ by means of the following mapping

$$G_m\ni x\mapsto\sum_{j=0}^{\infty}\frac{x_j}{M_{j+1}}\in[0,1].$$

It is easy to see that this correspondence is almost one-to-one and measure-preserving.

The system of characters of G_m can be given in the following way. For $k\in\mathbf{N}$ define the function r_k as

$$r_k(x):=\exp\frac{2\pi i x_k}{m_k}\quad (x\in G_m,\ i:=\sqrt{-1})$$

and arrange the finite products of r_k's as follows. If $n\in\mathbf{N}$, then there exists a unique representation

$$n=\sum_{k=0}^{\infty}n_kM_k\quad (0\leqq n_k<m_k,\ n_k\in\mathbf{N},\ k\in\mathbf{N}).$$

Let $\psi_n:=\prod_{k=0}^{\infty}r_k^{n_k}$, then the functions ψ_n are uniformly bounded and form a complete orthonormal system in L_1, which is called Vilenkin system (generated by the sequence m).

It is known, [7] [8], [16] that every Vilenkin system is a Schauder basis in L_p $(1<p<+\infty)$, from which it follows by means of interpolation the same statement for all reflexive Orlicz spaces. Taking into account Theorem 1 and the fact that the Vilenkin systems are bases in L_p $(1<p<\infty)$ we get

Theorem 2. *The assumptions* $\varphi\in\Phi$, $\varphi(L)\not\subset L_1$ *imply that all Vilenkin systems are systems of representation in* $\varphi(L)$. *(The representation is not unique.)*

In the case $\varphi(L)\subset L_1$ the Vilenkin systems may be at most Schauder bases in $\varphi(L)$, since they are uniformly bounded systems of functions. In this connection P. Oswald [9] showed that if a complete orthonormal system of uniformly bounded functions is basis in $\varphi(L)$ (for some $\varphi\in\Phi$), then $\varphi(L)$ is equivalent to an Orlicz space. (We consider L_1 as Orlicz space too.) It remains to answer only the question, in what Orlicz spaces are the Vilenkin systems bases? We know that the reflexivity of the space is sufficient for this. The next theorem shows that this condition is also necessary.

Theorem 3. *The Vilenkin systems are Schauder bases in a separable Orlicz space if and only if the space is reflexive.*

Furthermore, it follows from Theorem 2 and 3 the next statement.

Theorem 4. *If* $\varphi\in\Phi$, *then the Vilenkin systems are systems of representation in* $\varphi(L)$ *if and only if either* $\varphi(L)\not\subset L_1$ *or* $\varphi(L)$ *is equivalent to a reflexive Orlicz space.*

2. To the proof of Theorem 1 we need the following lemma.

Lemma 1. *Let* $1\leqq p<\infty$ *and the orthogonal system* $(g_n,\ n\in\mathbf{N})$ *be basis in* L_p *and* $\varphi\in\Phi$ *such that* $\liminf\limits_{x\to+\infty}\dfrac{\varphi(x)}{x}=0$, $\limsup\limits_{x\to+\infty}\dfrac{\varphi(x)}{x^p}<+\infty$. *Then for all* $f\in\varphi(L)$, $\varepsilon>0$ *and* $N\in\mathbf{N}$ *there exist* $R\in\mathbf{N}$ *and a polynomial* $P=\sum\limits_{k=N}^{R}a_kg_k$ *with respect to the system* $(g_n,\ n\in\mathbf{N})$, *for which*

$$\|f-P\|_\varphi\leqq\varepsilon \tag{1}$$

and

$$\Big\|\sum_{k=N}^{M}a_kg_k\Big\|_\varphi\leqq A_\varphi\|f\|_\varphi+\varepsilon\quad(N\leqq M\leqq R), \tag{2}$$

where the constant $A_\varphi>0$ *depends only on* φ.

Proof. It suffices to show that the statement is valid for the function $f=\alpha\chi$, where $\alpha\in\mathbf{R}$ and χ is the characteristic function of an arbitrary closed subinterval $[a,b]$ of $[0,1]$. To this end define the functions u_n $(n\in\mathbf{N})$ as follows

$$u_n(x):=\begin{cases}-n & \left(x\in\left[0,\ \dfrac{1}{n+1}\right)\right)\\[2mm] 1 & \left(x\in\left[\dfrac{1}{n+1},\ 1\right)\right)\end{cases}$$

and let $u_n(x+1)=u_n(x)$ $(x\in\mathbf{R})$. Thus

$$(3)\qquad \int_0^1 u_n = 0,\ \|u_n\|_p = \Big(\int_0^1 |u_n|^p\Big)^{1/p} \leqq 2n^{1-1/p}.$$

We can suppose (see [10]) that

$$(4)\qquad \varphi(x+y) \leqq C_\varphi(\varphi(x)+\varphi(y)),\ \int_0^1 \varphi(f(x))\,dx \leqq \psi(\|f\|_p) \quad (x, y \geqq 0,\ f\in L_p),$$

where $\psi\in\Phi$ is a suitable function and $C_\varphi>0$ is a constant depending only on φ. Since $\liminf_{x\to+\infty}\frac{\varphi(x)}{x}=0$, thus for all $\varepsilon>0$ there exits $n\in\mathbf{N}$ such that

$$(5)\qquad \frac{\varphi(\alpha(n+1))}{n+1} \leqq \frac{\varepsilon}{4C_\varphi}.$$

Denote C_p the Banach constant in L_p with respect to the system $(g_n, n\in\mathbf{N})$, i.e. for all series $\sum\alpha_k g_k$ we have that

$$\|\sum_{k=0}^{M}\alpha_k g_k\|_p \leqq C_p\|\sum_{k=0}^{\infty}\alpha_k g_k\|_p \quad (M\in\mathbf{N}).$$

Choose $j\in\mathbf{N}$ so that

$$(6)\qquad \psi\left(4^{1/p}C_p\alpha\frac{n^{1-1/p}}{j^{1/p}}\right) \leqq \varepsilon.$$

Let $\bigcup_{k=1}^{j}\Delta_k$ be a decomposition of $[a, b]$, where Δ_k's are disjoint intervals and the length of Δ_k $(k=1, ..., j)$ is $\frac{b-a}{j}$. Furthermore, denote χ_k the characteristic function of Δ_k $(k=1, ..., j)$. If $t_1, ..., t_j$ are natural numbers having the property $t_k\geqq T(j)$ $(k=1, ..., j)$ with some $0<T(j)\in\mathbf{N}$, then applying the Fejér lemma [11] (p. 77) we get from (5) that

$$(7)\qquad \begin{aligned}\int_0^1 \varphi\Big(\alpha\chi(x)-\sum_{k=1}^{j}\alpha\chi_k(x)u_n(t_kx)\Big)\,dx &= \sum_{k=1}^{j}\int_{\Delta_k}\varphi(\alpha(1-u_n(t_kx)))\,dx \leqq \\ &\leqq 2\sum_{k=1}^{j}\frac{b-a}{j}\int_0^1\varphi(\alpha(1-u_n(x)))\,dx \leqq 2\,\frac{\varphi(\alpha(n+1))}{n+1} \leqq \frac{\varepsilon}{2C_\varphi}.\end{aligned}$$

In virtue of (3) we have for fixed $s\in\mathbf{N}$ that

$$\lim_{t_k\to+\infty}\int_0^1 \chi_k u_{nk} g_s = 0 \quad (1 \leqq k \leqq j),$$

where the function u_{nk} is defined by $u_{nk}(x):=u_n(t_kx)$ $(x\in\mathbf{R})$. Because of this and since the system $(g_k,\, k\in\mathbf{N})$ is a basis in L_p, there exist natural numbers R_k; N_k and polynomials

$$p_k := \sum_{s=N_k}^{R_k} a_s g_s \quad (N_k < R_k < N_{k+1},\, k\in\mathbf{N})$$

such that if $N_k \leqq M \leqq R_k$ $(M\in\mathbf{N})$, then

$$\| \sum_{s=N_k}^{M} a_s g_s\|_p^p \leqq C_p^p \|\alpha\chi_k u_{nk}\|_p^p \leqq$$

(8)
$$\leqq C_p^p |\alpha|^p \int_{\Delta_k} |u_{nk}|^p \leqq 2C_p^p |\alpha|^p \frac{b-a}{j} \int_0^1 |u_n|^p \leqq 4C_p^p |\alpha|^p \frac{n^{p-1}}{j}$$

and

$$\|\alpha\chi_k u_{nk} - p_k\|_\varphi \leqq \frac{\varepsilon}{2jC_\varphi^j}.$$

We shall show that

$$P := \sum_{k=1}^{j} p_k = \sum_{k=1}^{j} \sum_{s=N_k}^{R_k} a_s g_s$$

is the desired polynomial. Indeed, in virtue of (4), (7) and (8) we have that

$$\|\alpha\chi - \sum_{k=1}^{j} p_k\|_\varphi \leqq C_\varphi \big(\|\alpha\chi - \sum_{k=1}^{j} \alpha\chi_k u_{nk}\|_\varphi + C_\varphi^{j-1} \sum_{k=1}^{j} \|\alpha\chi_k u_{nk} - p_k\|_\varphi\big) \leqq \varepsilon \tag{10}$$

and thus inequality (1) is proved.

Let S_M be the Mth $(M\in\mathbf{N})$ partial sum of P, i.e.

$$S_M := \sum_{k=1}^{q-1} p_k + \sum_{s=N_q}^{M} a_s g_s \quad (2 \leqq q \leqq j,\, N_q \leqq M \leqq R_q,\, q\in\mathbf{N}).$$

Then

$$\|S_M\|_\varphi \leqq C_\varphi \big(\|\sum_{k=1}^{q-1} p_k\|_\varphi + \|\sum_{s=N_q}^{M} a_s g_s\|_\varphi\big) =: C_\varphi (J_1 + J_2). \tag{11}$$

As in the proof of (1) we obtain

$$J_1 = \|\sum_{k=1}^{q-1} p_k\|_\varphi \leqq$$

(12)
$$\leqq C_\varphi^2 \big(\|\alpha \sum_{k=1}^{q-1} \chi_k\|_\varphi + \|\alpha \sum_{k=1}^{q-1} \chi_k (1-u_{nk})\|_\varphi + \|\sum_{k=1}^{q-1} \alpha\chi_k u_{nk} - p_k\|_\varphi\big) \leqq C_\varphi^2 (\|\alpha\chi\|_\varphi + 2\varepsilon).$$

From (4), (6) and (9) it follows that

$$J_2 = \|\sum_{s=N_q}^{M} a_s g_s\|_\varphi \leqq \psi\big(\|\sum_{s=N_q}^{M} a_s g_s\|_p\big) \leqq \varepsilon. \tag{13}$$

Using the estimations (11), (12) and (13) we get (2), which completes the proof of Lemma 1.

Proof of Theorem 1. Let $f\in\varphi(L)$. Applying Lemma 1 we consider the series $\sum a_s g_s=\sum p_k$, where

$$\|f-\sum_{k=0}^{n} p_k\|_\varphi \leqq 2^{-n} \quad (n\in \mathbf{N}) \quad \text{and} \quad \|\sum_{s=N_n}^{M} a_s g_s\|_\varphi \leqq A_\varphi(\|f-\sum_{k=0}^{n-1} p_k\|_\varphi+2^{-n})$$

$$(N_n \leqq M \leqq R_n,\ M\in\mathbf{N}).$$

It is not hard to see that this series converges to f in $\varphi(L)$. Theorem 1 is proved.

3. Let n be a natural number not less than 2. Denote Z_n the discrete cyclic group of order n, i.e. $Z_n:=\{0, 1, \dots, n-1\}$. Furthermore, let

$$p_{s,n}(t) = \sum_{j=0}^{s} c_j \exp\frac{2\pi ijt}{n} \quad (t,\ s\in Z_n)$$

be a discrete trigonometric polynomial of order s defined on Z_n (c_j's are arbitrary complex numbers) and $\|p_{s,n}\|_\infty:=\max_{t\in Z_n}|p_{s,n}(t)|$. We introduce the discrete measure on Z_n, i.e. let mes $\{t\}:=1/n$ $(t\in Z_n)$.

Lemma 2. *For all* $0<\alpha<1$ *and for all discrete trigonometric polynomials* $p_{s,n}$ $(0<s\in Z_n,\ n\in\mathbf{N})$ *the inequality*

$$\operatorname{mes}\{t\in Z_n\colon\ |p_{s,n}(t)| \geqq \alpha\|p_{s,n}\|_\infty\} \geqq \frac{1-\alpha}{2\pi s}$$

is true.

Proof. We denote by $P_{s,n}$ the following trigonometric polynomial

$$P_{s,n}(t):= \sum_{j=0}^{s} c_j \exp\frac{2\pi ijt}{n} \quad (t\in\mathbf{R}),$$

where

$$p_{s,n}(t) = \sum_{j=0}^{s} c_j \exp\frac{2\pi ijt}{n} \quad (t\in Z_n,\ n\in\mathbf{N},\ 0<s\in Z_n)$$

is a given discrete trigonometric polynomial. Let

$$\|P_{s,n}\|_\infty := \max_{t\in\mathbf{R}} |P_{s,n}(t)|.$$

On account of the well-known Bernstein inequality we have for the derivative of $P_{s,n}$ that

$$\|P'_{s,n}\|_\infty \leqq \frac{2\pi s}{n}\,\|P_{s,n}\|_\infty.$$

If $t_0 \in [0, n)$ is a point for which $|P_{s,n}(t_0)| = \|P_{s,n}\|_\infty$, then

$$|P_{s,n}(t)| = \left|P_{s,n}(t_0) + \int_{t_0}^{t} P'_{s,n}\right| \geqq \|P_{s,n}\|_\infty \left(1 - \frac{2\pi s}{n}|t - t_0|\right) \quad (t \in [0, n)).$$

Hence there exists an interval $\Delta \subset [0, n]$, the measure of which is not less than $(1-\alpha)n/\pi s$ such that

$$|P_{s,n}(t)| \geqq \|P_{s,n}\|_\infty \alpha \quad (t \in \Delta).$$

The number of the integers being in Δ is at least $[(1-\alpha)n/\pi s]$ (where $[x]$ denotes the integer part of the real number x) and since $\|P_{s,n}\|_\infty \geqq \|p_{s,n}\|_\infty$, therefore

$$\text{mes}\,\{t \in Z_n \colon |p_{s,n}(t)| \geqq \alpha \|p_{s,n}\|_\infty\} \geqq \max\left\{\frac{1}{n}, \frac{1}{n}\left[\frac{(1-\alpha)n}{\pi s}\right]\right\} \geqq \frac{1-\alpha}{2\pi s}.$$

Thus Lemma 2 is proved.

We shall show that the analogue of Lemma 2 is true for the Vilenkin systems too.

Lemma 3. *For all* $0 < \alpha < 1$ *and for all Vilenkin polynomials*

$$p_n = \sum_{k=0}^{n} c_k \psi_k$$

of order $(0<)n \in \mathbf{N}$ *(c_k's are arbitrary complex numbers) the inequality*

$$\text{mes}\,\{x \in G_m \colon |p_n(x)| \geqq \alpha \|p_n\|_\infty\} \geqq \frac{1-\alpha}{2\pi n}$$

is true, where $\|p_n\|_\infty := \max_{x \in G_m} |p_n(x)|$.

Proof. If p_n is the above Vilenkin polynomial and $jM_s \leqq n < (j+1)M_s$ ($n \in \mathbf{N}$, $j \in Z_{m_s}$), then

$$p_n = \sum_{k=0}^{M_s-1} c_k \psi_k + \sum_{t=1}^{j-1} \sum_{k=tM_s}^{(t+1)M_s-1} c_k \psi_k + \sum_{k=jM_s}^{n} c_k \psi_k =$$

$$= \sum_{k=0}^{M_s-1} c_k \psi_k + \sum_{t=1}^{j-1} r_s^t \sum_{k=0}^{M_s-1} c_{tM_s+k} \psi_k + r_s^j \sum_{k=0}^{n-jM_s} c_{k+jM_s} \psi_k =: P_0 + \sum_{t=1}^{j} r_s^t P_t,$$

where the Vilenkin polynomial P_t $(t = 0, \ldots, j)$ depends only on the first s coordinates of the argument. Let $z \in G_m$ such that $|p_n(z)| = \|p_n\|_\infty$, then $|p_n(x)| = \|p_n\|_\infty$ $(x \in I_{s+1}(z))$ and $p_n(y) = P_0(z) + \sum_{t=1}^{j} \exp \frac{2\pi i t y_s}{m_s} P_t(z)$ $(y \in I_s(z))$. Denote p_{j,m_s} the following discrete trigonometric polynomial

$$p_{j,m_s}(t) := P_0(z) + \sum_{t=1}^{j} P_t(z) \exp \frac{2\pi i t v}{m_s} \quad (v \in Z_{m_s}),$$

then $\|p_{j,m_s}\|_\infty=\|p_n\|_\infty$ and $|p_{j,m_s}(z_s)|=\|p_{j,m_s}\|_\infty$. On the other hand we have by Lemma 2 that

$$\operatorname{mes}\{v\in Z_{m_s}\colon |p_{j,m_s}(v)| \geqq \alpha\|p_{j,m_s}\|_\infty\} \geqq \frac{1-\alpha}{2\pi j}.$$

Hence

$$\operatorname{mes}\{x\in G_m\colon |p_n(x)| \geqq \alpha\|p_n\|_\infty\} \geqq \operatorname{mes}\{x\in I_s(z)\colon |p_n(x)| \geqq \alpha\|p_n\|_\infty\} =$$

$$= \frac{m_s}{M_{s+1}}\operatorname{mes}\{v\in Z_{m_s}\colon |p_{j,m_s}(v)| \geqq \alpha\|p_{j,m_s}\|_\infty\} \geqq \frac{1-\alpha}{2\pi j M_s} \geqq \frac{1-\alpha}{2\pi n},$$

which proves our lemma.

We get by standard argument from Lemma 3 the next

Corollary. *If* $0<q\leqq p\leqq +\infty$, *then for all Vilenkin polynomials* p_n *of order* $(0<)n\in\mathbf{N}$ *the following inequality is valid*

$$\|p_n\|_p \leqq C_{p,q} n^{\frac{1}{q}-\frac{1}{p}}\|p_n\|_q,$$

where $C_{p,q}>0$ *depends only on* p *and* q.

We remark that the special case $1\leqq q\leqq p\leqq +\infty$ can be found in [12].

Let $n, s\in\mathbf{N}$, $n\geqq 2$, $1\leqq s<n$ and

$$K_{s,n}(t) := \sum_{j=1}^{s}\exp\frac{2\pi i j t}{n} \quad (t\in Z_n).$$

Since for $0\neq t\in Z_n$ we have $|K_{s,n}(t)|=\dfrac{\left|\sin\frac{\pi t s}{n}\right|}{\sin\frac{\pi t}{n}}$ and $(2/\pi)x\leqq\sin x\leqq x$ $(0\leqq x\leqq\pi/2)$, therefore by $|K_{s,n}(t)|=|K_{s,n}(n-t)|$ it follows that

$$\text{(14)}\qquad \operatorname{card}\left\{t=1,\dots,n-1\colon |K_{s,n}(t)| \geqq \frac{2}{\pi}s\right\} \geqq 2\left[\frac{n}{2s}\right]-1 \quad \left(1\leqq s\leqq\left[\frac{n}{2}\right]\right).$$

A simple calculation shows the existence of an absolute constant $A\geqq 1$, such that

$$\text{(15)}\qquad \operatorname{card}\{t=1,\dots,n-1\colon |K_{s,n}(t)|\geqq y\} \geqq \frac{n}{Ay} \quad \left(1\leqq s\leqq\left[\frac{n}{2}\right],\ 1\leqq y\leqq\frac{s}{A}\right).$$

Define the numbers α_k $(k\in\mathbf{N})$ as follows. If $m_k\geqq 6A$, then let $\alpha_k=1$. If k, h are natural numbers such that $m_k\geqq 6A$, $m_{k+h}\geqq 6A$ but $m_{k+j}<6A$ $(0<j<h)$, then let $\alpha_{k+j}=0$ (if j is even) and $\alpha_{k+j}=1$ (if j is odd). Let us consider now the set of natural

numbers having the form

(16) $$N_n := \sum_{k=0}^{n-1} \alpha_k \left[\frac{m_k}{2}\right] M_k + a_n M_n \quad \left(1 \leqq n \in \mathbf{N},\ 1 \leqq a_n \leqq \left[\frac{m_n}{2}\right]\right),$$

where in the case $m_n \geqq 6A$ let $a_n \geqq 3A$. Thus $a_n M_n \leqq N_n < (a_n+1) M_n$ and

(17) $$\frac{N_{n+1}}{N_n} \leqq \max_{k \in \mathbf{N}} \frac{(3A+1) M_{k+1}}{\left[\frac{m_k}{2}\right] M_k} \leqq 3(3A+1).$$

Let $D_n := \sum_{k=0}^{n-1} \psi_k$ $(n \in \mathbf{N})$ the nth Dirichlet kernel with respect to the Vilenkin system. To the proof of Theorem 3 we need the following lemma.

Lemma 4. *If N_n $(n \in \mathbf{N})$ is of the form as in* (16), *then*

$$\lambda_{N_n}(x) := \operatorname{mes}\{z \in G_m \colon |D_{N_n}(z)| \geqq x\} \geqq \frac{C}{x} \quad \left(1 \leqq x \leqq \frac{N_n}{\pi}\right).$$

(Here and later on $C > 0$ denotes an absolute constant.)

Proof. If $z \in G_m$, then (see e.g. [7])

(18) $$|D_{N_n}(z)| = \left| \sum_{k=0}^{n} \sum_{j=1}^{t_k} \exp \frac{2\pi i j z_k}{m_k} D_{M_k}(z) \right|,$$

where $t_k := \alpha_k \left[\frac{m_k}{2}\right]$ $(0 \leqq k \leqq n-1)$ and $t_n := a_n$. It is also known [6] that

(19) $$D_{M_k}(z) = \begin{cases} M_k & (z \in I_k) \\ 0 & (z \in G_m \setminus I_k) \end{cases} \quad (k \in \mathbf{N}).$$

(I_k stands for $I_k(0) = \{y \in G_m \colon y_0 = 0, \dots, y_{k-1} = 0\}$.) Let $jM_s \leqq x < (j+1) M_s$ $\left(s = 0, 1, \dots, n,\ 1 \leqq j \leqq \frac{1}{A} \left[\frac{m_s}{2}\right[-2 \text{ for } s < n \text{ and } 1 \leqq j \leqq \frac{a_n}{A} - 2 \text{ for } s = n,\ j \in \mathbf{N}\right)$, where we assume as the first case that $m_s \geqq 6A$. Then by (15), (18) and (19) it follows for suitable $z \in I_s \setminus I_{s+1}$ that

$$|D_{N_n}(z)| \geqq M_s \left| \sum_{t=1}^{t_s} \exp \frac{2\pi i t z_s}{m_s} \right| - \sum_{k=0}^{s-1} \alpha_k \left[\frac{m_k}{2}\right] M_k \geqq (j+2) M_s - M_s \geqq (j+1) M_s \geqq x$$

and

(20) $$\lambda_{N_n}(x) \geqq \operatorname{mes}\{z \in I_s \setminus I_{s+1} \colon |D_{N_n}(z)| \geqq (j+1) M_s\} \geqq \frac{1}{M_{s+1}} \frac{m_s}{A(j+2)} \geqq \frac{C}{x}.$$

Now, let $jM_n \leqq x < (j+1) M_n$, $x \leqq \frac{N_n}{\pi}$, $m_n \geqq 6A$ and $\frac{a_n}{A} - 1 \leqq j \leqq a_n$. Then for suitable

$z \in I_n \setminus I_{n+1}$ we get by (14), (18) and (19) that

$$|D_{N_n}(z)| \geqq \frac{2}{\pi} a_n M_n - M_n \geqq \frac{a_n+1}{\pi} M_n \geqq \frac{N_n}{\pi} \geqq x$$

and

$$\lambda_{N_n}(x) \geqq \text{mes}\left\{z \in I_n \setminus I_{n+1}\colon |D_{N_n}(z)| \geqq \frac{N_n}{\pi}\right\} \geqq \left(2\left[\frac{m_n}{2a_n}\right]-1\right)\frac{1}{M_{n+1}} \geqq \frac{C}{x}.$$

If $M_n \leqq x \leqq \frac{N_n}{\pi}$ and $m_n < 6A$, then

$$\lambda_{N_n}(x) \geqq \text{mes}\,\{z \in G_m\colon |D_{N_n}(z)| \geqq N_n\} \geqq \frac{1}{M_{n+1}} \geqq \frac{C}{x}. \tag{21}$$

Finally, let $jM_s \leqq x < (j+1)M_s$, $s \leqq n-1$, $m_s < 6A$ and $1 \leqq j \leqq m_s - 1$. If $\alpha_s = 0$, then there exist five cases: 1) $s \leqq n-1$ and $m_{s+1} \geqq 6A$, 2) $s \leqq n-2$, $m_{s+1} < 6A$ and $m_{s+2} \geqq 6A$, 3) $s = n-1$ and $m_n < 6A$, 4) $s \leqq n-3$, $m_{s+1} < 6A$ and $m_{s+2} < 6A$, 5) $s = n-2$, $m_{n-1} < 6A$ and $m_n < 6A$. In the case 1) we get by (20)

$$\lambda_{N_n}(x) \geqq \text{mes}\,\{z \in I_{s+1} \setminus I_{s+2}\colon |D_{N_n}(z)| \geqq 2M_{s+1}\} \geqq \frac{1}{M_{s+1}} \geqq \frac{C}{x}.$$

The case 2) follows by same argument. In the case 3) it follows from (21) that $\lambda_{N_n}(x) \geqq 1/M_{n+1} \geqq C/x$. We get similarly the case 5). Hence it remains only the case ·). Since $\alpha_{s+1} \neq 0$, $\alpha_{s+2} = 0$ and for $z \in I_{s+2} \setminus I_{s+3}$

$$D_{N_n}(z) = \sum_{k=0}^{s+1} \alpha_k \left[\frac{m_k}{2}\right] M_k \geqq M_{s+1} \geqq x$$

is true, therefore it follows $\lambda_{N_n}(x) \geqq 1/2M_{s+2} \geqq C/x$.

If $\alpha_s = 1$, then $\alpha_{s+1} = 0$ or $m_{s+1} \geqq 6A$ and these cases can be examined as above. Since we showed already that $\lambda_{N_n}(M_s) \geqq C/M_s$ $(0 \leqq s \leqq n)$, therefore for $jM_s \leqq x < (j+1)M_s$, $0 \leqq s \leqq n-1$, $m_s \geqq 6A$ and $\frac{1}{A}\left[\frac{m_s}{2}\right] - 1 \leqq j \leqq m_s - 1$ we get $\lambda_{N_n}(x) \geqq \lambda_{N_n}(M_{s+1}) \geqq C/x$. This completes the proof of Lemma 4.

Proof of Theorem 3. It is well-known that the Vilenkin systems are not bases in L_1. (This follows from Lemma 4 too.) Let L_M be a separable Orlicz space generated by the N-function M and let $p := M'$. Furthermore, let N be the conjugate function of M in Young's sense and

$$\|f\|_M := \sup \int_0^1 fg \quad (f \in L_M)$$

where the supremum is taken over all g, for which $\int_0^1 N(g) \leqq 1$ is true. (For more details see e.g. [13].) If the Vilenkin system is a basis in the space L_M, then applying

Lemma 3 for $\alpha:=1/2$ it can be shown by same argument as in [14] that for the Dirichlet kernels the following estimation holds

$$\|D_n\|_M \leqq C \inf_x \frac{n+M(x)}{x} \leqq \tilde{C} \frac{n}{M^{-1}(n)} \quad (n \in \mathbf{N}).$$

On the other hand, we get by Lemma 4 (as in [14] again) for the indices N_n $(n\in\mathbf{N})$

$$\|D_{N_n}\|_M \geqq Cp(x) \ln \frac{N_n}{xp(x)} \quad (xp(x) \geqq 1).$$

Therefore $xp(x)\geqq 1$ implies $p(x) \ln \frac{N_n}{xp(x)} \leqq C \frac{N_n}{M^{-1}(N_n)}$ $(n\in\mathbf{N})$. In virtue of the Δ_2-condition and (17) this estimation holds for all $n\in\mathbf{N}$, from which the reflexivity of L_M follows by similar method as in [15]. Thus Theorem 3 is proved.

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