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Rate of approximation of linear processes

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Dedicated to Professor K. Tandori on the occasion of his 60th birthday

The well-known Korovkin theorem [8] established that a positive linear process, (or a sequence of positive linear operators), on $C[a, b]$ that approximates the functions 1, x and x^2 (for instance) also approximates any continuous function. An offspring of that result is the MOND-SHISHA [11] theorem that yields the rate of approximation of a function with certain smoothness to the rate of approximation of 1, x and x^2. The rate of approximation in the Mond-Shisha theorem and in many results that followed were forcibly uniform, that is independent of the point at which the function was approximated. In other words, the rate of approximation prescribed does not take into account that $L_n(\varphi_i, t)$ could tend to φ_i (φ_i being $1, x$ and x^2) at different rates for different points t. Recently, ESSER [6] and STRUKOV and TIMAN [13] proved that if L_n are positive linear operators on $C(I)$, $L_n(1, t)=1$, $L_n(x, t)=t$ and $L_n(x^2, t)=t^2+D_n(t)$ (in which case $D_n(t)\geqq 0$), we have $|L_n(f, t)-f(t)|\leqq$ $\leqq 15\omega_2\big(f, 1/2\sqrt{D_n}(t)\big)$ where

$$(1.1)\qquad \omega_2(f, h) \equiv \operatorname*{Sup}_{\eta\leqq h}\Big\{\operatorname*{Sup}_{x}\big(\Delta_\eta^2 f(x)|;\ [x-\eta, x+\eta]\subset I\big)\Big\},$$

I is $[a, b]$ or R^+ or R, $\Delta_h^r f\equiv\Delta_h(\Delta_h^{r-1}f)$ and $\Delta_h f(x)=f(x+h/2)-f(x-h/2)$. (The result mentioned here is that of STRUKOV and TIMAN [13]; ESSER [6] proved somewhat less but earlier).

Examining the situation on the particular but significant example of the Bernstein polynomials given by

$$(1.2)\qquad B_n(f, t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k},$$

we have the following two results which do not imply each other:

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a) For $0<\alpha<2$, $\operatorname{Sup}_{h<x<1-h} |\Delta_h^2 f(x)| \leqq Mh^\alpha$ if and only if $|B_n(f,t)-f(t)| \leqq \leqq M_1\left(\frac{t(1-t)}{n}\right)^{\alpha/2}$, BERENS and LORENTZ [1].

b) For $0<\alpha<2$, $\operatorname{Sup}_{h<x<1-h} |(x(1-x))^{\alpha/2}\Delta_h^2 f(x)| \leqq Mh^\alpha$ if and only if

$$\|B_n(f,\cdot)-f(\cdot)\| \leqq Mn^{-\alpha/2},$$

[2]. (In [5] it was shown that $\operatorname{Sup}_{h^2<x<1-h^2} |\Delta^2_{h\varphi(x)} f(x) \leqq Mh^\alpha$ for $\varphi(x)=\sqrt{x}$ when $x\in[0, 1/2]$ and, $\varphi(x)=\sqrt{1-x}$ when $x\in(1/2, 1]$, is also equivalent to the above).

In this paper we will be concerned only with the direct theorem, that is, in the particular cases (a) and (b) with the "only if" aspect. For positive operators for which $L_n(x, t)=t$ Strukov and Timan settled the "only if" question analogous to (a) and for positive operators with several conditions on $D_n(t)$ TOTIK [16] settled the question analogous to (b).

We will not have the restriction $L_n(x, t)=t$ (see application in §6) nor will we require that the operators be positive, and, therefore, higher moduli of smoothness can enter into the discussion (see applications in §7 and §8). We will impose relatively simple conditions on the moments and the result will be applied to many operators. The present result will provide some new applications even for positive operators. Its main additional strength, however, will be its uses for some non-positive operators, for instance combinations of "Exponential-type operators" introduced by C. P. MAY [10]. We will be aided by results using interpolation of spaces and Peetre K functionals and will introduce those concepts when needed.

2. Rate of convergence using moduli of continuity. In this section we will establish a direct theorem analogous to example (a) in the introduction. For a positive operator satisfying $L_n(1, x)=1$ we have the representation $L_n(f, x)=\int_I f(t)\,d\alpha_{n,x}(t)$ where $\alpha_{n,x}(t)$ is increasing and $\int_I d\alpha_{n,x}(t)=1$. The operators which we will treat in this paper will have the representation

$$L_n(f, x) = \int_I f(t)\, d\alpha_{n,x}(t) \quad \text{where} \quad \int_I d\alpha_{n,x}(t) = 1, \tag{2.1}$$

$$\int_{u\leqq t} |d\alpha_{n,x}(u)| \equiv v_{n,x}(t) \quad \text{and} \quad v_{n,x}(t) \leqq M.$$

The domain I represents a finite interval, semi-infinite ray or the whole real line and with no loss of generality we may assume that I is [0, 1], R^+ or R.

We can now state our first result.

Theorem 2.1. *Let* $I=[0,1]$, $I=R^+$ or $I=R$, $L_n(f,t)$ *defined by* (2.1), $\int_I t^i d\alpha_{n,x}(t)=x^i$ *for* $i=0,1,\dots,2m-1$, and $\int_I (t-x)^{2m} dv_{n,x}(t)\leqq D_n(x)$. *Then*

$$|L_n(f,x)-f(x)| \leqq \left(M+1+\frac{(4m)^{2m}}{2m!}\right)\omega_{2m}(f, D_n^{1/2m}(x))$$

for $I=R^+$ *or* $I=R$ *and*

$$|L_n(f,x)-f(x)| \leqq \left(M+1+\frac{(4m)^{2m}}{2m!}+LD_n^{1/2m}(x)\right)\omega_{2m}(f, D_n^{1/2m}(x))$$

for $I=[0,1]$ *where L depends only on m.*

Remark 2.2. The theorem is interesting only when $D_n(x)=o(1)$, $n\to\infty$. Obviously, $D_n(x)\geqq 0$ and at a point x_0 at which $D_n(x_0)=0$, $L_n(f,x_0)=f(x_0)$. In applications commonly met $D_n(x_0)=0$ only at the boundary of I. One can construct a sequence of linear (and even positive) operators for which $D_n(x)$ has a zero at a point internal to I, but in general those examples seem contrived and not very interesting.

Proof. We define for $h>0$ the Steklov-type averages

$$f_h(x)=\left(\frac{2m}{h}\right)^{2m}\int_0^{h/2m}\dots\int_0^{h/2m}\left\{\sum_{k=1}^{2m}\binom{2m}{k}(-1)^{k+1}f(x+k(u_1+\dots+u_{2m}))\,du_1\dots du_{2m}\right.$$

and for x such that $[x, x+2mh]$ is in I we have $|f(x)-f_h(x)|\leqq\omega_{2m}(f,h)$ and f_h has $2m$ continuous derivatives ($f_h^{(j)}$ is absolutely continuous for $j<2m$) and

$$|f_h^{(2m)}(x)|\leqq\left(\frac{2m}{h}\right)^{2m}\sum_{k=1}^{2m}\binom{2m}{k}\omega_{2m}\left(f,\frac{k}{2m}h\right)\leqq\left(\frac{4m}{h}\right)^{2m}\omega_{2m}(f,h).$$

We now show that for g with $2m$ derivatives we have

$$|L_n(g,x)-g(x)|\leqq\frac{D_n(x)}{2m!}\|g^{(2m)}\|.$$

Using Taylor's expansion we have

$$|L_n(g,x)-g(x)|\leqq\sum_{i=1}^{2m-1}\frac{1}{i!}|g^{(i)}(x)||L_n((t-x)^i,x)+$$

$$+\frac{1}{2m!}|g^{(2m)}(\xi)|\int_I(x-t)^{2m}dV_{n,x}(t)$$

$$\leqq\frac{D_n(x)}{2m!}\|g^{(2m)}\|_{C(I)}.$$

For $I=\overset{\circ}{R^+}$ or R we write

$$|L_n(f,x)-f(x)| \leqq |L_n(f-f_h,x)-(f(x)-f_h(x))+|L_n(f_h,x)-f_h(x)| \leqq$$

$$\leqq (M+1)\omega_{2m}(f,h)+\frac{D_n(x)}{h^{2m}}\frac{(4m)^{2m}}{2m!}\omega_{2m}(f,h)$$

and choose

$$h=(D_n(x))^{1/2m}.$$

For $I=[0,1]$ we choose a function $\psi\in C^\infty$ such that $\psi(x)=1$ on $[0,1/3]$, $\psi(x)=0$ on $[1/3,1]$ and $\psi(x)$ is decreasing. We define $F_h(x)=f_h(x)\psi(x)+f_{-h}(x)(1-\psi(x))$ where f_{-h} is the same as f_h but using $-h$ instead of h. We have $|F_h(x)-f(x)|\leqq\omega_{2m}(f,h)$ and $F_h^{(2m)}(x)=f_h^{(2m)}(x)$ in $[0,1/3]$ while $F_h^{(2m)}(x)=f_{-h}^{(2m)}$ in $(2/3,1]$. To calculate $F_h^{(2m)}(x)$ in $[1/3,2/3]$ we write

$$F_h^{(2m)}(x)=f_h^{(2m)}(x)-\{(f_h(x)-f_{-h}(x))(1-\varphi(x))\}^{(2m)}$$

and

$$\{(f_h(x)-f_{-h}(x))(1-\psi(x))\}^{(2m)}=\{(f_h^{(2m)}(x)-f_{-h}^{(2m)}(x))\}(1-\psi(x))-$$

$$-\sum_{k=0}^{2m-1}\binom{2m}{k}(f_h(x)-f_{-h}(x))^{(k)}\psi^{(2m-k)}(x)\equiv I_1+I_2.$$

By earlier consideration $|f_h^{(2m)}(x)-I_1|\leqq((4m)/h)^{2m}\omega_{2m}(f,h)$. To estimate I_2 we write $K\equiv\max_{x,i\leqq 2m}|\psi^{(i)}(x)|$, and so $|I_2|\leqq K\sum_{k=0}^{2m-1}\binom{2m}{k}|(f_h(x)-f_{-h}(x))^{(k)}|$, and therefore, it is enough to estimate for $k<2m$

$$|(f_h(x)-f_{-h}(x))^{(k)}(x)|=\left|\left(\frac{2m}{h}\right)^{2m}\left(\frac{d}{dx}\right)^k\int_0^{h/2m}\cdots\int_0^{h/2m}\bar\Delta^{2m}_{u_1+\dots+u_{2m}}f(x)\,du_1\dots du_{2m}-\right.$$

$$\left.-\left(\frac{2m}{h}\right)^{2m}\left(\frac{d}{dx}\right)^k\int_0^{-h/2m}\cdots\int_0^{h/2m}\bar\Delta^{2m}_{-u_1\dots-u_{2m}}f(x)\,du_1\dots du_{2m}\right|\equiv J(k;h),$$

where $\bar\Delta_\eta f(x)\equiv f(x+\eta)-f(x)$ and $\bar\Delta_\eta^l f(x)\equiv\bar\Delta(\bar\Delta_\eta^{l-1}f(x))$. We can now estimate $J(k,h)$ by estimating $J_+(k,h)$ and $J_-(k,h)$ being the first and second terms in the sum defining $J(k,h)$ respectively.

$$J_+(k,h)=\left|\left(\frac{2m}{h}\right)^{2m}\frac{d^k}{dx^k}\int_0^{h/2m}\cdots\int_0^{h/2m}\bar\Delta^{2m}_{u_1+\dots+u_{2m}}f(x)\,du_1\dots du_{2m}=\right.$$

$$=\left|\left(\frac{2m}{h}\right)^{2m}\int_0^{h/2m}\cdots\int_0^{h/2m}\bar\Delta^k_{h/2m}\bar\Delta^{2m}_{u_{k+1}+\dots+u_{2m}}f(x)\,du_{k+1}\dots du_{2m}\right|\leqq$$

$$\leqq\left(\frac{2m}{h}\right)^k 2^k\omega_{2m}\left(f,\frac{2m-k}{2m}h\right)$$

and $J_-(k,h)$ is evaluated similarly. Therefore

$$|I_2| \leqq 2h^{-2m+1}K\sum_{k=0}^{2m-1}\binom{2m}{k}(4m)^k\omega_{2m}(f, h) \leqq h^{-2m+1}2K(4m+1)^{2m}\omega_{2m}(f, h),$$

and choosing h as in the earlier case, we obtain our results.

Remark 2.3. We could have used Whitney's extension theorem (see STEIN [12, Ch. VI]) to find a function $F(x)$ defined on R that is identical with f in the domain I and whose $2m$ modulus of continuity $\omega_{2m}(F, h)$ in R is bounded by $K\omega_{2m}(f, h)$ in I. However, this method would still leave us with the need to estimate K and we will still need the Steklov averaging functions and almost all the steps of the present proof.

Remark 2.4. For $m=1$ and positive opeartors Strukov and Timan have a better estimate for the constant in the theorem, as extension F of f defined on [0, 1] or R is easily shown to satisfy $\omega_2(F, h)$ on R is smaller than $5\omega_2(f, h)$ on I (see TIMAN [14, p. 122]). That method is valid for $m=1$ even without the positivity but will yield a somewhat different constant.

It is obvious that instead of $\int_I t^i d\alpha_{n,x}(t)=x^i$ for $i=0, 1, \ldots, 2m-1$ we could have imposed $\int_I (t-x)^i d\alpha_{n,\mu}(t)=0$ for $i=1, \ldots, 2m-1$ and $\int d\alpha_{n,x}(t)=1$. We can now derive from Theorem 2.1 a generalization relaxing the conditions on the moments that would be useful for applications. We note that the next theorem would yield an estimate of the rate of convergence for positive operators for which $\int_I t\, d\alpha_{n,x}(t)\neq x$.

Theorem 2.2. *Suppose* $I=[0, 1]$, $I=R^+$ *or* $I=R$, $\int d\alpha_{n,x}(t)=1$ *and* $\int_I (t-x)^i\, d\alpha_{n,x}(t) \equiv R_{n,i}(x)$, $i=1, \ldots, 2m-1$ *where* $R_{n,i}(x) = o(1)$ $n\to\infty$, $\int_{u\leqq t} |d\alpha_{n,x}(u)|=v_{n,x}(t)\leqq M$ *and* $\int_I (t-x)^{2m}dv_{n,x}(t)\leqq D_n(x)$, *then*

$$|L_n(f, x)-f(x)| \leqq C\sum_{i=1}^{2m-1}\omega_i(f, |R_{n,i}(x)|^{1/i})+C\omega_{2m}(f, R_n(x)^{1/2m})$$

where $R_n(x)\equiv D_n(x)+K\sum_{i=1}^{2m-1}|R_{n,i}(x)|^{2m/i}$ *and where* C *and* K *depend only on* m.

Proof. To prove our theorem we will construct a new operator $A_n(f, x)$ that will satisfy the assumptions on the operator in Theorem 2.1. For that we add operators to obtain a new operator $A_n(f, x)$ such that on the one hand $A_n\big((t-x)^j, x\big)=0$ for $j\leqq 2m-1$ and on the other hand $\int_I (t-x)^{2m}dV_{n,x}(t)\leqq R_n(x)$ where $V_{n,x}(t)$ is the vaiiation up to t of $\beta_{n,x}(u)$, the measure describing $A_n(f, x)$, that is $A_n(f, x)\equiv$

$\equiv \int_I f(u)\,d\beta_{n,x}(u)$, and $R_n(x)$ would be as stated in the theorem. The function $R_n(x)$ will replace $D_n(x)$ and the operator $A_n(f,x)$ will replace $L_n(f,x)$ when we apply Theorem 2.1 to the present situation. We will write $\bar{\Delta}_h f(x) \equiv f(x+h)-f(x)$ and $\bar{\Delta}_h^m \equiv \bar{\Delta}_h(\bar{\Delta}_h^{m-1})$ and define $L_{n,i}(f,x)=((-1)/i!)\operatorname{Sgn} R_{n,i}(x)\bar{\Delta}^i_{|R_{n,i}(x)|^{1/i}}f(x)$ for all x in case $I=R_+$ or $I=R$, and for $0\le x\le 1/2$ in case $I=[0,1]$, in which case $L_{n,i}(f,x)=(-1)^{i+1}/i!\operatorname{Sgn} R_{n,i}(x)\bar{\Delta}^i_{-|R_{n,i}(x)|^{1/i}}f(x)$ for $1/2<x<1$. Since a simple calculation will yield $L_{n,i}((t-x)^i,x)=-R_{n,i}(x)$, we can add this operator to eliminate the i moment. However, for $i<j\le 2m-1$

$$L_{n,i}((t-x)^j,x) = c_{i,j}|R_{n,i}|^{j/i}(x)\operatorname{Sgn} R_{n,i}(x)$$

where $c_{i,j}$ is a constant that depends on i and j but not on n and x. To cancel that effect for $j=j_1$ we add the operator

$$L_{n,i,j_1}(f,x) \equiv \frac{-c_{i,j_1}}{j_1!}\operatorname{Sgn} R_{n,i}(x)\bar{\Delta}^{j_1}_{|R_{n,i}(x)|^{1/i}}f(x)$$

(and a similar version for $1/2<x\le 1$ in case $I=[0,1]$). Of course for $j_1<j\le 2m-1$ we still have $L_{n,i,j_1}((t-x)^{j_2},x)=c_{i,j_1,j_2}|R_{n,i}(x)|^{j_2/i}\operatorname{Sgn} R_{n,i}(x)$, the effect of which we cancel by adding $L_{n,i,j_1,j_2}(f,x)$ given in a similar way. In general we will define $L_{n,i,j_1,\dots j_k}(f,x)$ by induction. We have

$$L_{n,i,j_1,\dots j_{k-1}}((t-x)^{j_k},x) = c_{i,j_1,\dots j_k}(\operatorname{Sgn} R_{n,i}(x))|R_{n,i}(x)|^{j_k/i}$$

and for $i<j_1<\dots<j_k\le 2m-1$ we define

$$L_{n,i,j_1\dots j_k}(f,x) = \frac{-c_{i,j_1\dots j_k}}{j_k!}\operatorname{Sgn} R_{n,i}(x)\,\bar{\Delta}^{j_k}_{|R_{n,i}(x)|^{1/i}}f(x)$$

together with an appropriate modification for $1/2<x\le 1$ in case $I=[0,1]$. The operator $A_n(f,x)$ is given by

$$A_n(f,x) = L_n(f,x)+\sum_{i=1}^{2m-1}\{L_{n,i}(f,x)+ + \sum_{1\le i<j_1<\dots<j_k\le 2m-1} L_{n,i,j_1\dots j_k}(f,x)\}$$

where the second sum is taken on all finite sequences $j_1,\dots,j_k$ for which $1<j_1<\dots$ $\dots<j_k\le 2m-1$. To calculate the variation of the measure defining $A_n(f,x)$, we simply estimate its norm as an operator on $C(I)$. We can write

$$\|A_n\| \le M+\sum_{i=1}^{2m-1}\left\{\frac{2^i}{i!}+\sum_{1\le i<j_1<\dots<j_k\le 2m-1}|c_{i,j_1,\dots,j_k}|\frac{2^{j_k}}{j_k!}\right\},$$

and since $c_{i,j_1,\dots,j_k}$ are just constants that do not depend on our operator at all just on the i and j's (in case $R_{n,i}(x)=0$ they do not count at all being multiplied by 0), we have a bound for the variation $V_{n,x}(t)$ of the measure $\beta_{n,x}(t)$ given for $A_n(f,x)\equiv \equiv \int\limits_I f(t)\,d\beta_{n,x}(t)$. To estimate $J(n,x)=\int\limits_I (t-x)^{2m}dV_{n,x}(t)$, we write

$$J(n,x) \leqq D_n(x)+\sum_{i=1}^{2m-1}\Big\{|R_{n,i}(x)|^{2m/i}m_i+$$

$$+\sum_{1\leqq i<j_1<\ldots<j_k\leqq 2m-1}|c_{i,j_1\ldots j_k}||R_{n,i}(x)|^{2m/i}m_{i,j_1,\ldots,j_k}\leqq$$

$$\leqq D_n(x)+K\sum_{i=1}^{2m-1}|R_{n,i}(x)|^{2m/i}$$

since m_i and $m_{i,j_1,\ldots,j_k}$ are numbers independent of the particular operator.

Finally, we use theorem 2.1 and obtain $|A_n(f,x)-f(x)|\leqq C\omega_{2m}(f,R_n(x)^{1/2m})$ where an estimate for C is given in that theorem. To obtain the estimate for $|L_n(f,x)-f(x)|$, we estimate $L_{n,i}$ and $L_{n,i,j_1,\ldots,j_k}$ by $|L_{n,i}(f,x)|\leqq(1/i!)\,\omega_i(f,|R_{n,i}(x)|^{1/i})$ and

$$|L_{n,i,j_1,\ldots,j_k}(f,x)| \leqq \frac{|c_{i,j_1,\ldots,j_k}|}{j_k!}\,\omega_{j_k}(f,|R_{n,i}(x)|^{1/i})\leqq$$

$$\leqq \frac{|c_{i,j_1,\ldots,j_k}|}{j_k!}2^{j_k-i}\,\omega_i(f,|R_{n,i}(x)|^{1/i})$$

and this completes the proof.

3. Some preliminary Lemmas. For the result involving interpolation spaces we will need a few preliminary Lemmas which may be of interest by themselves.

Lemma 3.1. *Suppose* $f\in C[0,3/4]$, $f^{(i)}(x)$ *for* $0\leqq i<2m-1$ *is locally absolutely continuous in* $(0,3/4)$ *and* $\|x^{2m\gamma}f^{(2m)}(x)\|_{C[0,(3/4)]}=\Phi(f)<\infty$ *where* $0<\gamma<1$, *then we have*

(3.1) $\|x^{2m\gamma-2m+i}f^{(i)}(x)\|_{C\left[0,\frac{3}{4}\right]}\leqq K\big(\Phi(f)+\|f\|_{C\left[0,\frac{3}{4}\right]}\big)$ for $2m\gamma-2m+i>0$

and

(3.2) $\|f^{(i)}(x)\|_{C\left[0,\frac{3}{4}\right]}\leqq K\big(\Phi(f)+\|f\|_{C\left[0,\frac{3}{4}\right]}\big)$ for $2m\gamma-2m+i<0$.

Proof of Lemma 3.1. The proof follows to some extent a proof of a special case proved earlier by the author [2, p. 280]. We have $|x^{2m\gamma}f^{(2m)}(x)|\leqq\Phi(f)$ in $(0,3/4]$ and therefore $|f^{(2m)}(x)|\leqq M\Phi(f)$ in $[1/4,3/4]$. Consequently, $|f^{(2m-r)}(x)|\leqq \leqq M_1\big(M\Phi(f)+\|f\|_{C[0,(3/4)]}\big)$ for $1/4\leqq x\leqq 3/4$ and in particular for $x=1/2$. (This result follows a Kolmogorov-type inequality in a finite interval where the best constant is not known.) Assuming by induction on i $|x^{2m\gamma-i}f^{(2m-i)}(x)|\leqq K\big(\Phi(f)+$

$+\|f\|)$ $i=0, \dots, j-1$ and $2m\gamma-j>0$, then

$$\left|f^{(2m-j)}(x)-f^{(2m-j)}\left(\frac{1}{2}\right)\right| \leqq \left|\int_x^{1/2} f^{(2m-j-1)}(u)\,du\right| \leqq$$

$$\leqq K(\Phi(f)+\|f\|)\left|\int_x^{1/2} u^{-2m\gamma+j-1}\,du\right| \leqq K_1(\Phi(f)+\|f\|)x^{j-2m\gamma}$$

or

$$f^{(2m-j)}(x) \leqq K_1(\Phi(f)+\|f\|)x^{j-2m\gamma}+M_1(M\Phi(f)+\|f\|) \leqq$$
$$\leqq K_2(\Phi(f)+\|f\|)x^{j-2m\gamma}$$

which concludes the proof for $2m\gamma-j>0$. For j satisfying $2m\gamma-j=0$ we obtain the estimate $|f^{(2m-j)}(x)|\leqq K_1(\Phi(f)+\|f\|)\,|\log x|$. Since $\int_x^{1/2} |\log u|\,du \leqq M$ and $\int_x^{1/2} \frac{du}{u^\alpha} \leqq M$ for $\alpha<1$, we obtain the estimate $|f^{(2m-j_0)}(x)|\leqq K_2(\Phi(f)+\|f\|)$ in $[0, 3/4]$, for the first j satisfying $2m\gamma-2m+j<0$ which we denote by j_0. Therefore, also for $j_0\leqq j<2m$, we have

$$|f^{(2m-j)}(x)| \leqq \left(\|f^{(2m-j_0)}\|^{\frac{j-j_0}{2m-j_0}}\|f\|^{\frac{2m-j}{2m-j_0}}\right) \leqq K_3\left(\Phi(f)+\|f\|_{C\left[0,\frac{3}{4}\right]}\right).$$

We will now define the interpolation space which we will need in this paper. A will be the space of functions with $2m$ continuous derivatives in the interior of I whose i derivative for $i<2m$ is locally absolutely continuous, and for which the seminorm $\Phi_A(f)=\|(\varphi(x))^{2m}f^{(2m)}(x)\|_{C(I)}<\infty$ for some fixed weight function $\varphi(x)$. Recall that the Peetre K functional for the pair of spaces (C, A) is $K(f, t)= =\inf_{f=f_1+f_2}\{\|f_1\|_{C(I)}+t\Phi_A(f_2)\}$ and the interpolation space $(C, A)_\alpha$ (or $(C, A)_{\alpha,\infty}$) is the collection of all functions for which $\sup_t \frac{K(f, t)}{t^\alpha} \leqq M_f$ for some constant M_f.

Lemma 3.2. *Let A_i and A be the spaces for which*

$$\|x^{2m\gamma-2mi}f^{(i)}(x)\|_{C\left[0,\frac{3}{4}\right]}<\infty \quad \textit{and} \quad \|x^{2m\gamma}f^{(2m)}(x)\|_{C\left[0,\frac{3}{4}\right]}<\infty$$

respectively (the derivatives of lower order being absolutely continuous locally), then for $0<\gamma<1$ and $2m\gamma-2m+i>0$ we have $f\in(C[0, 3/4], A)_\beta$ implies $f\in(C[0, 3/4], A_i)_\beta$.

Proof. The Lemma follows immediately from (3.1).

Lemma 3.3. *Let $A(j+1)$ be the space of functions whose derivatives up to $f^{(j)}$ are locally absolutely continuous in $(0, 3/4)$ and for which $\|x^\sigma f^{(j+1)}(x)\|_{C[0,3/4]}= =\Phi_j(f)<\infty$ for some $0<\sigma<j+1$, then $f\in(C, A(j+1))_\beta$ where $C=C[0, 3/4]$*

for some $0<\beta<1$ *implies* $|\Delta_{hx^{\sigma/j+1}}^{j+1} f(x)| \leqq M h^{(j+1)\beta}$ *and in case* $\sigma \leqq 1$ *it also implies* $f \in \mathrm{Lip}^* j\beta$.

Actually for $\sigma<1$ the second part of the Lemma is an immediate corollary of Lemma 3.1 and is not too interesting. The interesting part is when $\sigma=1$. The first part of the Lemma for even $j+1$ was proved in [5] and in fact the proof is very similar.

Proof of Lemma 3.3. For $f\in(C, A(j+1))_\beta$ we have for any τ functions f_1 and f_2 such that $f_1 \in C[0, 3/4]$, $\|f_1\|_{C[0,3/4]} \leqq M\tau^\beta$ and $f_2 \in A(j+1)$ such that $\Phi_j(f_2) \leqq \leqq M\tau^{\beta-1}$ and $f=f_1+f_2$ and M does not depend on τ. For

$$\left[x-\frac{j+1}{2} hx^{\sigma/j+1},\ x+\frac{j+1}{2} hx^{\sigma/j+1}\right] \subset \left[0, \frac{3}{4}\right]$$

we have

$$|\Delta_{hx^{\sigma/j+1}}^{j+1} f(x)| \leqq |\Delta_{hx^{\sigma/j+1}}^{j+1} f_1(x)| + |\Delta_{hx^{\sigma/j+1}}^{j+1} f_2(x)| = I_1+I_2.$$

Choosing $\tau=h^{j+1}$, $I_1 \leqq 2^{j+1} M h^{(j+1)\beta}$. Using Taylor's formula, we have

$$|I_2| \leqq M_1 \max_{0\leqq l\leqq j+1} \left| \int_{x-\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}}^{x} \left(u-x+\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}\right)^j |f_2^{(j+1)}(u)|\,du \equiv \right.$$

$$\equiv M_1 \max_{0\leqq l\leqq j+1} I(l),$$

and we will estimate $I(l)$ for $l<(j+1)/2$ and $l>(j+1)/2$ separately. For $l>(j+1)/2$ we have

$$I(l) \leqq \int_{x}^{x-\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}} \frac{\left|u-\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}\right|^j}{u^\sigma} |u^\sigma f_2^{(j+1)}(u)|\,du \leqq$$

$$\leqq M\Phi_j(f_2) h^{j+1} x^{-\sigma} x^\sigma \leqq M\Phi_j(f_2)\tau.$$

For $l<(j+1)/2$ we have the same estimate provided $x/2>((j+1)/2)hx^{\sigma/j+1}$. If however $x/2<((j+1)/2)hx^{\sigma/j+1}$ then

$$I(l) \leqq \int_{x-\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}}^{x} \frac{\left(u-\left(\frac{j+1}{2}-l\right)hx^{\sigma/j+1}\right)^j}{u^\sigma} |u^\sigma f_2^{(j+1)}(u)|\,du \leqq$$

$$\leqq M\Phi_j(f_2) \int_0^x u^{j-\sigma}\,du \leqq M\Phi_j(f_2) X^{j+1-\sigma} \leqq M\Phi_j(f_2)(hx^{\sigma/j+1})^{j+1} x^{-\sigma} = M\Phi_j(f_2)\tau.$$

This completes the proof that $|\Delta_{hx^{\sigma/j+1}}^{j+1} f(x)| \leqq M h^{(j+1)\beta}$.

We are now ready to prove the second (and probably the more important) contention of our Lemma. Using a well-known result [14, p. 105], we have

$$\bar{\Delta}^j_{2\eta} f(x) - 2^j \bar{\Delta}^j_\eta f(x) = \sum_{\nu=0}^{j-1} \sum_{\mu=\nu+1}^{j} \binom{j}{\mu} \bar{\Delta}^{j+1}_\eta f(x+\nu\eta)$$

where $\bar{\Delta}_\eta f(x)$ are forward differences. We now use the former estimate on $\bar{\Delta}^{j+1}_\eta f(x) \equiv \equiv \bar{\Delta}^{j+1}_\eta f(x+(j+1)\eta/2)$ and obtain

$$|\bar{\Delta}^j_\eta f(x)| \leqq \frac{1}{2^j} |\bar{\Delta}^j_{2\eta} f(x)| + \frac{j}{2} \max_{0 \leqq \nu \leqq j-1} |\bar{\Delta}^{j+1}_\eta f(x+\nu\eta)| \leqq$$

$$\leqq \frac{1}{2^j} |\bar{\Delta}^j_{2\eta} f(x)| + \frac{j}{2} M \left(\frac{\eta}{\left[x + \left(\nu + \dfrac{j+1}{2}\right)\eta\right]^{\sigma/j+1}} \right)^{(j+1)\beta} \leqq$$

$$\leqq \frac{1}{2^j} |\bar{\Delta}^j_{2\eta} f(x)| + \frac{j}{2} M \eta^{(1-\sigma/j+1)(j+1)\beta},$$

and for $\sigma \leqq 1$ we have

$$|\bar{\Delta}^j_\eta f(x)| \leqq \frac{1}{2^j} |\bar{\Delta}^j_{2\eta} f(x)| + \frac{j}{2} M \eta^{j\beta}.$$

Repeating the above process l times,

$$|\bar{\Delta}^j_\eta f(x)| \leqq \frac{1}{2^{jl}} |\bar{\Delta}^j_{2^l\eta} f(x)| + \frac{j}{2} M \sum_{k=0}^{l-1} 2^{-jk} (2^k \eta)^{j\beta} \leqq \frac{1}{2^{jl}} |\bar{\Delta}^j_{2^l\eta} f(x)| + \frac{j}{2} M_1 \eta^{j\beta}.$$

Choosing l such that $1/8 \leqq 2^l \eta \leqq 1/4$ and using the elementary estimate $|\bar{\Delta}^j_{2^l\eta} f(x)| \leqq 2^j \|f\|$, we complete the proof. (Actually we proved the Lemma for $0 < x < 1/4$ but apart from the singularity near zero the Lemma and an even better estimate are well-known).

Lemma 3.4. For $\|x^{2m\beta} f^{(2m)}(x)\|_{C[1/2,\infty)} < \infty$ and $\|f\|_{C[1/2,\infty]} < \infty$ we have

$$\|x^{i\beta} f^{(i)}(x)\|_{C\left[\frac{1}{2},\infty\right]} \leqq C\left(\|x^{2m\beta} f^{(2m)}\|_{C\left[\frac{1}{2},\infty\right]} + \|f\|_{C\left[\frac{1}{2},\infty\right]}\right).$$

Proof. In $[A, A+A^\beta]$ the Lemmas follows [5, p. 311] with $b-a=A^\beta$. Our Lemma follows patching together pieces of this type.

4. Rate of convergence for the intermediate space. In this section we will be interested in the analogue to the direct theorem that shows that for $0<\alpha<2$ $((x(1-x))^{\alpha/2} \Delta^2_h f(x)) \leqq M h^\alpha$ or $|\Delta^2_{h\varphi} f(x)| \leqq M h^\alpha$ where $\varphi(x) = (x(1-x))^{1/2}$ which is equivalent (see [5, p. 312]) we have $\|B_n(f,x) - f(x)\| = O(1/n^{\alpha/2})$. The results here are not corollaries of the results in section 2 and this is best illustrated by the fact that in the particular case Bernstein polynomials after proving the analogue of Theorem 2.1 for positive operators, STRUKOV and TIMAN [13] show with a relatively lengthy computation that $\|B_n(x^\gamma, t)-t^\gamma\|=0(1/n^\gamma)$ for $0<\gamma<1$ which would have followed from $((x(1-x))^\gamma \Delta_h^2(x^\gamma))\leqq Mh^{2\gamma}$ (a result which follows observing that for $x(1-x)\leqq 5h$ the estimate is obvious and for $x(1-x)\geqq 5h$ the mean value theorem yields the estimate).

Definition 4.1 (a) A function $\varphi(x)$ defined on $I=[0, 1]$ satisfies the γ condition for some $0\leqq\gamma\leqq 1$ if $0<Ax^\gamma\leqq\gamma(x)\leqq Bx^\gamma$ for $0<x\leqq 1/2$ and $0<A(1-x)^\gamma\leqq \leqq\varphi(x)\leqq B(1-x)^\gamma$ for $1/2\leqq x<1$. (b) A function on $I=R^+$ or R satisfies the (γ, β) condition for some $0\leqq\gamma\leqq 1$, $0\leqq\beta\leqq 1$ if $0<Ax^\gamma\leqq\varphi(x)\leqq Bx^\gamma$ for $0<|x|\leqq 1/2$ and $0<A|x|^\beta\leqq\varphi(x)\leqq B|x|^\beta$ elsewhere.

Theorem 4.1. *Suppose for a sequence of linear operators on $C(I)$ where $I=[0, 1]$, $I=R^+$ or R given by $L_n(f, x)=\int_I f(t)\,d\alpha_{n,x}(t)$ we have*

a)
$$\int_I t^i\, d\alpha_{n,x}(t) = x^i \quad \text{for} \quad i = 0, 1, \ldots, 2m-1,$$
$$v_{n,x}(t) \equiv \int_{u\leqq t} |d\alpha_{n,x}(u)| \leqq M$$

and

b)
$$\int_I (t-x)^{2m}\, dv_{n,x}(t) \leqq \sigma_n^{2m}(\varphi(x)+\eta_n)^{2m}$$

where $\sigma_n=o(1)$, $\varphi(x)$ satisfies condition γ or condition (γ, β) (Definition 4.1(a) and (b)), $\eta_n=O(\sigma_n^{\gamma/1-\gamma})$ if $0<\gamma<1$ while $\eta_n=0$ for $\gamma=0$ and $\gamma=1$. Then for f $f\in(C, A)_\alpha$ we have $\|f(\cdot)-L_n(f, \cdot)\|_{C(I)}=O(\sigma_n^{2m\alpha})$.

The spaces $(C, A)_\alpha$ were characterized in [5] for $\varphi(x)$ given here. (See also Lemma 3.3).

Remark 4.1 (a) The addition of the term η_n is important for some applications, though it looks at first glance somewhat artificial. Of course the theorem is valid (and easier to prove) with $\eta_n=0$.

(b) One could have different γ near 0 and 1 which we call γ_0 and γ_1 respectively in which case the theorem would still be valid provided that $0\leqq\gamma_i\leqq 1$ and $\int_I (t-x)^{2m}dv_{n,x}(t)\leqq\sigma_n^{2m}(\varphi(x)+\eta_n(i))^{2m}$ with $\eta_n(0)=O(\sigma_n^{\gamma_0/1-\gamma_0})$ for $0\leqq x\leqq 1/2$ and $\eta_n(1)=O(\sigma_n^{\gamma_1/1-\gamma_1})$ for $1/2\leqq x\leqq 1$ if $\gamma_i\neq 0, 1$, and $\eta_n(i)=0$ otherwise. Since the proof will concentrate at the boundary points one at a time, no other change will be required.

(c) The special case of Theorem 4.1 dealing with positive operators was treated by V. Totik who had somewhat different (and more involved) conditions on $\varphi(x)$ [16]. It can be noticed that differentiability, convexity etc. of $\varphi(x)$ are not the issue here. However, it should be noted that Totik treated, the inverse as well as the direct theorem for positive operators.

(d) For $I=R$ generally $\gamma=0$ (in applications).

Proof. For $f\in(C,A)_\alpha$ there exists for each τ f_1 and f_2 such that $f=f_1+f_2$ and $\|f_1\|_{C(I)}+\tau\Phi(f_2)\leqq K\tau^\alpha$ or $\|f_1\|\leqq K\tau^\alpha$ and $\tau\Phi(f_2)\leqq K\tau^\alpha$. Choosing $\tau=\sigma_n^{2m}$, we have $\|L_n(f,x)-f(x)\|\leqq\|L_n(f_1,x)-f_1(x)\|+\|L_n(f_2,x)-f_2(x)\|\leqq(M+1)K\sigma_n^{2m\alpha}+$ $+\|L_n(f_2,x)-f_2(x)\|$.

We will now show that for $f_2\in A$ $\|L_n(f_2,x)-f_2(x)\|\leqq N\sigma_n^{2m}\Phi(f_2)$ which is the crucial step in the proof and which with the estimates above will complete the proof our theorem. For $g(t)\in A$ we write the Taylor expansion

$$g(t)=g(x)+(t-x)g'(x)+\ldots+\frac{(t-x)^{(2m-1)}}{(2m-1)!}g^{(2m-1)}(x)+$$

$$+\frac{1}{(2m-1)!}\int_x^t (u-t)^{2m-1}g^{(2m)}(u)\,du.$$

For $0<\gamma<1$, $B\sigma_n^{1/1-\gamma}\leqq x\leqq 1/2$ and for $I=R^+$ (or R) $x<t$, (while in case $I=[0,1]$ $x<t\leqq 3/4$) we have

$$\left|\int_x^t (u-t)^{2m-1}g^{(2m)}(u)\,du\right|\leqq\frac{c(x-t)^{2m}}{\varphi(x)^{2m}}\Phi(g)\leqq\frac{c_1(x-t)^{2m}}{(\varphi(x)+\eta_n)^{2m}}\Phi(g).$$

For $t<x$, $0<\gamma<1$ and $B\sigma_n^{1/1-\gamma}\leqq x\leqq 1/2$ we have

$$\frac{x}{2^k}\leqq t<\frac{x}{2^{k-1}}\quad\text{and therefore for } k=1,\ \text{or}\quad \frac{x}{2}\leqq t<x,$$

$$\left|\int_t^x (u-t)^{2m-1}g^{(2m)}(u)\,du\right|\leqq C\Phi(g)\int_t^x\frac{(u-t)^{2m-1}}{u^{2m\gamma}}\,du\leqq C\frac{2^{2m\gamma}}{2m}(x-t)^{2m}\Phi(g)\leqq$$

$$\leqq C_1\frac{\Phi(g)}{(\varphi(x)+\eta_n)^{2m}}(x-t)^{2m}.\quad\text{For}\quad k>1,\quad \frac{x}{2^k}\leqq t<\frac{x}{2^{k-1}}\quad\text{we have,}$$

$$\left|\int_t^x (x-t)^{2m-1}g^{(2m)}(u)\,du\right|\leqq C\Phi(g)\left|\int_t^x\frac{(u-t)^{2m-1}}{u^{2m\gamma}}\,du\right|\leqq$$

$$\leqq C\Phi(g)\int_{x/2^k}^x\frac{(u-x/2^k)^{2m-1}}{u^{2m\gamma}}\,du\leqq C\Phi(g)\sum_{l=0}^{k-1}\int_{x/2^{l+1}}^{x/2^l}\frac{|u-x/2^k|^{2m-1}}{u^{2\gamma m}}\,du\leqq$$

$$\leqq C\Phi(g)\sum_{l=0}^{k-1}\frac{\left(x\left|\frac{1}{2^l}-\frac{1}{2^k}\right|\right)^{2m-1}\frac{x}{2^{l+1}}}{(x/2^{l+1})^{2m\gamma}}\leqq C_1\Phi(g)\frac{(x-t)^{2m}}{x^{2m\gamma}}\leqq C_2\Phi(g)\frac{(x-t)^{2m}}{(\varphi(x)+\eta_n)^{2m}}.$$

Therefore, we use the estimate

$$L_n(g(t)-g(x), x) = \frac{1}{(2m-1)!} L_n\Big(\int_x^t (u-t)^{2m-1} g^{(2m)}(u)\,du,\ x\Big)$$

for $I=R^+$ or R and $B\sigma_n^{1/1-\gamma} \leqq x \leqq 1/2$ to get

$$L_n(g(t)-g(x), x) \leqq C\Phi(g)\frac{(\varphi(x)+\eta_n)^{2m}}{(\varphi(x)+\eta_n)^{2m}}\sigma_n^{2m} \leqq C\Phi(g)\sigma_n^{2m}.$$

For $I=[0, 1]$, $B\sigma_n^{1/1-\gamma} \leqq x \leqq 1/2$ we have

$$|L_n(g(t)-g(x), x) \leqq \int_0^{3/4} \Big|\int_x^t (u-t)^{2m-1} g^{(2m)}(u)\Big|\, dv_{n,x}(t) +$$

$$+\sum_{i=0}^{2m-1} \frac{1}{i!} |g^{(i)}(x)| \int_{3/4}^{1} |t-x|^i\, dv_{n,x}(t) \leqq$$

$$\leqq C\Phi(g)\sigma_n^{2m} + C\sum_{i=0}^{2m-1} |g^{(i)}(x)| \int_0^1 |t-x|^{2m}\, dv_{n,x}(t)$$

which, using Lemma 3.1, implies for the x in question $|L_n((g(t)-g(x)), x)| \leqq \leqq C\Phi(g)\sigma_n^{2m}$. For $x < B\sigma_n^{1/1-\gamma}$ we observe that $2m(1-\gamma)$ is either an integer or not. If $2m(1-\gamma)=i$, then $2m\gamma-2m+i=0$ and therefore, using Lemma 3.3, $f\in \mathrm{Lip}^*\,\alpha i$ in $[0, 3/4]$. Therefore, for a given $\tau, f=f_1+f_2$ such that $\|f_1\| \leqq M\tau^\alpha$ and $\|f_2^{(i)}\| \leqq \leqq M\tau^{\alpha-1}$, and we write again $\tau=\sigma_n^{2m}$. We observe now that

$$|L_n(f_2, x)-f_2(x)| \leqq \frac{1}{i!}\|f_2^{(i)}\| \int_I |t-x|^i\, dv_{n,x}(t) \leqq$$

$$\leqq \frac{1}{i!}\|f_2^{(i)}\| \Big\{\int_I (t-x)^{2m}\, dv_{n,x}(t)\Big\}^{i/2m} \Big\{\int_I dv_{n,x}(t)\Big\}^{1-(i/2m)} \leqq K\|f_2^{(i)}\|(\varphi(x)+\eta_n)^i \sigma_n^i \leqq$$

$$\leqq K_1\|f_2^{(i)}\|\sigma_n^{\gamma i/1-\gamma}\sigma_n^i \leqq K_1\|f_2^{(i)}\|\sigma_n^{2m} \leqq K_2\sigma_n^{2m\alpha}.$$

For $2m(1-\gamma)$ not an integer, choose i such that $0<2m\gamma-2m+i<1$ and, using Lemma 3.2, $f\in(C, A)_\alpha$ implies $f\in(C, A_i)_\alpha$ where $A_i=\{f, \|x^{2m\gamma-2+i}f^{(i)}(x)\|_{C[0,3/4]}<\infty\}$.

We write $f=f_1+f_2$ where $\|f_1\| \leqq M\tau^\alpha$ and $\Phi_i(f_2) \leqq M\tau^{\alpha-1}$ and set $\tau=\sigma_n^{2m}$. Now

$$|L_n(f_2, x)-f_2(x)| \leqq \frac{1}{(i-1)!}\Phi_i(f_2)\int_I \Big|\int_t^x \frac{(u-t)^{i-1}}{u^{2m\gamma-2m+i}}\,du\Big|\, dv_{n,x}(t).$$

For $x<B\sigma_n^{1/1-\gamma}$ we have for $t>x$

$$\Big|\int_x^t \frac{(u-t)^{i-1}}{u^{2m\gamma-2m+i}}\,du\Big| \leqq C|t-x|^{i-1}\int_0^t u^{2m-2m\gamma-i}\,du \leqq C_1|t-x|^{i-1}t^{2m-2m\gamma-i+1} \leqq$$

$$\leqq C_1|t-x|^{2m-2m\gamma}$$

and for $t<x$

$$\left|\int_t^x \frac{(u-t)^{i-1}}{u^{2m\gamma-2m+i}}\,du\right| \leqq \left|\int_t^x \frac{|u-t|^{i-1}}{|u-t|^{2m\gamma-2m+i}}\,du\right| \leqq C|x-t|^{2m-2m\gamma}.$$

But, using Hölder's inequality,

$$J=\int_I |x-t|^{2m-2m\gamma}\,dv_{n,x}(t) \leqq \Big\{\int_I |x-t|^{2m}\,dv_{n,x}(t)\Big\}^{1-\gamma}\Big\{\int_I dv_{n,x}(t)\Big\}^{\gamma} \leqq$$
$$\leqq C\big[\sigma_n\big(\varphi(x)+\eta_n\big)\big]^{2m-2m\gamma}$$

which for $x\leqq B\sigma_n^{1/1-\gamma}$ implies $J\leqq C_1\sigma_n^{2m-2m\gamma}(\sigma_n^{\gamma/1-\gamma})^{2m-2m\gamma}=C_1\sigma_n^{2m}$. With the above choice of f_2 and τ, we have our estimate for $0<x\leqq 1/2$ and $0<\gamma<1$. For $\gamma=0$ the estimate is actually trivial. For $\gamma=1$ we write

$$\left|\int_t^x (u-t)^{2m-1}g^{(2m)}(u)\,du\right| \leqq \Phi(g)\frac{4^m}{x^{2m}}\frac{1}{2m}(u-t)^{2m}$$

for $t\geqq x/2$ and therefore

$$|L_n(g,x)-g(x)| \leqq \left|\int_0^{x/2} g(t)\,d\alpha_{n,x}(t)-g(x)\int_0^{x/2} d\alpha_{n,x}(t)\right|+$$
$$+\left|\frac{1}{(2m-1)!}\int_{I\cap\{t>(x/2)\}}\Big\{\int_t^x (u-t)^{2m-1}g^{(2m)}(u)\,du\right| \leqq 2\|g\|_C\int_0^{x/2} dv_{m,x}(t)+$$
$$+\Phi(g)\frac{4^m}{x^{2m}}\frac{1}{(2m)!}\int_{I\cap\{t>(x/2)\}}(u-t)^{2m}\,dv_{n,x}(t) \leqq$$
$$\leqq \left(\|g\|_C 2\Big(\frac{2}{x}\Big)^{2m}\int_I (u-t)^{2m}\,dv_{n,x}(t)\right)+$$
$$+\Phi(g)\frac{4^m}{x^{2m}}\frac{1}{2m!}\int_I (t-x)^{2m}\,dv_{n,x}(t)\Big\} \leqq C(\|g\|_C+\Phi(g))\sigma_n^{2m}.$$

One can note that $\|g\|_C\leqq\|f\|_C+1$ and therefore the estimate follows). For $I=[0,1]$ near $x=1$ the estimate is similar to the above. We now have to estimate the rate for x bounded away from 0 for R^+ or R. For $t>x\geqq 1/2$ (in R^+ say)

$$\left|\int_x^t (u-t)^{2m-1}g^{(2m)}(u)\,du\right| \leqq \frac{C}{\varphi(x)^{2m}}(x-t)^{2m}\Phi(g).$$

Otherwise we distinguish the two cases $x-x^\beta/4<t\leqq x$, $x\geqq 1/2$ and $t<x-x^\beta/4$, $x\geqq 1/2$. In the first case we have $\left|\int_x^t |u-t|^{2m}g^{(2m)}(u)\,du\right|\leqq\dfrac{C}{\varphi(x)^{2m}}(x-t)^{2m}\Phi(g)$

and in the second case we just write $g(t)$. Consequently

$$|L_n(g, x)-g(x)| \leqq \frac{C}{\varphi(x)^{2m}} \int (x-t)^{2m} \Phi(g)\, dv_{n,x}(t)+$$

$$+C \sum_{i=0}^{2m-1} |g^{(i)}(x)| \int_{t \leqq x-(1/4)} |t-x|^i\, dv_{n,x}(t) \leqq \frac{C}{\varphi(x)^{2m}} \varphi(x)^{2m} \sigma_n^{2m} \Phi(g)+$$

$$+C_1 \sum_{i=0}^{2m-1} |g^{(i)}(x)| (x^\beta)^{-2m+i} \int_I |t-x|^{2m}\, dv_{n,x}(t) \leqq$$

$$\leqq C\sigma_n^{2m} \Phi(g)+C_2 \sum_{i=0}^{2m-1} |g^{(i)}(x)| x^{\beta i} \sigma_n^{2m},$$

and using Lemma 3.4, we complete the proof of our theorem.

5. Rate of convergence, continued. In this section we will deal with the situation in which moments of lower order are different from 0. We denote

$$\int_I (t-x)^i\, d\alpha_{n,x}(t) = R_{n,i}(x). \tag{5.1}$$

A different result for approximation operators for which $R_{n,i}(x) \neq 0$ for some of the i's was given in theorem 2.2. We will first itemize what conditions the functions $R_{n,i}(x)$ have to satisfy and while these conditions are not very simple to state, they are relatively simple to verify in applications.

Definition 5.1. For $I=[0, 1]$, $0<\gamma<1$, $R_{n,i}(x)$ satisfies the $(\gamma, 2m, i, \sigma_n)$ condition if

$$|R_{n,i}(x)| \leqq M\sigma_n^{2m} \min\{\max(x(1-x), \sigma_n^{1/1-\gamma})^{2m\gamma-2m+i}, 1\};$$

for $\gamma=0$ $|R_{n,i}(x)| \leqq M\sigma_n^{2m}$ and for $\gamma=1$ $|R_{n,i}(x)| \leqq M\sigma_n^{2m}(x(1-x))^i$.

Definition 5.2. For $I=R^+$ (or R) $R_{n,i}(x)$ satisfies the $(\gamma, \beta, 2m, i, \sigma_n)$ condition if for $|x| \leqq 1/2$ it satisfies the condition in Definition 5.1 (x may replace $x(1-x)$ but that would not change the situation) and for other x, $|R_{n,i}(x)| \leqq$ $\leqq M\sigma_n^{2m} |x|^{\beta i}$.

We are now ready to state and prove our theorem about rate of convergence.

Theorem 5.1. *Suppose a sequence of linear operators on* $C(I)$, $L_n(f, x)=$ $= \int_I f(t)\, d\alpha_{n,x}(t)$ *satisfy the conditions of Theorem* 4.1 *except that* $R_{n,i}(x)$ *are not necessarily* 0 *but satisfy the conditions in definition* 5.1 *and* 5.2 *with the same* γ $0 \leqq \gamma \leqq 1$ and β $0 \leqq \beta \leqq 1$ *given in theorem* 4.1, *then for* $f \in (C, A)_\alpha$, $0<\alpha<1$, $\|L_n(f, k)-f(\cdot)\|=$ $=0(\sigma_n^{2m\alpha})$, *where* $A=\{f;\ \Phi(f)=\|\varphi(x)^{2m} f^{(2m)}(x)\|_{C(I)}<\infty$ *and* f *has* $2m-1$ *absolutely continuous derivatives locally in the interior of* $I\}$.

Proof. The process that we use is the same as that of Theorem 2.2 and we construct a new operator $A_n(f,x)=\int f(t)\,d\beta_{n,x}(t)$. In order to complete the proof we have to show two things: (a) that the behaviour of $\int_I (t-x)^{2m}dV_{n,x}(t)$, where $V_{n,x}(t)$ is the variation of $\beta_{n,x}(t)$, is the same as $\int (t-x)^{2m}dv_{n,x}(t)$ (required in Theorem 4.1) where $v_{n,x}(t)$ is the variation of $\alpha_{n,x}(t)$; (b) that for $f\in(C,A)_\alpha$ the operators we added contribute at most $M\sigma_n^{2m\alpha}$.

To prove (a) let us recall that we have introduced the operators $C\bar{\Delta}^j_{|R_{n,i}(x)|^{1/i}}$ $i\leqq j<2m$ for $x<1/2$ or in general in case I is not $[0,1]$ and $C\bar{\Delta}^j_{-|R_{n,i}(x)|^{1/i}}$ for $x>1/2$ and $I=[0,1]$. Each term of this kind will add to the variaton of $\alpha_{n,x}(t)$, that is to $v_{n,x}(t)$, to produce eventually (after the process is completed) the operator $A_n(f,x)=\int_I f(t)\,d\beta_{n,x}(t)$ where we denote the variation of $\beta_{n,x}(t)$ by $V_{n,x}(t)$. The amount added to $\int_I (t-x)^{2m}dv_{n,x}(x)$ to get $\int_I (t-x)^{2m}dV_{n,x}(t)$ is for each i a constant times $|R_{n,i}(x)|^{2m/i}$. We will now show that these additions with $R_{n,i}(x)$ restricted as in the conditions of our theorem will leave us with a new operator $\int_I f(t)\,d\beta_{n,x}(t)$ that satisfies the restriction in Theorem 4.1.

For $\gamma=0$ or for $0<\gamma<1$ and $2m\gamma-2m+i\leqq 0$ we have

$$|R_{n,i}(x)|^{2m/i}\leqq (M\sigma_n^{2m})^{2m/i}\leqq M\sigma_n^{(2m)^2/i}=M\sigma_n^{2m}\sigma_n^{2m((2m/i)-1)}\leqq M\sigma_n^{2m}(\sigma_n^{\gamma/1-\gamma})^{2m}$$

since $\gamma/(1-\gamma)\leqq 2m/i-1=(2m-i)/i$ which follows $2m-i\geqq 2m\gamma$ and $i\leqq 2m(1-\gamma)$. For $0<\gamma<1$ and $2m\gamma-2m+i>0$ we have to distinguish two possibilities: (I) $x(1-x)\leqq A\sigma_n^{1/1-\gamma}$ for $I=[0,1]$ (and $x\leqq A\sigma_n^{1/1-\gamma}$ for $I=R^+$ or $I=R$); (II) $x(1-x)\geqq A\sigma_n^{1/1-\gamma}$ for $I=[0,1]$ (and $x\geqq A\sigma_n^{1/1-\gamma}$ for $I=R^+$ or $I=R$). For the situation (I) we have

$$|R_{n,i}(x)|^{2m/i}\leqq M\big(\sigma_n^{2m/i}(\sigma_n^{1/1-\gamma})^{(2m\gamma-2m+i)/i}\big]^{2m}=M\sigma_n^{2m/1-\gamma}=M\sigma_n^{2m}\sigma_n^{2m(\gamma/1-\gamma)}.$$

For the estimate in case II we will be concerned with the case $0<x<1/2$ (the other case being similar) and obtain

$$|R_{n,i}(x)|^{2m/i}\leqq M\sigma_n^{(2m)^2/i}x^{2m(2m\gamma-2m+i)/i}=M\sigma_n^{2m}[\sigma_n^{2m(2m-i)/i}x^{2m(2m\gamma-2m+i)/i}]\leqq$$
$$\leqq M_1\sigma_n^{2m}[x^{(1-\gamma)2m(2m-i)/i}x^{2m(2m(\gamma-1)+i)/i}]=M_1\sigma_n^{2m}x^{2m\gamma}.$$

For $\gamma=1$ we have near $x=0$ $|R_{n,i}(x)|^{2m/i}\leqq M(\sigma_n^{i/i}|x|^{i/i})^{2m}\leqq M\sigma_n^{2m}x^{2m}$ (and for x near 1 in case $I=[0,1]$ the same type of estimate follows too). We are left with the estimate for other x but there $|R_{n,i}(x)|^{2m/i}\leqq M\sigma_n^{(2m)^2/i}|x|^{2m\beta i/i}\leqq M\sigma_n^{2m}|x|^{2m\beta}$.

We will now prove (b), that is, we will show that near 0 $f\in(C,A)_\alpha$ implies for $i\leqq j<2m$ $|\bar{\Delta}^j_{|R_{m,i}(x)|^{1/i}}f(x)|\leqq M\sigma_n^{2m\alpha}$. It is a similar situation near $x=1$ in case $I=[0,1]$ and for other x it is substantially simpler. It is enough to prove the above contention for $j=i$. First we see that for $f\in(C,A)_\alpha$ we have $f(x)=f_1(x)+f_2(x)$

where $\|f_1\|_{C(I)} \leqq K\sigma_\alpha^{2m\alpha}$ and $\Phi(f_2) \leqq K\sigma_n^{2m(\alpha-1)}$ where K does not depend on n. We can now write

$$|\bar{\Delta}^i_{|R_{n,i}(x)|^{1/i}} f(x)| \leqq |\bar{\Delta}^i_{|R_{m,i}(x)|^{1/i}} f_1(x)| + |\bar{\Delta}^i_{|R_{m,i}(x)|^{1/i}} f_2(x)| \equiv I_1 + I_2.$$

Obviously, $I_1 \leqq 2^i K\sigma_n^{2m\alpha}$ and $\|f_2\|_C \leqq \|f\|_C + 1$. For $2m\gamma - 2m + i < 0$ Lemma 3.1, yields $\|f_2^{(i)}\| \leqq M(\Phi(f_2) + \|f_2\|_C)$ and $|R_{n,i}(x)|^{1/i} \leqq M\sigma_n^{2m/i}$ or

$$|\bar{\Delta}^i_{|R_{n,i}(x)|^{1/i}} f_2(x)| \leqq |R_{n,i}(x)| \, \|f_2^{(i)}(x)\| \leqq M'\sigma_n^{2m}\sigma_n^{2m(\alpha-1)} = M'\sigma_n^{2m\alpha}.$$

For $2m\gamma - 2m + i > 0$ we estimate first for $x \geqq A\sigma_n^{1/1-\gamma}$ or for $\gamma = 1$, and $x \leqq 1/2$ and write using Lemma 3.1

$$|I_2| = |R_{n,i}(x)|\, |f_2^{(i)}(\xi)| \leqq M\sigma_n^{2m} x^{2m\gamma-2m+i} |f_2^{(i)}(\xi)| \leqq M\sigma_n^{2m}\big(\xi^{2m\gamma-2m+i}|f_2^{(i)}(\xi)|\big) \leqq$$
$$\leqq M_1\sigma_n^{2m}\big(\Phi(f_2) + \|f_2\|\big) \leqq M_2\sigma_n^{2m}\sigma_n^{2m(\alpha-1)} = M_2\sigma_n^{2m\alpha}.$$

For $x \leqq A\sigma_n^{1/1-\gamma}$ we observe that $|R_{n,i}(x)| \leqq M\sigma_n^{2m}(\sigma_n^{1/1-\gamma})^{2m\gamma-2m+i} = M\sigma_n^{i/1-\gamma}$ or $|R_{n,i}(x)|^{1/i} \leqq M_1\sigma_n^{1/1-\gamma}$. Writing $\theta = |R_{n,i}(x)|^{1/i}$, we have $|\bar{\Delta}^i_\theta f_2(x)| \leqq \Delta^i_\theta f_2(x + (i\theta/2))|$ and we can use the Taylor formula with integral remainder to expand around $x + (i\theta)/2$ and obtain

$$|\bar{\Delta}^i_\theta f_2(x)| \leqq M \max_{0 \leqq l \leqq i} \Big| \int_{x+l\theta}^{x+(i/2)\theta} (u - x - l\theta)^{i-1} \Big| f_2^{(i)}(u)\, |du| = M \max_{0 \leqq l \leqq i} J(l).$$

For $l > i/2$ we have

$$J(l) \leqq \int_{x+(i/2)\theta}^{x+l\theta} (x + l\theta - u)^{i-1} |f_2^{(i)}(u)|\, du \leqq$$

$$\leqq \int_{x+(i/2)\theta}^{x+l\theta} \frac{(x + l\theta - u)^{i-1}}{\left(x + \dfrac{i}{2}\theta\right)^{2m\gamma-2m+i}} |u^{2m\gamma-2m+i} f_2^{(i)}(u)|\, du \leqq$$

$$\leqq M\frac{\Phi(f_2) + \|f_2\|}{\theta^{2m\gamma-2m+i}}\theta^i \leqq M'\big(\Phi(f_2) + \|f_2\|\big)\sigma_n^{2m} \leqq M''\sigma_n^{2m\alpha}.$$

For $l < i/2$ we have

$$J(l) \leqq M \int_{x+l\theta}^{x+(i/2)\theta} \frac{(u - x - l\theta)^{i-1}}{u^{2m\gamma-2m+i}} |u^{2m\gamma-2m+i} f_2^{(i)}(u)|\, du \leqq$$

$$\leqq M_1\Big(\int_0^{x+(i/2)\theta} u^{2m-2m\gamma+i}\, du\Big)\big(\Phi(f_2) + \|f_2\|\big)$$

and since $2m - 2m\gamma > 0$, as we already treated $\gamma = 1$, we have

$$J(l) \leqq M_2\theta^{2m-2m\gamma}\sigma_n^{2m(\alpha-1)} \leqq M_3\sigma_n^{2m}\sigma_n^{2m(\alpha-1)} = M_3\sigma_n^{2m\alpha}.$$

We now turn our attention to the case $2m - 2m\gamma + i = 0$ by first observing that $f \in (C, A)_\alpha$ implies quite easily $f \in (C, A_{i+1})_\alpha$ where $A_{i+1} = \{f;\, f, \dots, f^{(i)}$ are locally absolutely continuous in $(0, 3/4)$ and $\|xf^{(i+1)}\|<\infty\}$. Using Lemma 3.3 with $i=j$, $\sigma=1$ and $\beta=\alpha$, we have

$$|\bar{\Delta}^i_{|R_{n,i}(x)|^{1/i}} f(x)| \leqq M|R_{n,i}(x) \leqq M_1 \sigma_n^{2m\alpha}.$$

One need now only observe that near $x=1$ (in case $I=[0, 1]$) the proof is similar and for other x we actually just use $|\bar{\Delta}^i_\eta f(x)| \leqq \eta^i f^{(i)}(\xi)$ and obtain our result.

6. Application, some positive operators. **(a)** The Kantorovich operator given by

$$K_n(f, t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} (n+1) \int_{k/n+1}^{k+1/n+1} f(u)\, du \tag{6.1}$$

or by $K_n(f, t)=(d/dt)B_{n+1}(F, t)$ where $F(u)=\int_0^u f(v)dv$ and $B_n(f, t)$ are then Bernstein polynomials. It is known that $K_n(1, t)=1$, $K_n(\cdot-t, t)=\dfrac{1-2t}{2(n+1)}$ and $K_n((\cdot-t)^2, t)=\dfrac{t(1-t)}{n}+0(n^{-2})$. Using Theorem 2.2. with

$$R_n(t) = \frac{t(1-t)}{n} + \frac{1}{4}\left(\frac{1-2t}{n}\right)^2 + O\left(\frac{1}{n^2}\right) = \frac{t(1-t)}{n} + O\left(\frac{1}{n^2}\right)$$

and $R_{n,1}(t)=\dfrac{1-2t}{2(n+1)}$, we have:

Theorem 6.1. *For $f \in C[0, 1]$ and $K_n(f, t)$ defined by* (6.1), *we have*

$$|K_n(f, t)-f(t)| \leqq M\omega_2\left(f, \left(\frac{t(1-t)}{n}+\frac{L}{n^2}\right)^{1/2}\right) + \omega_1\left(f, \frac{|1-2t|}{n}\right) \tag{6.2}$$

(and the theorems 2.1 *and* 2.2 *can yield a reasonable estimate on M while L can be estimated by* 1*).*

Using Theorems 4.1 and 5.1, we obtain with $\gamma=1/2$, $\sigma_n=1/\sqrt{n}$, $\eta_n=1/\sqrt{n}$ and $m=2$ the following result.

Theorem 6.2. For $f \in (C[0, 1], A)_\alpha$, $0<\alpha<1$, where $A=\{f; t(1-t)f''(t) \in \in C[0, 1]$ and f, f' are locally absolutely continuous in the interior of $(0, 1\}$, then $\|K_n(f, \cdot)-f(\cdot)\|_{C[0,1]}=0(1/n^\alpha)$.

Remarks. (I) We cannot omit the second term in Theorem 6.1 as is obvious when we observe the effect of the function x. (II) In [15, p. 54], in an added in proof remark, V. Totik indicated that the analogous result (to Theorem 6.2 is valid for L_p, $1<p<\infty$.

(b) The integral version of the Szász and Baskakov operators are given by

$$S_n^*(f,x)=\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}n\int_{k/n}^{k+1/n} f(u)\,du \tag{6.2}$$

and

$$V_n^*(f,x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k(1+x)^{-n-k}n\int_{k/n}^{k+1/n} f(u)\,du \tag{6.3}$$

which can also be given by

$$S_n^*(f,x)=\frac{d}{dx}S_n(F,x)\quad V_n^*(f,x)=\frac{d}{dx}V_{n-1}(F,x)\quad\text{and}\quad F(u)=\int_0^u f(u)\,dv \tag{6.4}$$

where $S_n(f,x)$ and $V_n(f,x)$ are the Szász and Baskakov operators given respectively by

$$S_n(f,x)=\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right) \tag{6.5}$$

and

$$V_n(f,x)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k(1+x)^{-n-k}f\left(\frac{k}{n}\right). \tag{6.6}$$

Theorem 6.3. *For* $f\in C(R^+)$

$$|S_n^*(f,x)-f(x)|\leqq M\omega_2\left(f,\sqrt{\frac{x}{n}+\frac{1}{n^2}}\right)+\omega_1\left(f,\frac{1}{2n}\right) \tag{6.7}$$

and

$$|V_n^*(f,x)-f(x)|\leqq M\omega_2\left(f,2\sqrt{\frac{x(1+x)}{n}+\frac{1}{n^2}}\right)+\omega_1\left(f,\frac{x}{n}+\frac{1}{2n}\right). \tag{6.8}$$

Proof. One can calculate $D_n(x)$ $R_{n,1}(x)$ and $R_n(x)$ as $D_n(x)=x/n+1/3n^2$, $R_{n,1}(x)=1/2n$ and $R_n(x)=x/n+(7/12)(1/n^2)$ for the Szász operator.

We will calculate in detail for the integral version $V_n^*(f,x)$ of the Baskakov operator $D_n(x)$, $R_n(x)$ and $R_{n,1}(x)$,

$$D_n(x)=V_n^*\left((t-x)^2,x\right)=$$

$$=\sum_{k=0}^{\infty}\binom{n-k-1}{k}x^kx^{-n-k}\left[\frac{n}{3}\left(\frac{k+1}{n}\right)^3-\frac{n}{3}\left(\frac{k}{n}\right)^3-2x\left(\frac{n}{2}\left(\frac{k+1}{n}\right)^2-\frac{n}{2}\left(\frac{k}{n}\right)^2\right)+x^2\right]=$$

$$=V_n(t^2,x)+\frac{1}{n}V_n(t,x)+\frac{1}{3n^2}V_n(1,x)-2xV_n(t,x)-\frac{x}{n}V_n(1,x)+x^2V_n(1,x)=$$

$$=\frac{x(1+x)}{n}+\frac{1}{3n^2},$$

$R_{n,1}(x) = V_n^*((t-x), x) = x/n + 1/(2n)$ and therefore $R_n(x) \leqq D_n(x) + R_{n,1}(x)^2 \leqq \leqq 2x(1+x)/n + 1/n^2$. Substituting the above in Theorem 2.1, we have Theorem 6.3.

Remarks. One cannot omit the second term from formulae (6.7) and (6.8) as the result would fail then for the function x. The corresponding results for the operators $S_n(f, x)$ and $V_n(f, x)$ would already follow the theorem of Strukov and Timan [13] and therefore are not stated here. Similarly, one can prove the following corollary of Theorems 4.1 and 5.1.

Theorem 6.4. *For* $f \in (C(R^+), A)_\alpha$ *where* $A = \{f;\ x \cdot f \in C(R^+)\}$

$$\|S_n^*(f, x) - f(x)\|_{C(R^+)} \leqq M \frac{1}{n^\alpha} \tag{6.9}$$

and

$$\|S_n(f, x) - f(x)\|_{C(R^+)} \leqq M \frac{1}{n^\alpha}. \tag{6.10}$$

Theorem 6.5. *For* $f \in (C(R^+), A)_\alpha$ *where* $A = \{f;\ x(1+x)f'' \in C(R^+)\}$

$$\|V_n^*(f, x) - f(x)\|_{C(R^+)} \leqq M \frac{1}{n^\alpha} \tag{6.11}$$

and

$$\|V_n(f, x) - f(x)\|_{C(R^+)} \leqq M \frac{1}{n^\alpha}. \tag{6.12}$$

Proof. We simply adjust the moments already calculated to the moments and functions in Theorems 4.1 and 5.1. We observe that $\gamma = 1/2$, $\sigma_n = 1/\sqrt{n}$ and $\eta_n = 1/\sqrt{n}$ in both Theorems, but $\beta = 1/2$ in Theorem 6.4 and $\beta = 1$ in Theorem 6.5.

Theorems 6.4 and 6.5 could be adjusted to exponential behaviour as x tends to infinity following the treatment in [3] for instance but it is the goal here to get corollaries of the general theorems preceding this section rather than deal with particular behaviour.

One should note that in Theorems 6.2 and 6.4 and 6.5 we have $\eta_n \neq 0$ and while it looked redundant to allow such η_n in the beginning, from the point of view of the applications it would appear quite important.

(c) The Post-Widder Laplace transform inversion formula.

The Post-Widder Laplace transform inversion formula is in face an approximation operator given by [17, Ch. 7]

$$P_n(f, t) = \frac{1}{n!} \left(\frac{n}{t}\right)^{n+1} \int_0^\infty e^{-nu/t} u^n f(u)\, du. \tag{6.13}$$

It is an inversion of the Laplace transform given by

$$(6.14)\quad P_n(f,t) = (-1)^n F^{(n)}\left(\frac{n}{t}\right)\left(\frac{n}{t}\right)^{n+1}\frac{1}{n!} \quad \text{where} \quad F(u) = \int_0^\infty e^{-ut} f(t)\,dt.$$

The difference between this and earlier examples is that here γ (that corresponds to Theorem 4.1) is equal to 1 rather than 1/2 in (a) and (b). Since $P_n(1,t)=1$, $P_n((\cdot-t),t)=0$ and $P_n((\cdot-t)^2,t)=t^2/n$, we have:

Theorem 6.6. *For $P_n(f,t)$ defined by* (5.13) *on* $C(R^+)$

$$(6.15)\qquad |P_n(f,t)-f(t)| \leqq 15\omega_2\left(f, \frac{t}{\sqrt{n}}\right)$$

and for $f\in(C(R^+),A)_\alpha$ *where* $A=\{f;\ t^2f''(t)\in C(R^+)\}$

$$(6.16)\qquad \|P_n(f,t)-f(t)\|_{C(R^+)} \leqq M\frac{1}{n^\alpha}.$$

Again one can modify the result for exponential growth.

(d) the Meier—König and Zeller operator given by

$$(6.17)\qquad M_n(f,t) = (1-t)^n \sum_{k=0}^{k}\binom{k+n}{k} t^k f\left(\frac{k}{n+k}\right)$$

can also be treated using theorem 4.1 and it can be shown that:

Theorem 6.7. *For* $f\in(C[0,1],A)_\alpha$, $0<\alpha<1$ *where*

$$A = \{f;\ \|x(1-x)^2f''(x)\|_{C[0,1]}<\infty \quad f, f'\in AC_{\mathrm{loc}}(0,1)\},$$

we have

$$\|M_n(f,t)-f(t)\| \leqq M\frac{1}{n^{\alpha/2}}.$$

Proof. This immediately follows the calculation of the moments.

The interesting part about this operator is that the γ's near zero and near one are different (1/2 and 1 respectively), a possibility mentioned in Remark 4.1(b).

For a similar operator

$$M_n^+(f,t) = (1-t)^n \sum_{k=0}^{\infty}\binom{n+k-1}{k} t^k f\left(\frac{k}{k+n}\right),$$

we have $M_n^+(f,t)=V_n(f_1, t(1-t))$ where $f_1(u)=f(u/(1+u))$ and V_n is the Baskakov operator given in (6.6). It is not easy to translate the behaviour of V_n to that of M_n^+ or M_n and it better done directly.

7. Non-positive approximation processes, combinations of Bernstein polynomials. In sections 7 and 8 approximation processes that are not positive but that converge faster depending on higher degrees of smoothness, will be discussed. In particular in section 8 we apply our theorems to combinations of „Exponential-type" operators introduced by C. P. May in [9] and [10]. In [2] the author proved a global direct and inverse theorem for combinations of Bernstein polynomials. We will see first how the direct part of [2] follows from the general theorems of this paper. Actually the results in Theorems 4.1 and 5.1 were motivated by the result on Bernstein polynomials and it seems interesting how those general theorems apply.

Combinations of Bernstein polynomials that would yield faster rates of convergence are given by

$$(7.1)\qquad (2^r-1)B_n(f,r,x) \equiv 2^r B_{2n}(f,r-1,x)-B_n(f,r-1,x)$$

and $B_n(f,0,x)\equiv B_n(f,x)$. Other combinations are possible (see [2, p. 278]) but these seem to be the simplest form with the given rate of convergence. To establish results as corollaries of the theorems of this paper we have to compute moments of $B_n(f,r-1,x)$. (We choose $B_n(f,r-1,x)$ with $r-1$ use the same notation used in [2]). First we observe that

$$(7.2)\qquad B_n(f,r-1,x) = \sum_{j=0}^{r-1} C_j B_{2^j n}(f,x)$$

and C_i are constants independent of n which among other properties satisfy

$$(7.3)\qquad \sum_{i=0}^{r-1} C_j = 1 \quad\text{and}\quad \sum_{l=0}^{r-1} C_j n^{-l} = 0 \quad\text{for}\quad l=1,\dots,r-1.$$

We set $2m=2r$ and calculate $D_n(x)$. Using (6.2) and [2, (4.2) p. 285], we have

$$\begin{aligned} D_n(x) &\leqq \sum_{i=0}^{r-1} |C_i| B_{2^i n}((t-x)^{2r},x) \leqq \Big(\sum_{i=1}^{r-1}|C_i|\Big)\max_i \big|B_{2^i n}((t-x)^{2r},x)\big| \leqq \\ &\leqq \Big(\sum_{i=0}^{r-1}|C_i|\Big)B_n((t-x)^{2r};\ x) = \\ &= \Big(\sum_{i=1}^{r-1}|C_i|\Big)\left(\frac{1}{n^r}\left(A_1(x(1-x))^r+A_2(x)\frac{(x(1-x))^{r-1}}{n}+\dots+A_r(x)\frac{x(1-x)}{n^{r-1}}\right)\right) \leqq \\ &\leqq Kn^{-r}\left[(x(1-x))+\frac{1}{n}\right]^r \leqq K_1 n^{-r}\left[\sqrt{x(1-x)}+n^{-1/2}\right]^{2r}. \end{aligned}$$

To calculate $R_{n,i}(x)$ we use formulae (4.2) and (4.3) of [2, p. 285] together with formula (7.3) here to obtain first $R_{n,i}(x)=0$ for $i=1,\dots,r$ and then for $i\geqq r+1$ we have

$$R_{n,i}(x) = \frac{1}{n^r}\sum_{j=1}^{i-r}\frac{(x(1-x))^j}{n^{i-r-j}}B_{j,i}(x)$$

or for $i \geqq r+1$,

$$|R_{n,i}(x)| \leqq B\frac{1}{n^r}\{(x(1-x))+n^{-1}\}^{i-r} \leqq B_*\frac{1}{n^r}[\max(x(1-x), n^{-1})]^{r-2r+i}.$$

We can estimate $R_n(x)$ by $R_n(x) \leqq K_2 n^{-r}[\sqrt{x(1-x)}+n^{-1/2}]^{2r}$ and this implies the following theorem.

Theorem 7.1. *For* $B_n(f, r-1, x)$ *defined by* (6.1) *and* $f(x) \in C[0, 1]$ *we have*

$$|B_n(f, r-1, x)-f(x)| \leqq K\left\{\omega_{2r}\left(f, \left(\frac{x(1-x)}{n}+n^{-2}\right)^{1/2}\right)+\right. \tag{7.4}$$

$$\left.+\sum_{j=r+1}^{2r-1}\omega_j\left(f, n^{-r/j}\left[x(1-x)+\frac{1}{n}\right]^{(j-r)/j}\right)\right\}.$$

This result is new and was not proved in [2]. In particular for $B_n(f, 1, x) \equiv \equiv 2B_{2n}(f, x)-B_n(f, x)$ we have

$$|2B_{2n}(f, x)-B_n(f, x)-f(x)| \leqq K\left\{\omega_4\left(f, \left(\frac{x(1-x)}{n}+n^{-2}\right)^{1/2}\right)+\right. \tag{7.5}$$

$$\left.+\omega_3\left(f, n^{-2/3}\left[x(1-x)+\frac{1}{n}\right]^{1/3}\right)\right\}.$$

We recall that for x^3 $\omega_3(f, h) \sim Kh^3$ which will fit exactly here in view of the fact that, as we observed in [2, p. 279],

$$|2B_n(f, x)-B_n(x)-f(x)| \leqq M\left(\frac{x(1-x)}{n}\right)^{\alpha/2}$$

is not equivalent to $f \in \mathrm{Lip}^* \alpha$. As a corollary of Theorem 5.1 we have:

Theorem 7.2. *For* $f \in (C, A_{2r})_\alpha$ *where*

$$A_{2r} = \{f;\ f, \dots, f^{2r-1} \in A.C._{\mathrm{loc}}(0, 1) \quad \text{and} \quad \|(x(1-x))^r f^{(2r)}(x)\| < \infty\}$$

and for $B_n(f, r-1, x)$ *given by* (6.1), *we have*

$$\|B_n(f, r-1, x)-f(x)\|_{C[0,1]} \leqq M\frac{1}{n^{r\alpha}}. \tag{6.6}$$

This theorem is the direct theorem proved in [2, p. 284].

8. Combinations of exponential-type operators. Exponential-type operators were defined first by C. P. May in [9] and [10] by

$$S_\lambda(f, x) = \int_A^B W(\lambda, t, u)f(u)\,du \tag{8.1}$$

where A and B may be infinite and $W(\lambda, t, u)$ is a measure in u satisfying

$$\frac{\partial}{\partial t} W(\lambda, t, u) = \frac{\lambda}{p(\lambda)} W(\lambda, t, u)(u-t) \tag{8.2}$$

(where the derivative is taken in the distribution sense). C. P. May restricted himself to $p(t) \geqq 0$ being a polynomial of degree less than or equal to two for which many well-known applications are valid (Bernstein, Baskakov, Szász, Post-Widder and Gauss-Weierstrass). Later Ismail and May [7] showed that if $p(t) \geqq 0$ is analytic in (A, B), we still have some of the properties and results of [10]. May [9], [10] proved that for combinations of exponential-type operators local, direct and inverse theorems are valid and Ismail and May [7] showed that a local direct and inverse theorem is valid for $S_\lambda(f, x) - f(x)$ (no combinatons). We will show that in those cases global direct theorems follows Theorems 4.1 and 5.1. (The global result in this case is new.) The result in section 6(c) and the result in section 7 about Bernstein polynomials are included in this but the result in section 7 is important, being the motivating result for much of this paper; and in fact Bernstein polynomials were the motivation for exponential-type operators.

We are now ready to define the combinations of $S_\lambda(f, x)$ for finite, fixed but arbitrary constants $d_0, \ldots, d_k$:

$$S_\lambda(f, k, x) = \sum_{j=0}^{k} C(j, k) S_{d_j \lambda}(f, x) \tag{8.3}$$

where

$$C(j, k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i} \quad k \neq 0 \quad \text{and} \quad C(0, 0) = 1. \tag{8.4}$$

We are now in a position to state and prove our result.

Theorem 8.1. *For $f \in C[A, B]$ abd $f \in (C, A(k+1))_\alpha$ where $A(k+1) = \{f; f, \ldots, f^{(2k+1)}$ are absolutely continuous locally in (A, B) and $\Phi_k(f) = \|p(x)^{k+1} f^{(2k+2)}(x)\|_{C[A,B]} < \infty\}$, when $p(x) \geqq 0$ is a polynomial of degree 2, we have*

$$\|S_\lambda(f, k, x) - f(x)\|_{C[A,B]} \leqq M \lambda^{-(k+1)\alpha}. \tag{8.5}$$

For other analytic posititve $p(x)$ where $\sqrt{p(x)}$ behaves like the $\varphi(x)$ of definition 4.1 near the boundary points A and B, we have (8.5) for $k=0$.

Remark 8.2. (a) Ismail and May [7] do not deal with the convergence of combinations of the operators there but, following their properties 2.2 of [7, p. 448], some of these results will still be valid. Here we just want to show the applicability of our earlier result and not get involved in various generalizations of particular situations.

(b) As MAY [10] and ISMAIL and MAY [7] observed, and as was also observed earlier in this paper, exponential behaviour of the functions is allowed in case A or B (or both) are not finite.

Proof. The key to the proof is proposition 3.2 of May's paper [10], p. 227]. The moments

$$A_m(\lambda, t) = \lambda^m \int_A^B W(\lambda, t, u)(u-t)^m \, du \tag{8.6}$$

are studied and, using the recursion relation

$$A_{m+1}(\lambda, t) = \lambda m p(t) A_{m-1}(\lambda, t) + p(t) \frac{d}{dt} A_m(\lambda, t), \tag{8.7}$$

May showed that $A_m(\lambda, t)$ are polynomials in λ (and in t when $p(t)$ is a polynomial of degree less than or equal to 2) of degree [m/2] in λ and that the coefficient of λ^m in $A_{2m}(\lambda, t)$ is $cp(t)^m$ and in $A_{2m+1}(\lambda, t)$ is $c(t)p(t)^m$. What is not exactly stated but still follows from (8.7) is that $\lambda^{-2k} A_{2k}(\lambda, t)$ is a sum of the type

$$\frac{p(t)^k}{\lambda^k} + c_1(t) \frac{p(t)^{k-1}}{\lambda^{k-1}} + c_2(t) \frac{p(t)^{k-2}}{\lambda^{k-2}} + \ldots$$

and that $\lambda^{-2k-1} A_{2k}(\lambda, t)$ is a sum of the type $\frac{p(t)^k}{\lambda^{k+1}} + c_1(t) \frac{p(t)^{k-1}}{\lambda^{k-1}} + \ldots$ where if the corresponding γ in Theorem 4.1 is 1, $c_1(t)$ may have a zero at the boundary ($c_2(t)$ a double zero, etc.). Observing that the combinations in (8.3) will cause $\sum_{j=0}^{k} c(j, k) \frac{1}{(d_j k)^l} = 0$ for $l \leqq k$, we will following the Bernstein polynomials case, obtain the correct estimate on the moments. For $p(t)$ analytic we just claim that if $\sqrt{p(t)}$ satisfies the condition on $\varphi(x)$ in theorem 4.1, then the result is valid, which is obvious as $\lambda^{-1} p(t) = D_\lambda(t)$ and the first moment is equal to 0.

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