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Norm convergence of generalized martingales in L^p-spaces over von Neumann algebras

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Dedicated to Professor Károly Tandori on his 60th birthday

Introduction

After scattered partial results the norm convergence of martingales in L_p-spaces over von Neumann algebras has been proved by GOLDSTEIN [10]. The main difference between his approach and our one is twofold. While in [10] (as well as in [3], [6], [7], [15]) the martingale sequence is formed by means of conditional expectations (i.e. state preserving projections of norm one onto subalgebras) we use ω-conditional expectations introduced in [1] (which are not projections in general but they always exist). On the other hand, the L^p-norm we shall use is different from the L_p-norm used in [10] when restricted to L^∞. So [10] does not cover our results even in the case in which all the conditional expectations involved are norm one projections.

All the theorems are proved for a von Neumann algebra with a faithful normal state on it. The framework is the theory of $L(p)$ spaces as complex forms rather than operators developed in [4] which are, very roughly speaking, representations of the spaces of TERP [18], and so closely connected to the spaces of CONNES and HILSUM [5], [14].

The results of this paper are contained in Theorem 9 and Theorem 10. Their forerunner (the strong convergence of bounded martingales with ω-conditional expectations) was obtained in [16], [17] and independently in [13].

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Preliminaries

Let M be a von Neumann algebra acting on a Hilbert space H. We denote by M' the commutant of M and by ω' a faithful normal state on M'. The triple $(\pi_{\omega'}, H_{\omega'}, \Omega')$ is the result of the GNS-construction with ω'.

We summarize some results and notations contained in [5]. As usually we set $D(H, \omega')=\{\xi\in H\colon \|a\xi\|\leqq c\omega'(a^*a)^{1/2}$ for all $a\in M'$ and some $c>0\}$. The space $D(H, \omega')$ is a dense vector space in H and for each $\xi\in D(H, \omega')$ there is a unique bounded linear operator $R_{\omega'}(\xi)\colon H_{\omega'}\to H$ such that

$$R_{\omega'}(\xi)\pi_{\omega'}(a)\Omega' = a\xi.$$

The correspondence $\xi\mapsto R_{\omega'}(\xi)$ is linear and for all $\xi, \eta\in D(H, \omega')$ the operator $R_{\omega'}(\xi)R_{\omega'}(\eta)^*$ is in M. If $\varphi\in M_*^+$ then the equality

$$q_\varphi(\xi) = \varphi\big(R_{\omega'}(\xi)R_{\omega'}(\xi)^*\big)$$

defines a lower semicontinuous positive form on $D(H, \omega')$ to which a positive self-adjoint operator $(d\varphi)/(d\omega')$ (the spatial derivative of φ with respect to ω') is associated ([5]).

Now we are in a position to define the spaces $L^p(M, \omega')$ for $1\leqq p<\infty$ as in [14]. $L^p(M, \omega')$ is the set of all closed densely defined operators on H with polar decomposition $T=u|T|$ such that

$$u\in M \quad \text{amd} \quad |T|^p = \frac{d\varphi}{d\omega'}$$

for some $\varphi\in M_*^+$. If $\psi\in M_*$ has a polar decomposition $\psi=u|\psi|$ then we define

$$T_{\omega'}(\psi) = u\frac{d|\psi|}{d\omega'} \quad \text{and} \quad T_{\omega'}(\psi)\, d\omega' = \psi(1).$$

The spaces $L^p(M, \omega')$ $(1\leqq p<\infty)$ are Banach spaces endowed with the norm

$$\|T\|_p = \Big(\int |T|^p\, d\omega'\Big)^{1/p}$$

if by sum (and later by product) of unbounded operators we take the strong sum (and strong product).

Let us now fix a faithful normal state ω on M and shorten $(d\omega)/(d\omega')$ in d. For $1\leqq p<\infty$ we define $H(p, \omega, \omega')$ as the Hilbert space completion of the domain of $d^{-1/2p}$ under the inner product

$$\langle\xi, \eta\rangle_p = \langle d^{-1/2p}\xi, d^{-1/2p}\eta\rangle$$

and $H(\infty, \omega, \omega')=H$. There is a unique unitary operator $V(\omega, \omega', P_2, P_1)$: $H(p_1, \omega, \omega')\to H(p_2, \omega, \omega')$ such that

$$V(\omega, \omega', P_2, P_1)\xi = d^{-(p_1^{-1}-p_2^{-1})/2}\xi$$

for $\xi \in D(H, \omega)$ and for $1 \leqq p_1 < p_2 \leqq \infty$. (Here $D(H, \omega)$ is defined and has the same properties as $D(H, \omega')$ above by reversing the roles of M and M'.)

When ω and ω' are fixed we shall shorten our notation to $H(p)$ for the Hilbert spaces and to $V(p_2, p_1)$ for the unitaries introduced above.

Let $1 \leqq p < \infty$. We set $L(p, M, \omega, \omega')$ for the set of all complex forms (i.e. complex linear combinations of positive forms) defined on $D(H, \omega)$ and having the form

$$q(T)(\xi) = \langle |T|^{1/2} V(p, \infty)^* u^* V(p, \infty)\xi, |T|^{1/2}\xi \rangle_p$$

when T is a closed densely defined operator on $H(p)$ with a polar decomposition

$$V(\infty, p)^* u V(\infty, p) |T|$$

such that u is a partial isometry in M and

$$V(\infty, p) T V(\infty, p)^*$$

is in $L^p(M, \omega')$.

For $p = \infty$ we set $L(\infty, M, \omega, \omega') = \{q(a) \colon a \in M\}$ where $q(a)(\xi) = \langle \xi, a\xi \rangle$ $(\xi \in D(H, \omega))$.

We define a norm on $L(p, M, \omega, \omega')$ by requiring the linear bijection $\lambda_p \colon L(p, M, \omega, \omega') \to L^p(M, \omega')$, $\lambda_p \colon q(T) \mapsto V(\infty, p) T V(\infty, p)^*$ to be an isometry for $1 \leqq p \leqq \infty$. In [4] it was shown that the spaces $L(p, M, \omega, \omega')$ do not depend on the auxiliarly state ω' used in their construction (ω' can even be taken to be a normal semifinite weight).

We note so that $L(1, M, \omega)$ is isometrically isomorphic to M_* and we denote this isomorphism by ι_ω. Explicitely,

$$\iota_\omega(\psi)(\xi) = \psi(|R_{\omega'}(d^{-1/2}\xi)^*|^2) \qquad (\psi \in M_*, \xi \in D(H, \omega)),$$

since $d^{-1/2}\xi \in D(H, \omega')$.

If $1 \leqq p_1 < p_2 \leqq \infty$ then $L(p_2, M, \omega) \subset L(p_1, M, \omega)$ and $L(p_2, M, \omega)$ is norm dense in $L(p_1, M, \omega)$. For $q \in L(p_2, M, \omega)$ we have

$$\|q\|_{L(p_2, M, \omega)} \geqq \|q\|_{L(p_1, M, \omega)}.$$

These properties will be used without reference.

Let M_0 be a subalgebra of M and $\omega_0 = \omega | M_0$. The ω-conditional expectation $E^\omega \colon M \to M_0$ defined in [1] is an ω-preserving completely positive contraction and it turns out to be the dual of the embedding of M_0 into M when suitable embeddings of the algebras into their preduals are considered (see [2] and [17]). In [4] it was proved that there exists a contraction $\varepsilon^\omega \colon L(1, M, \omega) \to L(1, M_0, \omega_0)$ such that

$$\varepsilon^\omega q(a)(\xi) = \langle \xi, E^\omega(a)\xi \rangle$$

($\xi \in D(H, \omega_0)$, $a \in M$). Interpolation techniques give that the restriction of ε^ω to $L(p, M, \omega)$ is also a contraction into $L(p, M_0, \omega_0)$ ($1 < p < \infty$, see [4] and [18]). Later we define a natural mapping $\varkappa$: $L(p, M_0, \omega_0) \to L(p, M, \omega)$ and we form the composition $\varkappa \circ \varepsilon^\omega$ in order to have a selfmapping of $L(p, M, \omega)$.

Results

The elements of the spaces $L(p, M, \omega)$ are complex forms on $D(H, \omega)$ so the pointwise convergence of forms can be defined in a natural way. We deal with the relation of this convergence to the norm convergence in $L(p, M, \omega)$. We need also the connection between the strong operator topology on M and the norm topology of $L(p, M, \omega)$.

Lemma. *Let* $(q_n) \subset L(1, M, \omega)$. *If* $\iota_\omega^{-1}(q_n) \to 0$ *weakly then for any* $\xi \in D(H, \omega)$

$$q_n(\xi) \to 0.$$

Moreover, if (q_n) *is bounded then the converse also holds.*

Proof. Since

$$q_n(\xi) = (\iota_\omega^{-1} q_n)(|R_{\omega'}(d^{-1/2}\xi)^*|^2)$$

the first part of the statement follows immediately. To get the converse it suffices to note that the linear hull of the set

$$\{|R_{\omega'}(d^{-1/2}\xi)^*|^2 \colon \xi \in D(H, \omega)\}$$

is dense in M.

Proposition 2. *Let* $(q_n) \subset L(p, M, \omega)$ *and* $1 \leqq p < \infty$. *If* $q_n \to q$ *in norm of* $L(p, M, \omega)$ *then*

$$q_n(\xi) \to q(\xi)$$

for every $\xi \in D(H, \omega)$.

Proof. $q_n \to q$ in the norm of $L(1, M, \omega)$ and so in the weak topology. Lemma 1 can be applied.

Now we prove technical lemmas on different norms. To simplify formulas we shall shorten $d^{1/2s}$ in D.

Lemma 3. *Let* $a \in M$ *and* s, k *be integers such that* $s \geqq 3$ *and* $0 \leqq k \leqq s-3$. *Then*

$$\|q(a)\|_{L(2^s, M, \omega)}^{2^s} \leqq \|a\|^{2^s + 2^{s-2-k} + 1} \|(Da^* Da)^{2^{s-2} - 2^k} d^{2^{-s+k+1}}\|_{L^2(M, \omega')}^{2^{-k-1}}.$$

Proof. We apply induction on k. First let $k=0$.

$$\|q(a)\|^{2^s}_{L(2^s,M,\omega)} = \|D^{1/2}aD^{1/2}\|^{2^s}_{L(2^s,M,\omega')} =$$
$$= \int (d^{1/2^{s+1}}a^*Dad^{1/2^{s+1}})^{2^s-1}\,d\omega' = \int (a^*DaD)^{2^{s-1}}\,d\omega' \leqq$$
$$\leqq \|(a^*DaD)^{2^{s-2}}a^*\|_{L^2(M,\omega')}\|DaD(a^*DaD)^{2^{s-2}-1}\|_{L^2(M,\omega')} \leqq$$
$$\leqq \|a\|^{2^{s-1}+1}\Big[\int (Da^*Da)^{2^{s-2}-1}Da^*D^2aD(a^*DaD)^{2^{s-2}-1}\,d\omega'\Big]^{1/2} \leqq$$
$$\leqq \|a\|^{2^{s-1}+1}\Big[\int a^*Da(Da^*Da)^{2^{s-2}-2}Da^*D^2aD(a^*DaD)^{2^{s-2}-1}\,d\omega'\Big]^{1/2} \leqq$$
$$\leqq \|a\|^{2^{s-1}+1}\Big[\int a^*Da(Da^*Da)^{2^{s-2}-2}Da^*D^2aD(a^*DaD)^{2^{s-2}-1}\,d\omega'\Big]^{1/2} \leqq$$
$$\|a\|^{2^{s-1}+1}\|a^*Da(Da^*Da)^{2^{s-2}-2}Da^*D^2a\|^{1/2}_{L^2(M,\omega')}\|D(a^*DaD)^{2^{s-2}-1}D\|^{1/2}_{L^2(M,\omega')} \leqq$$
$$\leqq \|a\|^{2^{s-1}+1}\|a\|^{2^{s-2}}\|D(a^*DaD)^{2^{s-2}-1}D\|^{1/2}_{L^2(M,\omega')} =$$
$$= \|a\|^{2^s-2^{s-2}+1}\|(Da^*Da)^{2^{s-2}-1}D^2\|^{1/2}_{L^2(M,\omega')}.$$

Here we have used the Hölder inequality repetedly. Now we carry out the induction step. We have:

$$\|(Da^*Da)^{2^{s-2}-2^k}\,d^{2^{-s+k+1}}\|_{L^2(M,\omega')} =$$
$$= \Big(\int d^{2^{-s+k+1}}(a^*DaD)^{2^{s-2}-2^k}(Da^*Da)^{2^{s-2}-2^k}\,d^{2^{-s+k+1}}\,d\omega'\Big)^{1/2} =$$
$$= \Big(\int (a^*DaD)^{2^{s-2}-2^k}(Da^*Da)^{2^{s-2}-2^k}\,d^{2^{-s+k+2}}\,d\omega'\Big)^{1/2} \leqq$$
$$\leqq \|(a^*DaD)^{2^{s-2}-2^k}(Da^*Da)^{2^k}\|^{1/2}_{L^2(M,\omega')}$$
$$\|(Da^*Da)^{2^{s-2}-2^{k+1}}\,d^{2^{-s+k+1}}\|^{1/2}_{L^2(M,\omega')} \leqq$$
$$\|a\|^{2^{s-2}}\|(Da^*Da)^{2^{s-2}-2^{k+1}}\,d^{-s+k+2}\|^{1/2}_{L^2(M,\omega')}.$$

So our hypothesis on k implies our claim for $k+1$.

Lemma 4. *Let a and s be as in the previous Lemma. Then*

$$\|q(a)\|^{2^s}_{L(2^s,M,\omega)} \leqq \|a\|^{m(s)}\|ad^{1/2}\|^{2^{-s+1}}_{L^2(M,\omega')}$$

where $m(s)=2^s-1+(2^{s-1}-1)2^{-s+3}$.

Proof. Using Lemma 3 with $k=s-3$ we can majorize as follows.

$$\|(Da^*Da)^{2^{s-3}}\,d^{1/4}\|_{L^2(M,\omega')} =$$
$$= \Big[\int d^{1/4}(a^*DaD)^{2^{s-3}}(Da^*Da)^{2^{s-3}}\,d^{1/4}\,d\omega'\Big]^{1/2} =$$
$$= \Big[\int (a^*DaD)^{2^{s-3}}(Da^*Da)^{2^{s-3}}\,d^{1/2}\,d\omega'\Big]^{1/2} \leqq$$
$$\|(a^*DaD)^{2^{s-3}}(Da^*Da)^{2^{s-3}-1}Da^*D\|^{1/2}_{L^2(M,\omega')}\|ad^{1/2}\|^{1/2}_{L^2(M,\omega')} \leqq$$
$$\|a\|^{2^{s-2}-1/2}\|ad^{1/2}\|^{1/2}_{L^2(M,\omega')}$$

Proposition 5. *Let* $(a_n)\subset M$ *be a bounded sequence. If* $a_n\to a$ *strongly then* $q(a_n)\to q(a)$ *in the norm of* $L(p, M, \omega)$ *for* $1\leq p<\infty$.

Proof. We may assume that $a=0$ and $p=2^s$. For arbitrary $a\in M$ we have

$$\|ad^{1/2}\|^2_{L^2(M,\omega')} = \int d^{1/2}a^*a\, d^{1/2}\, d\omega' = \int da^*a\, d\omega' = \omega(a^*a).$$

Now an application of Lemma 4 completes the proof.

Let M_0 be a subalgebra of M. We denote by ω_0 the restriction of ω to M_0. Clearly, $D(H,\omega)\subset D(H,\omega_0)$ and if q is a form on $D(H,\omega_0)$ then $\varkappa(q)$ will stand for $q|D(H,\omega)$.

Lemma 6. *Let* M_0, M, ω_0, ω *and* $\varkappa$ *be as above. Then* $\varkappa|L(1, M_0, \omega_0)$ *is a linear contraction from* $L(1, M_0, \omega_0)$ *to* $L(1, M_0, \omega)$.

Proof. Denote by H_ω (H_{ω_0}) the Hilbert space for the standard representation $\pi_\omega(M)$ $(\pi_{\omega_0}(M_0))$ with respect to $\omega(\omega_0)$ and Ω its cyclic and separating vector defining ω (and also ω_0 since H_{ω_0} is considered as a subspace of H_ω). Let J_ω (J_{ω_0}) be the usual canonical conjugation of the Tomita—Takesaki theory for the couple $(M,\omega)((M_0,\omega_0))$ and P the projection from H_ω onto H_{ω_0}. We define a partial isometry V as it was denote in [1].

$$VJ_{\omega_0}\pi_\omega(a)\Omega = J_\omega\pi_\omega(a)\Omega \quad \text{for} \quad a\in M_0$$

$$V\xi = 0 \quad \text{for} \quad \xi\perp H_{\omega_0}$$

From [4] we know that $J_\omega|R_\omega(\xi)|^2J_\omega\in\pi_\omega(M)$ $(\xi\in D(H,\omega))$ (and, for $\xi\in D(H,\omega_0)$, $J_{\omega_0}|R_{\omega_0}(\xi)|^2J_{\omega_0}\in\pi_{\omega_0}(M_0)$). Now if E^ω is the ω-conditional expectation from M to M_0 then

$$E^\omega(\pi_\omega^{-1}(J_\omega|R_\omega(\xi)|^2J\omega)) = \pi_{\omega_0}^{-1}(V^*J_\omega|R_\omega(\xi)|^2J_\omega V) =$$
$$= \pi_{\omega_0}^{-1}(J_{\omega_0}|R_{\omega_0}(\xi)|^2J_{\omega_0}).$$

The last equality follows from: $R_\omega(\xi)P\pi_\omega(a)\Omega=R_\omega(\xi)\pi_\omega(a)\Omega=a\xi=R_{\omega_0}(\xi)\pi_{\omega_0}(a)\Omega$ for $a\in M_0$ and $\xi\in D(H,\omega)$, which implies $R_\omega(\xi)P|H_{\omega_0}=R_{\omega_0}(\xi)$.

Let nos $\varphi\in M_*$. It is proved in [4] that $\iota_\omega(\varphi)(\xi)=\pi_\omega^{-1}(J_\omega|R_\omega(\xi)|^2J_\omega)$ for $\xi\in D(H,\omega)$ and the similar equality holds also for ι_{ω_0}. We have therefore, for $\xi\in D(H,\omega)$ and $\varphi\in(M_0)_*$,

$$\varkappa(\iota_{\omega_0}(\varphi)(\xi) = \iota_{\omega_0}(\varphi)(\xi) = \varphi(\pi_{\omega_0}^{-1}(J_{\omega_0}|R_{\omega_0}(\xi)|^2J_{\omega_0})) =$$
$$= \varphi(E^\omega(\pi_\omega^{-1}(J_\omega|R_\omega(\xi)|^2J_\omega))) = \iota_\omega(\varphi\circ\varepsilon),$$

and

$$\|\varkappa\circ\iota_{\omega_0}(\varphi)\|_{L(1,M,\omega)} = \|\iota_\omega(\varphi\circ\varepsilon)\|_{L(1,M,\omega)} = \|\varphi\circ\varepsilon\| \leq$$
$$\leq \|\varphi\|_{(M_0)^*} = \|\iota_{\omega_0}(\varphi)\|_{L(1,M,\omega)},$$

which proves our statement.

From the above Lemma, it is clear that $\varkappa \circ \iota_{\omega_0}(\varphi)$ depends only on the value of φ on the range of E^{ω}. This implies that $\varkappa$ in general is not injective on $L(1, M, \omega)$. More precisely, $\varkappa \circ \iota_{\omega_0}(\varphi)=0$ if $\varphi|E^{\omega}(M)\equiv 0$. This implies that $\varkappa$ is injective if and only if $E^{\omega}(M)$ is weak-operator dense in M_0, which is not the case in general (cf. [1], section 4).

Proposition 7. *Let M, M_0, ω, ω_0 and $\varkappa$ be as above. If $q\in L(p, M_0, \omega_0)$ then $\varkappa(q)\in L(p, M, \omega)$ for $1<p<\infty$. Moreover, $\varkappa$ is a contraction with respect to the $L(p)$ norms.*

Proof. It is straightforward that for $a\in M_0$ we have $\varkappa(q(a))\in L(\infty, M, \omega)$ and

$$\|\varkappa(q(a))\|_{L(\infty, M, \omega)} = \|q(a)\|_{L(\infty, M_0, \omega_0)}$$

where $q(a)(\xi)=\langle \xi, a\xi\rangle$ $(\xi\in D(H, \omega_0))$. On the other hand the statement has been proved in Lemma 6 for $p=1$. By the Calderon—Lions interpolation theorem ([4], [18]) for $1<p<\infty$ we have $\varkappa(q)\in L(p, M, \omega)$ and

$$\|\varkappa(q)\|_{L(p, M, \omega)} \leqq \|q\|_{L(p, M_0, \omega_0)}$$

whenever $q\in L(p, M, \omega)$.

Let us fix a von Neumann algebra M with a faithful normal state ω and an increasing sequence (M_n) of von Neumann subalgebras. Assume that M is generated by $\bigcup_{n=1}^{\infty} M_n$. We denote by ω_n the restriction of ω to M_n and E_n^{ω} will stand for the ω-conditional expectation $M\to M_n$. It is porved in [16], [17] and independently in [13] that $E_n^{\omega}(a)\to a$ strongly for every $a\in M$. As above we write ε_n^{ω} for the extension of E_n^{ω} to $L(1, M, \omega)$ and $\varkappa_n$: $L(1, M_n, \omega_n)\to L(1, M, \omega)$ is the restriction mapping.

Theorem 8. *With the notation above, for every $q\in L(p, M, \omega)$*

$$\varkappa_n \circ \varepsilon_n^{\omega}(q) \to q$$

in the norm of $L(p, M, \omega)$ $(1\leqq p<\infty)$.

Proof. Since the sequence $(\varkappa_n\circ\varepsilon_n^{\omega})$ is uniformly bounded it is sufficient to prove our statement on a dense set. We shall assume that $q\in L(\infty, M, \omega)$, that is $q=q(a)$ for some $a\in M$. So $E_n^{\omega}(a)\to a$ strongly and by Proposition 5 $q(E_n^{\omega}(a))\to q(a)$ in the norm of $L(p, M, \omega)$. However, $q\circ E_n^{\omega}=K_n\circ\varepsilon_n^{\omega}$ and the proof is complete.

Let $(q_n)\subset L(p, M, \omega)$ be a sequence such that

$$\varkappa_k\cdot\varepsilon_k^{\omega}(q_n) = q_k \quad (n>k).$$

Such a sequence (q_n) will be called (generalized) martingale (adapted to the sequence (M_n) of subalgebras). The martingale (q_n) is called regular if there is a $q\in L(p, M, \omega)$ such that $q_n=K_n\circ\varepsilon_n^{\omega}(q)$.

Theorem 9. *Let* $(q_n)\subset L(p, M, \omega)$ *be a martingale (adapted to the sequence* (M_n)*) and* $1<p<\infty$. *Then the following conditions are equivalent.*

(i) (q_n) *is regular.*

(ii) (q_n) *converges in the norm of* $L(p, M, \omega)$.

(iii) $\sup_n \|q_n\|_{L(P,M,\omega)}<\infty$

Proof. (i)→(ii) is just the previous Theorem. (ii)→(iii) is trivial. If (iii) holds then due to the reflexivity of $L(p, M, \omega)$ (see [4], [14], [18]) we can find a weakly convergent subsequence of (q_n), say $q_{k(n)}\to q$ weakly. If n is large enough then

$$\varkappa_m \varepsilon_m^\omega (q_{k(n)}) = q_m$$

and we have $q_m=\varkappa_m\varepsilon_m^\omega(q)$.

Theorem 10. *Let* $(q_n)\subset L(1, M, \omega)$ *be a martingale (adapted to the sequence* (M_n)*). Then the following conditions are equivalent.*

(i) (q_n) *is regular,*

(ii) (q_n) *converges in the norm of* $L(1, M, \omega)$.

(iii) $\{q_n\colon n\in\mathbb{N}\}$ *is relatively* $\sigma(L(1), L(\infty))$ *compact in* $L(1, M, \omega)$.

Proof. We can follow the proof of Theorem 9 but instead of reflexivity we may apply the Eberlein—Smulian theorem ([8]).

The reversed martingale convergence theorem does not hold if the sequence is formed with ω-conditional expectations. A counter example is contained in [1].

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