Acta Sci. Math., 48 (1985), 25—36

Factoring compact operator-valued functions

H. BERCOVICI, C. FOIAŞ, C. PEARCY, and B. SZ.-NAGY

Dedicated to the 60th anniversary of Prof. K. Tandori

1. Let $\mathfrak{H}$ be a separable Hilbert space and denote by $\mathscr{L}(\mathfrak{H})$, $\mathscr{K}(\mathfrak{H})$, $\mathscr{C}_1(\mathfrak{H})$ and $\mathscr{F}(\mathfrak{H})$ the sets of all bounded linear operators, compact operators, trace-class, and finite-rank operators acting on $\mathfrak{H}$, respectively. We set $C=\{\lambda\in\mathbf{C}: |\lambda|=1\}$ and, if $\mathfrak{X}$ is a Banach space and $1\leqq p\leqq\infty$, we denote by $L^p(\mathfrak{X})$ the space of (classes of) Bochner integrable functions f defined on C, with values in $\mathfrak{X}$, such that

$$\|f\|_p=\left(\frac{1}{2\pi}\int_0^{2\pi}\|f(e^{it})\|^p\,dt\right)^{1/p}<\infty\quad(p<\infty),\qquad \|f\|_\infty=\operatorname{ess\,sup}\|f(e^{it})\|<\infty.$$

Assume that $\mathfrak{H}$ is infinite dimensional. It is fairly easy to see that for any function $Z\in L^1(\mathscr{K}(\mathfrak{H}))$ one can find functions $X, Y\in L^2(\mathscr{K}(\mathfrak{H}))$ such that $Z=Y^*X$ in $L^1(\mathscr{K}(\mathfrak{H}))$, i.e., $Z(e^{it})=Y(e^{it})^*X(e^{it})$ almost everywhere on C.

In this paper we show that under certain conditions, the functions X and Y can be chosen such that the $\mathfrak{H}$-valued functions $X(e^{it})h$, $Y(e^{it})h$ ($e^{it}\in C$, $h\in\mathfrak{H}$) belong to a certain prescribed functional model space. The methods used here were developed successively in [8], [6], and [2]. Another factorization theorem for Hilbert—Schmidt valued functions is proved in [1], where the operator-theoretic implications of such factorizations are studied in some detail. We hope that these factorization theorems will prove to be relevant in the study of infinite dimensional systems.

The research in this paper was essentially completed in 1981 and partially inspired our subsequent work.

2. We will use the notation $H^p(\mathfrak{X})$ for the Hardy subspace of $L^p(\mathfrak{X})$. If $\mathfrak{X}=\mathbf{C}$ we write H^p and L^p for $H^p(\mathfrak{X})$ and $L^p(\mathfrak{X})$, respectively. If σ is a measurable subset of C, then $L^p(\sigma,\mathfrak{X})$ will denote the space of all functions $f\in L^p(\mathfrak{X})$ that vanish almost everywhere off σ.

Received July 15, 1983.

2.1. Definition. A subset S of the unit ball of H^2 is said to be *dominating* for the measurable subset σ of C if the closed absolutely convex hull of the set $\{|\varphi|^2\chi_\sigma:\ \varphi\in S\}$ coincides with the unit ball of $L^1(\sigma)$:

$$\overline{\text{aco}}\{|\varphi|^2\chi_\sigma:\ \varphi\in S\}=\{f\in L^1(\sigma):\ \|f\|_1\leqq 1\}. \tag{2.2}$$

Here, as usual, χ_σ denotes the characteristic function of the set σ.

In order to provide some motivation for the preceding definition let us compare it with the following one, suggested by one of the basic theorems of Brown, Shields and Zeller [4] (these authors only consider the case in which $\sigma=C$).

2.3. Definition. A subset A of the unit disc $D=\{\lambda:\ |\lambda|<1\}$ is said to be *dominating* for the measurable subset σ of C if almost every point of σ is a nontangential limit of a sequence of points in A.

To see the connection between the two definitions let us consider the functions

$$p_\mu(z)=(1-|\mu|^2)^{1/2}(1-\bar{\mu}z)^{-1},\quad z\in C,\quad \mu\in D, \tag{2.4}$$

which belong to H^2, and indeed to H^∞, and for which $|p_\mu(z)|^2$ equals the Poisson kernel function $P_\mu(z)=(1-|\mu|^2)|1-\bar{\mu}z|^{-2}$; $\|P_\mu\|_1=1$.

2.5. Proposition. *If a subset $A\subset D$ is dominating for $\sigma<C$, then $S=\{p_\mu:\ \mu\in A\}\subset H^2$ is also dominating for σ.*

(See Lemma 1.2 a) in [2].)

2.6. Lemma. *Assume that $\mathfrak{H}$ is a separable Hilbert space, σ is a measurable subset of C, and S is a subset of the unit ball of H^2, dominating for σ. Then the closure in $L^1(\sigma,\mathcal{K}(\mathfrak{H}))$ of the set*

$$\Sigma=\Big\{\sum_j|\varphi_j^2|\chi_\sigma C_j\ \ (\textit{finite sums}):\ \varphi_j\in S,\ \ C_j\in\mathcal{F}(\mathfrak{H}),\ \ \sum_j\|C_j\|\leqq 1\Big\}$$

coincides with the unit ball of $L^1(\sigma,\mathcal{K}(\mathfrak{H}))$.

Proof. The dual space of $L^1(\sigma,\mathcal{K}(\mathfrak{H}))$ can be identified with $L^\infty(\sigma,\mathcal{C}_1(\mathfrak{H}))$ *via* the bilinear form

$$\langle F,G\rangle=\frac{1}{2\pi}\int_0^{2\pi}\operatorname{tr}\big(F(e^{it})G(e^{it})\big)\,dt,\quad F\in L^\infty(\sigma,\mathcal{C}_1(\mathfrak{H})),\quad G\in L^1(\sigma,\mathcal{K}(\mathfrak{H})). \tag{2.7}$$

The set Σ is convex and balanced, and the Hahn—Banach theorem implies that it suffices to show that

$$\sup\{|\langle F,G\rangle|:\ G\in\Sigma\}=\|F\|_\infty\quad\text{for}\quad F\in L^\infty(\sigma,\mathcal{C}_1(\mathfrak{H})). \tag{2.8}$$

Since Σ is clearly contained in the unit ball of $L^1(\sigma, \mathscr{K}(\mathfrak{H}))$, we have

$$|\langle F, G\rangle| \leqq \|F\|_\infty \|G\|_1 \leqq \|F\|_\infty \quad \text{for} \quad G\in\Sigma,$$

and so it remains to check the opposite inequality. The hypothesis that S is dominating for σ shows that the closure of Σ contains all functions of the form fK with $f\in L^1(\sigma)$, $K\in\mathscr{K}(\mathfrak{H})$, $\|f\|_1\leqq 1$, $\|K\|\leqq 1$. Let $\{K_n\}$ be a dense sequence in the unit ball of $\mathscr{K}(\mathfrak{H})$. Then we have $\|R\|=\sup_n\{|\mathrm{tr}(RK_n)|\}$ for every $R\in\mathscr{C}_1(\mathfrak{H})$. Thus, if $F\in L^\infty(\sigma, \mathscr{C}_1(\mathfrak{H}))$, we have

$$\|F(z)\|_1 = \sup_n\{|\mathrm{tr}\,(F(z)K_n)|\}, \quad z\in C.$$

(Here and in the sequel we indicate by $z\in C$ a relation that holds almost everywhere on C.) The following calculation is justified for $z\in C$:

$$\|F(z)\| = \sup_n\{|\mathrm{tr}\,(F(z)K_n)|\} \leqq \sup_n\{\||\mathrm{tr}\,(F(\cdot)K_n)|\|_\infty\} =$$

$$= \sup_n \sup\left\{\left|\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,\mathrm{tr}\,(F(e^{it})K_n)\,dt\right| : f\in L^1(\sigma),\ \|f\|_1\leqq 1\right\} =$$

$$= \sup_n \sup\{|\langle F, fK_n\rangle| : f\in L^1(\sigma),\ \|f\|_1 \leqq 1\} =$$

$$\leqq \sup\{|\langle F, fK\rangle| : f\in L^1(\sigma),\ K\in\mathscr{K}(\mathfrak{H}),\ \|f\|_1\leqq 1,\ \|K\|\leqq 1\} =$$

$$\leqq \sup\{|\langle F, G\rangle| : G\in\Sigma^-\} = \sup\{|\langle F, G\rangle| : G\in\Sigma\}.$$

This yields $\|F\|_\infty \leqq \sup\{|\langle F, G\rangle| : G\in\Sigma\}$; hence (2.8) holds and the proof is completed.

3. We recall now some notation pertaining to functional model spaces; cf. [7]. Let $\mathfrak{F}$ and $\mathfrak{F}'$ be separable Hilbert spaces, and assume that $\Theta\colon D\to\mathscr{L}(\mathfrak{F}, \mathfrak{F}')$ is contractive and analytic: $\|\Theta(\lambda)\|\leqq 1$ and $\Theta(\lambda)=\sum_0^\infty \lambda^k\Theta_k$ $(\Theta_k\in\mathscr{L}(\mathfrak{F}, \mathfrak{F}'))$ for $\lambda\in D$. The strong limit $\Theta(z)=\lim_{r\to 1-0}\Theta(rz)$, and hence $\Delta(z)=(I-\Theta(z)^*\Theta(z))^{1/2}$ also exist for $z\in C$, are strongly measurable functions on C, and generate an analytic Toeplitz operator $T_\Theta\in\mathscr{L}(H^2(\mathfrak{F}), H^2(\mathfrak{F}'))$ and a multiplication operator $\Delta\in\mathscr{L}(L^2(\mathfrak{F}))$ by setting, for $z\in C$,

$$(T_\Theta u)(z) = \Theta(z)u(z), \quad u\in H^2(\mathfrak{F}), \quad \text{and} \quad (\Delta v)(z) = \Delta(z)v(z), \quad v\in L^2(\mathfrak{F}).$$

Next we construct the function space

$$\mathfrak{K}_+ = H^2(\mathfrak{F}')\oplus(\Delta L^2(\mathfrak{F}))^-,$$

the bar indicating norm closure, and observe that

$$Vu = T_\Theta u\oplus\Delta u, \quad u\in H^2(\mathfrak{F}), \tag{3.1}$$

defines an isometry from the space $H^2(\mathfrak{F})$ into the space $\mathfrak{K}_+$. As a consequence,

$$\mathfrak{G} = VH^2(\mathfrak{F}) \quad \text{and} \quad \mathfrak{H}(\Theta) = \mathfrak{K}_+ \ominus \mathfrak{G}$$

are subspaces of $\mathfrak{K}_+$; $\mathfrak{H}(\Theta)$ is the "functional model space" associated with the contractive analytic function Θ.

All these spaces will be regarded as subspaces of the Hilbert function space $L^2(\mathfrak{F}' \oplus \mathfrak{F})$.

Note the following relation between the adjoints of the operator T_Θ and the operator $\Theta \in \mathscr{L}(L^2(\mathfrak{F}), L^2(\mathfrak{F}'))$ of multiplication by the function $\Theta(z)$ on $L^2(\mathfrak{F})$:

$$(T_\Theta^* v)(z) = [\Theta^* v]_+(z), \quad v \in H^2(\mathfrak{F}'), \quad z \in C,$$

where we denote by $[\ \ ]_+$ the natural orthogonal projection of any (scalar- or vector-valued) function space of type L^2 onto its subspace H^2.

Let us also note that for any fixed $z_0 \in C$ for which $\Theta(z_0)$ has sense, $V(z_0)a = \Theta(z_0)a \oplus \Delta(z_0)a$, $a \in \mathfrak{F}$, defines an isometry of $\mathfrak{F}$ into $\mathfrak{F}' \oplus \mathfrak{F}$.

It will be of importance for us to consider elements of $\mathfrak{H}(\Theta)$ of the form $P_{\mathfrak{H}(\Theta)}(u \oplus 0)$, $u \in H^2(\mathfrak{F}')$. Straightforward calculation gives

$$P_{\mathfrak{H}(\Theta)}(u \oplus 0) = (u \oplus 0) - VT_\Theta^* u,$$

and hence,

$$\|P_{\mathfrak{H}(\Theta)}(u \oplus 0)\|^2 = \|u\|^2 - \|T_\Theta^* u\|^2, \tag{3.2}$$

where the norms are in the spaces $L^2(\mathfrak{F}' \oplus \mathfrak{F})$, $L^2(\mathfrak{F}')$, and $L^2(\mathfrak{F})$, respectively.

If $u(z) = \varphi(z)a$, $z \in C$, where $\varphi \in H^2$ and $a \in \mathfrak{F}'$, let us denote $P_{\mathfrak{H}(\Theta)}(u \oplus 0)$ by $\varphi \circ a$.[1]) Thus, we have

$$(\varphi \circ a)(z) = (\varphi(z)a \oplus 0) - V(z)\big(T_\Theta^*(\varphi a)\big)(z), \quad z \in C. \tag{3.3}$$

From (3.2) it is clear that

$$\|\varphi \circ a\|_2 \leqq \|\varphi\|_2 \, \|a\|. \tag{3.4}$$

On the other hand, we deduce from (3.3) that

$$\|(\varphi \circ a)(z)\| \leqq |\varphi(z)| \, \|a\| + \|[\Theta^* \varphi a]_+(z)\|, \quad z \in C.$$

In the special case of the functions $\varphi_m(z) = z^m$ $(m = 0, 1, \ldots)$ we have

$$[\Theta^* \varphi_m a]_+(z) = \sum_{k=0}^{m} z^{m-k} \Theta_k^* a.$$

Since $\Theta_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \Theta(e^{it})\, dt$ is a contraction we deduce that

$$\|[\Theta^* \varphi_m a]_+(z)\| \leqq (m+1)\|a\|, \quad z \in C,$$

---

[1]) In the paper [2] the notion $\varphi \circ a$ was only used for the functions $\varphi = p_\mu$ $(\mu \in D)$ defined in (2.4), and $p_\mu \circ a$ was then denoted in the shorter form $\mu \circ a$.

and we conclude:

$$\|(\varphi_m \circ a)(z)\| \leqq (m+2)\|a\|. \tag{3.5}$$

Let us notice, furthermore, that for any $f \in \mathfrak{F}$ and $f' \in \mathfrak{F}'$,

$$((\varphi_m \circ a)(z), f' \oplus f) = \left(a, z^{-m} f' - \sum_{k=0}^{m} z^{k-m} \Theta_k V(z)^* (f' \oplus f)\right), \quad z \in C.$$

Hence we see that if a runs through a sequence in $\mathfrak{F}'$ converging weakly to 0, then $(\varphi_m \circ a)(z)$ also converges weakly to 0 in $\mathfrak{F}' \oplus \mathfrak{F}$, $z \in C$. By virtue of linearity of $\varphi \circ a$ with respect to φ this statement extends to the finite linear combinations of the functions φ_m, that is, to all polynomials $p(z) = \sum_0^M c_m z^m$.

Assume now that $\mathfrak{H}$ is a separable Hilbert space and denote by $L^2(\mathcal{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$ the space of norm-square integrable functions Y with values $Y(z)$ compact operators in $\mathcal{L}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F})$. The adjoint Y^* of such a function, $Y^*(z) = (Y(z))^*$, $z \in C$, is in $L^2(\mathcal{K}(\mathfrak{F}' \oplus \mathfrak{F}, \mathfrak{H}))$. Since the function $(\varphi \circ a)(z)$ has its values in $\mathfrak{F}' \oplus \mathfrak{F}$ (indeed, $\varphi \circ a \in \mathfrak{H}(\Theta)$), the function

$$(Y^*(\varphi \circ a))(z), \quad z \in C,$$

makes sense, is measurable, satisfies

$$\|Y^*(\varphi \circ a)\|_1 = \frac{1}{2\pi} \int_0^{2\pi} \|Y^*(e^{it})(\varphi \circ a)(e^{it})\|\, dt \leqq$$

$$\leqq \frac{1}{2\pi} \int_0^{2\pi} \|Y^*(e^{it})\|\, \|(\varphi \circ a)(e^{it})\|\, dt \leqq \|Y^*\|_2 \|\varphi \circ a\|_2,$$

and, by virtue of (3.4),

$$\|Y^*(\varphi \circ a)\|_1 \leqq \|Y^*\|_2 \|\varphi\|_2 \|a\|. \tag{3.6}$$

3.7. Lemma. *For any given* $Y \in L^2(\mathcal{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$, $\varphi \in H^2$, *and for any sequence of elements* $a_n \in \mathfrak{F}'$ *weakly tending to* 0, *we have*

$$\|Y^*(\varphi \circ a_n)\|_1 \to 0 \quad as \quad n \to \infty.$$

Proof. As weakly convergent sequences are bounded, we may assume that $\|a_n\| \leqq 1$. Fix $\varepsilon > 0$ and choose a polynomial p such that $\|\varphi - p\|_2 < \varepsilon$ (any sufficiently large partial sum of the power series of φ does it). As we have already proved, $(p \circ a_n)(z)$ converges weakly to 0 in $\mathfrak{F}' \oplus \mathfrak{F}$ for $z \in C$. Compactness of the values of Y implies then that $Y^*(p \circ a_n)(z)$ converges strongly to 0 in $\mathfrak{H}$, i.e.,

$$\|Y^*(p \circ a_n)(z)\| \to 0, \quad z \in C.$$

On the other hand, we deduce from (3.5) that $\|(p\circ a_n)(z)\|\leqq M_p\|a_n\|\leqq M_p$ for a finite constant M_p only dependent on p; and therefore,

$$\|Y^*(p\circ a_n)(z)\| \leqq \|Y^*(z)\|\,M_p.$$

As $\|Y^*(z)\|$ is square integrable by assumption, it is L^1-integrable too, so we can apply the Lebesgue dominated convergence theorem to get

$$\|Y^*(p\circ a_n)\|_1 \to 0.$$

In its turn, (3.6) yields, when applied to $\varphi-p$ in place of φ:

$$\big\|Y^*((\varphi-p)\circ a_n)\big\|_1 \leqq \|Y^*\|_2\|\varphi-p\|_2\|a_n\| \leqq \|Y^*\|_2\varepsilon.$$

So we have

$$\|Y^*(\varphi\circ a_n)\|_1 \leqq \big\|Y^*((\varphi-p)\circ a_n)\big\|_1+\|Y^*(p\circ a_n)\|_1 \leqq \|Y^*\|_2\varepsilon+o(1).$$

As $\varepsilon>0$ was chosen arbitrary this concludes the proof of Lemma 3.7.

A function $Y\in L^2(\mathcal{K}(\mathfrak{H}, \mathfrak{F}'\oplus\mathfrak{F}))$ will be said to be $\mathfrak{H}(\Theta)$-*oriented* if $Yh\in\mathfrak{H}(\Theta)$ for every $h\in\mathfrak{H}$. The following examples will be of particular interest.

3.8. Lemma. *Let* $\varphi\in H^2$ *and* $A\in\mathcal{F}(\mathfrak{H}, \mathfrak{F}')$. *Then there exists a* $\mathfrak{H}(\Theta)$-*oriented function* $\varphi\circ A\in L^2(\mathcal{F}(\mathfrak{H}, \mathfrak{F}'\oplus\mathfrak{F}))$ *such that*

$$(\varphi\circ A)(z)h = (\varphi\circ Ah)(z), \quad z\in C, \quad h\in\mathfrak{H},$$

and

$$\|\varphi\circ A\|_2 \leqq \|\varphi\|_2(\operatorname{rank} A)^{1/2}\|A\|.$$

Proof. Choose an orthonormal basis $\{e_j\}_1^r$ in $(\ker A)^\perp$ ($r=\operatorname{rank} A$); then we have $Ah=\sum_1^r (h, e_j)a_j$ for $a_j=A\,e_j$ and for all $h\in\mathfrak{H}$; and hence,

$$(\varphi\circ Ah)(z) = \sum_1^r (h, e_j)(\varphi\circ a_j)(z), \quad z\in C. \tag{3.9}$$

Recall that each $\varphi\circ a_j$ is a vector in the space $L^2(\mathfrak{F}'\oplus\mathfrak{F})$ (indeed, in its subspace $\mathfrak{H}(\Theta)$), and therefore is a class of equivalent measurable functions. Choose representative functions from each of these classes, say $(\varphi\circ a_j)^\wedge$ ($j=1, \dots, r$), which are defined and finite valued at every point of C. The sum in (3.9) formed with these representatives is linear in $h(\in\mathfrak{H})$ and therefore yields, for every fixed z on C, a linear operator of rank $\leqq r$, which we may denote by $(\varphi\circ A)^\wedge(z)$. By virtue of the inequalities

$$\Big\|\sum_1^r (h, e_j)(\varphi\circ a_j)^\wedge(z)\Big\| \leqq \sum_1^r |(h, e_j)|\,\|(\varphi\circ a_j)^\wedge(z)\| \leqq$$

$$\leqq \Big(\sum_1^r |(h, e_j)|^2\Big)^{1/2}\Big(\sum_1^r \|(\varphi\circ a_j)^\wedge(z)\|^2\Big)^{1/2} \leqq \|h\|\Big(\sum_1^r \|(\varphi\circ a_j)^\wedge(z)\|^2\Big)^{1/2},$$

and also using (3.4) we get

$$\|(\varphi\circ A)^{\wedge}\|_2^2=\frac{1}{2\pi}\int_0^{2\pi}\|(\varphi\circ A)^{\wedge}(e^{it})\|^2\,dt\leqq$$

$$\leqq\sum_1^r\frac{1}{2\pi}\int_0^{2\pi}\|(\varphi\circ a_j)^{\wedge}(e^{it})\|^2\,dt\leqq\sum_1^r\|\varphi\|_2^2\|Ae_j\|^2\leqq\|\varphi\|_2^2 r\|A\|^2.$$

Different choices of the representatives $(\varphi\circ a_j)^{\wedge}$ yield equivalent functions $(\varphi\circ A)^{\wedge}$; denoting their equivalence class by $\varphi\circ A$, relation (3.9) tells us that $(\varphi\circ A)(z)h=(\varphi\circ Ah)(z)$ a.e. on C; and all the further requirements in the lemma are clearly satisfied also. This finishes the proof of Lemma 3.8.

3.10. Lemma. *The operator valued function* $\varphi A\oplus 0$ $(\varphi\in H^2,\ A\in\mathscr{F}(\mathfrak{H},\mathfrak{F}'))$ *defined by* $(\varphi A\oplus 0)(z)h=\varphi(z)Ah\oplus 0$ $(h\in\mathfrak{H},\ z\in C)$ *also belongs to* $L^2(\mathscr{F}(\mathfrak{H},\mathfrak{F}'\oplus\mathfrak{F}))$, *and so does the difference*

$$VT_\Theta^*\varphi A=(\varphi A\oplus 0)-(\varphi\circ A);$$

moreover, we have

$$\|VT_\Theta^*\varphi A\|_2\leqq\|T_\Theta^*|\varphi A\mathfrak{H}\|_2\|\varphi\|_2(\operatorname{rank}A)^{1/2}\|A\|.$$

Proof. The notation is motivated by the fact that, using relation (3.3), we get

$$(VT_\Theta^*\varphi A)(z)h=(\varphi A\oplus 0)(z)h-(\varphi\circ A)(z)h=$$

$$=\big(\varphi(z)Ah\oplus 0\big)-(\varphi\circ Ah)(z)=V(z)(T_\Theta^*\varphi Ah)(z),\quad z\in C.$$

The inequality follows by reasons analogous to those applied in the proof of Lemma 3.8.

Remark. Since $V(z)$ is an isometry, one easily sees that the same inequality holds for $T_\Theta^*\varphi A\in L^2(\mathscr{F}(\mathfrak{H},\mathfrak{F}))$ too.

We shall assume *from now on* that $\dim\mathfrak{F}'=\infty$.

3.11. Lemma. *Assume that* $\varphi\in H^2$ *and that* $A_j\in\mathscr{F}(\mathfrak{H},\mathfrak{F}')$ $(j=1,2,\dots)$ *are such that the norms and ranks are bounded, say*

$$\|A_j\|\leqq M,\quad \operatorname{rank}A_j\leqq N\quad (j=1,2,\dots).$$

If, in addition, the ranges $A_j\mathfrak{H}$ $(j=1,2,\dots)$ *are pairwise orthogonal in* $\mathfrak{F}'$, *then*

$$\lim_{j\to\infty}\|Y^*(\varphi\circ A_j)\|_1=\lim_{j\to\infty}\|(\varphi\circ A_j)^*Y\|_1=0$$

for every $Y\in L^2(\mathscr{K}(\mathfrak{H},\mathfrak{F}'\oplus\mathfrak{F}))$.

Proof. As $(Y^*(\varphi\circ A_j)(z))^*=(\varphi\circ A_j)^*Y(z)$, $z\in C$, equality of the limits is obvious, and therefore we may treat the first limit only.

Because rank $A_j \leqq N$, there exists an orthonormal sequence $\{e_{jn}\}_{n=1}^{N} \subset \mathfrak{H}$ such that

$$A_j h = \sum_{n=1}^{N} (h, e_{jn}) a_{jn} \quad \text{for} \quad a_{jn} = A_j e_{jn}, \quad h \in \mathfrak{H}, \quad \text{and} \quad j = 1, 2, \ldots.$$

It follows that

$$Y^*(\varphi \circ A_j h) = \sum_{n=1}^{N} (h, e_{jn}) Y^*(\varphi \circ a_{jn}).$$

As $|(h, e_{jn})| \leqq \|h\|$ this implies that

$$\|Y^*(\varphi \circ A_j)\|_1 \leqq \sum_{n=1}^{N} \|Y^*(\varphi \circ a_{jn})\|_1.$$

Now, for fixed n, the sequence $\{a_{jn}\}_{j=1}^{\infty}$ is orthogonal (because $a_{jn} \in A_j \mathfrak{H}$) and bounded (by M), and therefore weakly convergent to 0. Thus, by virtue of Lemma 3.7, we have $\lim_{j \to \infty} \|Y^*(\varphi \circ a_{jn})\|_1 = 0$. Summing for n and applying the above inequality we obtain that $\lim_{j \to \infty} \|Y^*(\varphi \circ A_j)\|_1 = 0$, and the proof is done.

4. We shall be keeping $\mathfrak{F}$, $\mathfrak{F}'$ and Θ fixed, with $\dim \mathfrak{F}' = \infty$. For every nonzero function $\varphi \in H^2$ the space $\varphi \mathfrak{F}'$ can be considered as a subspace of $H^2(\mathfrak{F}')$, and hence we can define the operator $A_\varphi \in \mathscr{L}(\varphi \mathfrak{F}', H^2(\mathfrak{F}))$ by

$$A_\varphi(\varphi a) = T_\Theta^*(\varphi a), \quad a \in \mathfrak{F}'.$$

Note that $\varphi \mathfrak{F}'$ is also infinite dimensional, so that the following notation makes sense:

$$\eta_\Theta(\varphi) = \inf \sigma_e((A_\varphi^* A_\varphi)^{1/2}),$$

where σ_e denotes the essential spectrum. This function is in the following relation with the function

$$\hat{\eta}_\Theta(\mu) = \inf \sigma_e((\Theta(\mu)\Theta(\mu)^*)^{1/2}), \quad \mu \in D$$

considered in [2]:

$$\eta_\Theta(p_\mu) = \hat{\eta}_\Theta(\mu), \quad \mu \in D.$$

Indeed, denoting by $\Phi(\mathfrak{F}')$ the set of finite codimensional subspaces of $\mathfrak{F}'$, and applying relation (2.13) of [2], we deduce:

$$\eta_\Theta(p_\mu) = \inf \sigma_e((A_{p_\mu}^* A_{p_\mu})^{1/2}) = \sup_{\mathfrak{F}_0 \in \Phi(\mathfrak{F}')} \inf_{\substack{a \in \mathfrak{F}_0 \\ \|a\| = 1}} \|A_{p_\mu}(p_\mu a)\|_2 =$$

$$= \sup_{\mathfrak{F}_0 \in \Phi(\mathfrak{F}')} \inf_{\substack{a \in \mathfrak{F}_0 \\ \|a\| = 1}} \|T_\Theta^* p_\mu a\|_2 = \sup_{\mathfrak{F}_0 \in \Phi(\mathfrak{F}')} \inf_{\substack{a \in \mathfrak{F}_0 \\ \|a\| = 1}} \|\Theta(\mu)^* a\| =$$

$$= \inf \sigma_e((\Theta(\mu)\Theta(\mu)^*)^{1/2}) = \hat{\eta}_\Theta(\mu).$$

The set

$$R_\Theta = \{\varphi \in H^2 \colon 0 < \|\varphi\|_2 \leqq 1, \ \eta_\Theta(\varphi) = 0\}$$

will play an important role in the sequel. Clearly,

$$R_\Theta \supset \{p_\mu : \mu \in D,\ \hat{\eta}_\Theta(\mu) = 0\}.$$

4.1. Lemma. *Assume that* $\{\varphi_j\}_1^\infty \subset R_\Theta$, *and that* $\{\varepsilon_j\}_1^\infty$ *and* $\{N_j\}_1^\infty$ *are sequences of positive reals and positive integers, respectively. Then there exists a sequence* $\{\mathfrak{F}'_j\}_1^\infty$ *of pairwise orthogonal subspaces of* $\mathfrak{F}'$ *satisfying the following conditions:*

(*i*) $\dim \mathfrak{F}'_j = N_j$, (ii) $\|T_\Theta^* | \varphi_j \mathfrak{F}'_j\| \leqq \varepsilon_j \quad (j = 1, 2, \ldots)$.

Proof. Denote by E_j the spectral measure of the self-adjoint operator $(A_{\varphi_j}^* A_{\varphi_j})^{1/2}$. Since $\eta_\Theta(\varphi_j) = 0$, the space $E_j[0, \varepsilon_j](\varphi_j \mathfrak{F}')$ must be infinite dimensional. A straightforward inductive argument proves the existence of orthogonal subspaces $\mathfrak{F}'_j \subset \mathfrak{F}'$ with $\dim \mathfrak{F}'_j = N_j$ and such that $\varphi_j \mathfrak{F}'_j \subset E_j[0, \varepsilon_j](\varphi_j \mathfrak{F}')$. These subspaces also satisfy condition (ii) so the proof is complete.

We are now almost ready to prove the factorization theorem which is our main aim in this paper. The proof will be by stepwise approximation, and the basic step is as follows.

4.2. Lemma. *Suppose that the set* R_Θ *is dominating for the measurable set* $\sigma \subset C$. *If we are given* $\eta > 0$, *a function* $Z \in L^1(\mathscr{K}(\mathfrak{H}))$, $\mathfrak{H}(\Theta)$*-oriented functions* $X, Y \in L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$, *and a positive number* ω *such that*

$$\|\chi_\sigma(Z - Y^* X)\|_1 < \omega,$$

then there exist $\mathfrak{H}(\Theta)$*-oriented functions* $X', Y' \in L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$ *such that*

(i) $\|\chi_\sigma(Z - Y'^* X')\|_1 < \eta$; *and* (ii) $\|X' - X\|_2 < \omega^{1/2}$, $\|Y' - Y\|_2 < \omega^{1/2}$.

Proof. Fix $\varepsilon > 0$ and ω' such that $\|\chi_\sigma(Z - Y^* X)\|_1 < \omega' < \omega$. An easy application of Lemma 2.6 shows that we can find an integer $n > 0$, functions $\varphi_1, \varphi_2, \ldots, \ldots, \varphi_n \in R_\Theta$, and operators $C_1, C_2, \ldots, C_n \in \mathscr{F}(\mathfrak{H})$ such that

$$\Big\|\chi_\sigma\Big(Z - Y^* X - \sum_{j=1}^n |\varphi_j|^2 C_j\Big)\Big\|_1 < \varepsilon \quad \text{and} \quad \sum_{j=1}^n \|C_j\| < \omega'. \tag{4.3}$$

Choose now a new positive number δ that will also depend on n. We apply Lemma 4.1 to produce a system $\{\mathfrak{F}'_{ij} : 1 \leqq i < \infty,\ 1 \leqq j \leqq n\}$ of pairwise orthogonal subspaces of $\mathfrak{F}'$ such that $\dim \mathfrak{F}'_{ij} = \operatorname{rank} C_j$ and

$$\|T_\Theta^* | \varphi_j \mathfrak{F}'_{ij}\| < \delta \quad (1 \leqq i < \infty,\ 1 \leqq j \leqq n). \tag{4.4}$$

We can then choose isometries $W_{ij} : C_j^* \mathfrak{H} \to \mathfrak{F}'_{ij}$, write the polar decompositions $C_j = U_j A_j$ with $A_j = (C_j^* C_j)^{1/2}$, and set

$$B_{ij} = W_{ij} A_j^{1/2}, \quad D_{ij} = W_{ij} A_j^{1/2} U_j^*.$$

We clearly have $C_j = D_{ij}^* B_{ij}$ and $\|B_{ij}\| = \|D_{ij}\| = \|C_j\|^{1/2} < \omega'^{1/2}$. In addition, the spaces $\{B_{ij}\mathfrak{H}: 1 \leqq i < \infty, 1 \leqq j \leqq n\}$ and $\{D_{ij}\mathfrak{H}: 1 \leqq i < \infty, 1 \leqq j \leqq n\}$ are pairwise orthogonal.

We make a final choice using Lemma 3.11: We choose integers $i_1, i_2, \ldots, i_n$ one by one in this order, such that, upon setting

$$B_j = B_{i_j, j}, \quad D_j = D_{i_j, j} \quad (1 \leqq j \leqq n)$$

we have

$$\|\Phi^*(\varphi_j \circ B_j)\|_1 < \delta, \quad \|\Phi^*(\varphi_j \circ D_j)\|_1 < \delta \quad \text{for} \quad \Phi = X \quad \text{and} \quad Y \ (1 \leqq j \leqq n). \tag{4.5}$$

With all these choices made (we still have to say what conditions ε and δ must satisfy!) we define

$$X' = X + \sum_1^n (\varphi_j \circ B_j), \quad Y' = Y + \sum_1^n (\varphi_j \circ D_j).$$

From the general relation $\varphi \circ A = (\varphi A \oplus 0) - VT_\Theta^* \varphi A$, where V is the isometry in (3.1), $\varphi \in H^2$, and $A \in \mathcal{F}(\mathfrak{H}, \mathfrak{F}')$, we have

$$\|X' - X\|_2 \leqq \Big\|\sum_1^n \varphi_j B_j\Big\|_2 + \sum_1^n \|T_\Theta^* \varphi_j B_j\|_2.$$

Since the operators B_j have pairwise orthogonal ranges, we have

$$\Big\|\sum_1^n \varphi_j B_j\Big\|_2 \leqq \Big(\sum_1^n \|B_j\|^2\Big)^{1/2} = \Big(\sum_1^n \|C_j\|\Big)^{1/2}.$$

On the other hand, using inequality (4.4) we deduce from Lemma 3.10 that

$$\|T_\Theta^* \varphi_j B_j\|_2 \leqq \delta (\text{rank } B_j)^{1/2} \|B_j\| = \delta (\text{rank } C_j)^{1/2} \|C_j\|^{1/2}. \tag{4.6}$$

We conclude:

$$\|X' - X\|_2 \leqq \omega'^{1/2} + \delta \omega'^{1/2} \sum_1^n (\text{rank } C_j)^{1/2}.$$

The same inequality obviously holds for $\|Y' - Y\|_2$ too. It is now clear that δ can be chosen so that the inequalities (ii) of the statement are verified (note that δ is chosen *after* the C_j, $1 \leqq j \leqq n$).

In order to verify (i) we first note that the orthogonality of the spaces $\mathfrak{F}'_{ij}$ implies that

$$(\varphi_i D_i)^* \varphi_j B_j = 0 \quad (i \neq j), \quad (\varphi_j D_j)^* \varphi_j B_j = |\varphi_j|^2 D_j^* B_j = |\varphi_j|^2 C_j.$$

We have

$$Y'^* X' = Y^* X + Y^* \Big(\sum_1^n \varphi_j \circ B_j\Big) + \Big(\sum_1^n \varphi_j \circ D_j\Big)^* X + Q,$$

where

$$Q = \sum_1^n \sum_1^n (\varphi_j \circ D_j)^*(\varphi_i \circ D_i) = \sum_1^n (\varphi_j D_j \oplus 0)^*(\varphi_j B_j \oplus 0) +$$

$$+ \sum_1^n \big((\varphi_j D_j \oplus 0)^* R + S^*(\varphi_j B_j \oplus 0)\big) + S^* R,$$

with

$$R = \sum_1^n VT_\Theta^*(\varphi_j B_j), \quad S = \sum_1^n VT_\Theta^*(\varphi_i D_i).$$

Comparing these relations and applying the obvious inegality $\|LM\|_1 \leqq \|L\|_2 \|M\|_2$ we obtain:

$$\|\chi_\sigma(Z-Y'^*X')\|_1 \leqq \Big\|\chi_\sigma(Z-Y^*X-\sum_1^n |\varphi|_j^2 C_j\Big\|_1 + \sum_1^n \big(\|Y^*(\varphi_j \circ B_j)\|_1 +$$

$$+\|(\varphi_j \circ D_j)^* X\|_1\big) + \sum_1^n (\|\varphi_j\|_2 \|D_j\| \cdot \|R\|_2 + \|S\|_2 \cdot \|\varphi_j\|_2 \|B_j\|) + \|S\|_2 \|R\|_2.$$

Applying inequalities (4.3), (4.5), (4.6) and the fact that $\|D_j\| = \|B_j\| = \|C_j\|^{1/2} < \omega$, we infer

$$\|\chi_\sigma(Z-Y'^*X')\|_2 \leqq \varepsilon + 2n\delta + 2\delta\omega \sum_1^n (\text{rank } C_j)^{1/2} + \delta^2\omega \big(\sum_1^n (\text{rank } C_j)^{1/2}\big)^2,$$

and it clearly follows that (i) is verified if ε and δ are chosen appropriately small. The lemma is proved.

We prove now the main result of this paper, which is a rather standard self-improvement of Lemma 4.2. It actually generalizes Theorem A of [2], case $\vartheta=0$.

4.7. Theorem. *Suppose that the set R_Θ is dominating for the measurable set $\sigma \subset C$. If we are given a function $Z \in L^1(\mathscr{K}(\mathfrak{H}))$, and $\mathfrak{H}(\Theta)$-oriented functions $X, Y \in L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$, then for any number ω for which $\|\chi_\sigma(Z-Y^*X)\|_1 < \omega$, there exist $\mathfrak{H}(\Theta)$-oriented functions $X', Y' \in L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$ such that*

(i) *$Z(z) = (Y'^*X')(z)$ for almost every $z \in \sigma$, and*

(ii) *$\|X'-X\|_2 < \omega^{1/2}$, $\|Y'-Y\|_2 < \omega^{1/2}$.*

Proof. Choose ω' such that $\|\chi_\sigma(Z-Y^*X)\|_1 < \omega' < \omega$, and a positive number $\vartheta < 1$ such that $(1-\vartheta^{1/2})^{-1}\omega'^{1/2} < \omega^{1/2}$. Set $X_0 = X$ and $Y_0 = Y$. An inductive application of Lemma 4.2 shows the existence of $\mathfrak{H}(\Theta)$-oriented functions X_n, Y_n $(n \geqq 1)$ satisfying for $n \geqq 0$ the inequalities

$$\|\chi_\sigma(Z-Y_n^* X_n)\|_1 < \vartheta^n \omega', \quad \text{and} \quad \|X_{n+1}-X_n\|_2,\ \|Y_{n+1}-Y_n\|_2 \leqq (\vartheta^n \omega')^{1/2}.$$

These inequalities show that $\{X_n\}$ and $\{Y_n\}$ are Cauchy sequences in $L^2(\mathscr{K}(\mathfrak{H}, \mathfrak{F}' \oplus \mathfrak{F}))$

and, upon setting $X'=\lim_{n\to\infty} X_n$, $Y'=\lim_{n\to\infty} Y_n$, we have

$$\|X'-X\|_2 \leqq \sum_{n=0}^{\infty} \|X_{n+1}-X_n\|_2 \leqq \sum_{n=0}^{\infty} (\vartheta^n\omega')^{1/2} = (1-\vartheta^{1/2})^{-1}\omega'^{1/2} < \omega^{1/2}$$

and, analogously, $\|Y'-Y\|_2<\omega^{1/2}$. It also follows that $\|\chi_\sigma(Z-Y'^*X')\|_1=0$, and this concludes the proof of our theorem.

We remind the reader of the fact from [2] that the relations in the introductory part of Section 4 imply that the assumption of Theorem 4.7 concerning the set R_Θ is certainly satisfied if the right essential spectrum of the model operator $S(\Theta)$ is dominating for the set σ.

A natural continuation of the circle of ideas in this paper takes place in [1], where applications to invariant subspaces and reflexivity are discussed.

The authors owe to Dr. L. Kérchy for some useful suggestions he has made when reading a draft of this paper.

References

[1] H. Bercovici, C. Foiaş, and C. Pearcy, Factoring trace-class operator-valued functions with applications to the class $A_{\aleph_0}$, *J. Operator Theory* (to appear).

[2] H. Bercovici, C. Foiaş, C. Pearcy, and B. Sz.-Nagy, Functional models and extended spectral dominance, *Acta Sci. Math.*, **43** (1981), 243—254.*)

[3] A. Brown and C. Pearcy, *Introduction to operator theory I: Elements of functional analysis*, Springer (New York, 1977).

[4] L. Brown, A. Shields, and K. Zeller, On absolutely convergent exponential sums, *Trans. Amer. Math. Soc.*, **96** (1960), 162—183.

[5] P. Duren, *Theory of H^p spaces*, Academic Press (New York, 1970).

[6] C. Foiaş, C. Pearcy and B. Sz.-Nagy, The functional model of a contraction and the space L^1, *Acta Sci. Math.*, **41** (1979), 403—410.

[7] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, Akadémiai Kiadó — North-Holland (Budapest—Amsterdam, 1970).

[8] B. Sz.-Nagy and C. Foiaş, The functional model of a contraction and the space L^1/H_0^1, *Acta Sci. Math.*, **41** (1979), 403—410.

(H. B.)
M.S.R.I.
2223 FULTON STREET
BERKELEY, CA 94720, U.S.A.

(C. F.)
DEPARTMENT OF MATH.
INDIANA UNIV.
BLOOMINGTON, IN 47405, U.S.A.

(C. P.)
DEPARTMENT OF MATH.
UNIVERSITY OF MICHIGAN
ANN ARBOR, MI 48109, U.S.A.

(B. SZ.-N.)
BOLYAI INSTITUTE
UNIVERSITY SZEGED
SZEGED, HUNGARY 6720

*) It should be remarked that the proof of Lemma 2.4 in [2], p. 249, needs some correction because the term $(\omega^{1/2}r^{1/2}+\omega^{1/2}r^{1/2}+\omega r)\varepsilon$ in the estimate of $\|\Omega\|$ contains the number r depending on ε, and therefore may not be small for ε small. However, this situation can be easily remedied by requiring in the choice of the sequence $\{b_m\}_1^r$ in (2.23), that $\|l(\mu_m\circ b_m)^*\|_1\leqq\varepsilon/r$, instead of $\leqq\varepsilon$. (The term in question changes then to $(2\omega^{1/2}+\omega)\varepsilon$.)

Further minor corrections for the same page in [2]: in the 2nd rows from above and from below change S for S_θ, and (2.19) for (2.20), respectively.