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Note on a theorem of Dieudonné

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DIEUDONNÉ [2] has proved that for any $f \in L^1(A)$, $f * L^1(A) \neq L^1(A)$, where A is a nondiscrete, locally compact abelian group. Applying Banach algebra methods we shall prove the same result for $L^1(G)$ over a compact, connected Lie group G.

Dieudonné has proved the above result by applying the methods of harmonic analysis on LCA groups. Later this theorem was proved by GOLDBERG and BURNHAM [3] by applying Banach algebra methods. We shall follow their ideas, but since in our case the algebra $L^1(G)$ is not commutative in general, the proof is much more difficult.

We start by recalling a few notions from Banach algebras. Let B be a complex Banach algebra.

Definition 1. We say that $b \in B$ is a divisor of zero, if $rb = br = 0$ for some $r \in B$, $r \neq 0$.

Definition 2. We say that $a \in B$ is a topological divisor of zero, if there exists a sequence $\{g_n\} \subset B$ such that $\|g_n\| \geqq \delta > 0$ $(n = 1, 2, \ldots)$ but $\|ag_n\| + \|g_n a\| \to 0$, as $n \to \infty$.

We have the following simple results on topological divisors of zero in Banach algebras.

(1) If $a \in B$ is a topological divisor of zero, but not a divisor of zero, then $aB \neq B$.

(2) Let D be a dense subset of B. Assume that for a certain sequence $\{x_n\} \subset B$, $\|x_n\| \geqq \delta > 0$ $(n = 1, 2, \ldots)$, $\|x_n d\| + \|dx_n\| \to 0$, as $n \to \infty$, for every $d \in D$. Then every element of B is a topological divisor of zero in B.

In what follows we assume that the reader is familiar with the basic theory of compact Lie groups, as is presented for example in [1]. Let G be a compact, connected Lie group. Denote by $\hat{G}$ its dual. For $h \in L^p(G)$ $(p \geqq 1)$ we denote by $\|h\|_p$ the L^p-norm. For $\alpha \in \hat{G}$ and $T_\alpha \in \alpha$ the character function $\varphi_\alpha(g) = \mathrm{Tr}\, T_\alpha(g)$ is continuous on G.

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Lemma. *Let G be a compact, connected, non-abelian Lie group. Then for every $h \in L^2(G)$ we have*

(i) $|h*\varphi_\alpha(g)| \leqq M_h, \ \forall \alpha \in \hat{G}$,

(ii) $h*\varphi_\alpha(g) = \varphi_\alpha * h(g) \to 0$ *as* $\alpha \to \{\infty\}$,

(iii) *there exists* $\delta > 0$ *such that* $\|\varphi_\alpha\| \geqq \delta$ *for a certain* $\alpha \to \{\infty\}$.

Proof. (i) $|h*\varphi_\alpha(g)| \leqq \int |h(x)\cdot\varphi_\alpha(gx^{-1})|dx \leqq \|h\|_2\|\varphi_\alpha\|_2 = \|h\|_2$.

(ii) Let $\hat{h}(\alpha) = \int h(x)\, T_\alpha(x)^*\, dx$; here $T_\alpha(x)^*$ denotes the adjoint of $T_\alpha(x) \in L(H_\alpha)$ ($L(H_\alpha)$ stands for all linear operators in H_α). Assume that $\dim H_\alpha = N_\alpha$. We have

$$\sum_{\alpha \in \hat{G}} N_\alpha \|\hat{h}(\alpha)\|_2^2 = \|h\|_2^2,$$

where $\|\hat{h}(\alpha)\|_2^2 = \operatorname{Tr} \hat{h}(\alpha)^* \hat{h}(\alpha)$.

Since

$$[\operatorname{Tr} \hat{h}(\alpha)^* \hat{h}(\alpha) \cdot N_\alpha]^{1/2} \to 0 \quad \text{as} \quad \alpha \to \{\infty\}$$

and

$$h*\varphi_\alpha(g) = \int h(s^{-1}g) \operatorname{Tr} T_\alpha(s)\, ds = \int h(x) \operatorname{Tr} T_\alpha(g)\, T_\alpha(x)^*\, dx = \operatorname{Tr} T_\alpha(g)\, \hat{h}(\alpha),$$

therefore

$$|h*\varphi_\alpha(g)| = |\operatorname{Tr} T_\alpha(g)\hat{h}(\alpha)| \leqq [N_\alpha \operatorname{Tr} \hat{h}(\alpha)^* \hat{h}(\alpha)]^{1/2} \to 0 \quad \text{as} \quad \alpha \to \{\infty\}.$$

(iii) Let T be a maximal torus in G. Since $\varphi_\alpha(g_1 g_2) = \varphi_\alpha(g_2 g_1)$, $\forall \alpha \in \hat{G}$, applying Weyl's theorem [1, Th. 6.1] we have

$$\int |\varphi_\alpha(g)|\, dg = \int_T |\varphi_\alpha(t)|\, u(t)\, dt,$$

where $u(t) = |p(t)|^2 |W|^{-1}$, $|p(t)|^2 = \prod_{j=1}^{m} 4\sin^2 \pi\theta_j(t)$, $|W| \in \mathbf{N}$ is a universal integer, and $\Theta_1, \ldots, \Theta_m$ are distinct roots of G. But T is commutative, so

$$\varphi_\alpha(t) = \sum_{k=1}^{N_\alpha} \exp\big(2i\pi\lambda_k^{(\alpha)}(t)\big),$$

where $\lambda_k^{(\alpha)}(t)$ are real. Assume that $\dim T = n$. Then we have

$$\lambda_k^{(\alpha)}(t_1, \ldots, t_n) = \sum_{p=1}^{n} a_{kp}^{(\alpha)} t_p, \quad a_{kp}^{(\alpha)} \in \mathbf{Z}, \ \forall k, p.$$

Thus

$$|W| \int |\varphi_\alpha(t)|\, u(t)\, dt = \int_{I_n} \int_0^1 |\exp(2\pi i a_{11}^{(\alpha)} t_1) A_1(\bar{t}) + \ldots + \exp(2\pi i a_{N_\alpha 1}^{(\alpha)} t_1) A_{N_\alpha}(\bar{t})|\, u(t)\, dt,$$

where $t=(t_1, \bar{t})$ and $|A_s(\bar{t})|=1$, $s=1, 2, \ldots, N_\alpha$, $I_n=[0, 1]^{n-1}$. Hence

$$|W|\int|\varphi_\alpha(t)|u(t)\,dt=$$
$$=\int_{I_n}\int_0^1\left|1+\exp\left(2\pi i(a_{21}^{(\alpha)}-a_{11}^{(\alpha)})t_1\right)A_2(\bar{t})\bar{A}_1(\bar{t})+\ldots\right.$$
$$\left.\ldots+\exp\left(2\pi i(a_{N_\alpha 1}^{(\alpha)}-a_{11}^{(\alpha)})t_1\right)A_{N_\alpha}(\bar{t})\bar{A}_1(\bar{t})\right|u(t_1, \bar{t})\,dt_1\,d\bar{t}.$$

Choose $\alpha\to\{\infty\}$ such that $a_{kp}^{(\alpha)}-a_{11}^{(\alpha)}\neq 0$ for every k, p. Applying Szegö's theorem we have

$$\int_{I_n}\int_0^1\left|1+\ldots+\exp\left(2\pi i(a_{N_\alpha 1}^{(\alpha)}-a_{11}^{(\alpha)})t_1\right)A_{N_\alpha}(\bar{t})\bar{A}_1(\bar{t})\right|u(t_1, \bar{t})\,dt_1\,d\bar{t}\geqq$$
$$\geqq\int_{I_n}\exp\int_0^1\log u(t_1, \bar{t})\,dt_1\,d\bar{t}.$$

Since

$$\int_0^1\log\sin^2 r\,dr>-\infty,$$

so

$$\exp\int_0^1\log u(t_1, \bar{t})\,dt_1\geqq\delta(\bar{t})>0$$

and is a continuous function of $\bar{t}\in I_n$. Hence

$$\int_{I_n}\int_0^1\left|1+\ldots+\exp\left(2\pi i(a_{N_\alpha 1}^{(\alpha)}-a_{11}^{(\alpha)})t_1\right)A_{N_\alpha}(\bar{t})\bar{A}_1(\bar{t})\right|u(t_1, \bar{t})\,dt_1\,d\bar{t}\geqq\delta,$$

for a certain $\delta>0$. Note also that the number

$$\int_{I_n}\exp\int_0^1\log u(t_1, \bar{t})\,dt_1\,d\bar{t}$$

does not depend on α, and so

$$|W|\int|\varphi_\alpha(t)|u(t)\,dt\geqq\delta$$

for every $\alpha\in\hat{G}$. The proof is complete.

As is well known, no $h\in L^1(G)$ $(h\neq 0)$ is a divisor of zero in $L^1(G)$. Hence applying Lemma, (1), and (2) we get

Theorem. *Let G be a compact, connected Lie group. Then for every $h\in L^1(G)$ the mapping $L^1(G)\ni g\to h*g\in L^1(G)$ is not surjective.*

Proof. If G is abelian, the result holds by the theorem of Dieudonné. Hence we can assume that G is not abelian. By (i) and (ii) of Lemma and the Lebesgue dominated convergence theorem we have $\|h*\varphi_\alpha\|_1 \to 0$ as $\alpha \to \{\infty\}$, for any $h \in L^2(G)$. Application of (1) and (2) ends the proof.

Remark 1. Since $L^p(G)$ is $L^1(G)$ module, for $p \geqq 1$, the above theorem can be easily extended to $L^p(G)$. Namely, for every $h \in L^1(G)$ the mapping $L^p(G) \ni g \to h * g \in L^p(G)$ is not surjective. The proof is the same as before (note that $\|\varphi_\alpha\|_p \geqq \|\varphi_\alpha\|_1$, $\forall \alpha \in \hat{G}$).

Remark 2. It seems that the above result is also true in the context of nilpotent Lie groups (this is true for the Heisenberg group of arbitrary dimension).

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