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An elementary minimax theorem

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A recent simple proof for von Neumann's minimax theorem by I. Joó [2] urged us to formulate a minimax principle as a direct property of the function in question. In consequence our approach omits the usual convextity requirements. However, our proof is simple by applying a finite dimensional separation result concerning convex sets. In fact we use a modified version of a proof taken from BALAKRISHNAN [1]. Our theorem generalizes some of the known results of this type.

Theorem. *Let $f(x, y)$ be a real-valued function on $X\times Y$ with the following three properties:*

$$\min_{y\in B}\sum_{x\in A}\lambda(x)f(x,y)\leqq\sup_{x\in X}\min_{y\in B}f(x,y), \tag{1^x}$$

where $A\subset X$ and $B\subset Y$ are finite subsets and $\lambda\colon A\to\mathbf{R}_+$ is a discrete probability measure on A.

$$\inf_{y\in Y}\sup_{x\in X}f(x,y)\leqq\sup_{x\in X}\sum_{y\in B}\mu(y)f(x,y), \tag{2^y}$$

where $B\subset Y$ is a finite subset and $\mu\colon B\to\mathbf{R}_+$ is a discrete probability measure on B.

(3) *There exist $y_0\in Y$ and $c_0\in\mathbf{R}$, $c_0<\inf_{y\in Y}\sup_{x\in X}f(x,y)=c^*$ such that if $D\subset\subset(c_0,\infty)\times Y$ is a subset with the property that for any $x\in X$, $f(x,y_0)\geqq c_0$, there exists $(t_x,y_x)\in D$ with $f(x,y_x)<t_x$ then there exists a finite subset in D with the same property.*

Then

$$c_*=\sup_{x\in X}\inf_{y\in Y}f(x,y)=\inf_{y\in Y}\sup_{x\in X}f(x,y)=c^*. \tag{4}$$

Proof. Since $\inf_{y\in Y}f(x,y)\leqq f(x,y)$ holds for any $x\in X$, $y\in Y$, the inequality $c_*\leqq\sup_{x\in X}f(x,y)$ follows for any $y\in Y$, showing that $c_*\leqq c^*$. To prove (4) we start with $c_*<c^*$ and for a c, $\max\{c_*, c_0\}<c<c^*$, write $H_y=\{x\in X: f(x,y)\geqq c\}$ for

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any $y \in Y$. Showing that some $x_0 \in X$ belongs to $\cap\{H_y: y \in Y\}$ we get a contradiction:

$$c \leqq \inf_{y \in Y} f(x_0, y) \leqq \sup_{x \in X} \inf_{y \in Y} f(x, y) = c_*.$$

To do this let first $B = \{y_1, \ldots, y_n\}$ be a finite subset in Y, and suppose that $\cap\{H_y: y \in B\}$ is empty. Then for any $x \in X$ there exists a $y \in B$ such that $f(x, y) < c$. As a consequence, the function $\varphi: X \to \mathbf{R}^n$, given by

$$\varphi(x) = (f(x, y_1) - c, \ldots, f(x, y_n) - c)$$

has the following property: $\varphi(A) \cap \mathbf{R}^n_+ = \emptyset$, where $\varphi(A)$ is the range of φ and $\mathbf{R}^n_+$ is the positive cone of vectors with nonnegative coordinates in $\mathbf{R}^n$. But then $\mathrm{Co}\varphi(A)$, the convex hull of the range of φ, does not meet $\mathrm{int}\,\mathbf{R}^n_+$, the interior of $\mathbf{R}^n_+$. There were otherwise a discrete probability measure $\lambda: X \to \mathbf{R}_+$ with finite support A, $A = \{x_1, \ldots, x_m\} \subset X$, such that $c < \sum_{j=1}^{m} \lambda_j f(x_j, y_i)$ holds for any $i = 1, \ldots, n$. But (1^x) implies then

$$c < \min_{1 \leqq i \leqq n} \sum_{j=1}^{m} \lambda_j f(x_j, y_i) \leqq \sup_{x \in X} \min_{1 \leqq i \leqq n} f(x, y_i),$$

contradicting the assumption that $\cap\{H_y: y \in B\}$ is empty. As a result we have a nonzero separating linear functional $\mu = (\mu_1, \ldots, \mu_n) \in \mathbf{R}^n$ (see e.g. [2, 2.5.1]) such that

$$\sum_{i=1}^{n} \mu_i f(x, y_i) - c \sum_{i=1}^{n} \mu_i \leqq \sum_{i=1}^{n} \mu_i t_i \quad \text{for any} \quad x \in X, \quad t = (t_1, \ldots, t_n) \in \mathbf{R}^n_+.$$

In this case $\mu \in \mathbf{R}^n_+$ is obvious so that we may assume that $\sum_{i=1}^{n} \mu_i = 1$ also holds. As a consequence

$$c^* = \inf_{y \in Y} \sup_{x \in X} f(x, y) \leqq \sup_{x \in X} \sum_{i=1}^{n} \mu_i f(x, y_i) \leqq c;$$

a contradiction follows by (2^y) and the choice of c. Summing up, we have proved that $\cap\{H_y: y \in B\}$ is nonempty for any finite subset B in Y. For $B = Y$ we get the same conclusion if we topologize X by chosing the subsets $\{x \in X: f(x, y) < t\}$ $(t \in \mathbf{R}, y \in Y)$ in X as a subbase for open sets such that $\{H_y\}_{y \in Y}$ are closed sets and $\{x \in X: f(x, y_0) \geqq c_0\}$ is compact by (3). Indeed, the finite intersection property of F. Riesz implies the desired conclusion. The proof is thus complete.

Corollary. *Let $f(x, y)$ be a real-valued function on $X \times Y$ with finite X such that (1^x) (with $A = X$) and (2^y) hold. Then (4) also holds.*

References

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