Acta Sci. Math. 47 (1984), 445—448

Normal composition operators

B. S. KOMAL and D. K. GUPTA

1. Preliminaries

Let $(X, \mathcal{S}, \lambda)$ be a σ-finite measure space and let T be a measurable non-singular $(\lambda T^{-1}(E)=0$ whenever $\lambda(E)=0)$ transformation from X into itself. Then the equation

$$C_T f = f \circ T \quad \text{for every} \quad f \in L^p(\lambda)$$

defines a composition transformation C_T from $L^p(\lambda)$ into the space of all complex valued functions on X. If the range of C_T is contained in $L^p(\lambda)$ and C_T turns out to be a bounded operator on $L^p(\lambda)$, then we call it a composition operator induced by T. Let $X=N$, the set of all (non-zero) positive integers and $\mathcal{S}=P(N)$, the power set of N. Suppose $w=\{w_n\}$ is a sequence of (non-zero) positive real numbers. Define a measure λ on $P(N)$ by

$$\lambda(E) = \sum_{n \in E} w_n \quad \text{for every} \quad E \in P(N).$$

Then for $p=2$, $L^p(\lambda)$ is a Hilbert space with the inner product defined on it by

$$\langle f, g \rangle = \sum_{n=1}^{\infty} w_n f(n) \overline{g(n)} \quad \text{for every} \quad f, g \in L^p(\lambda).$$

This space is called a weighted sequence space with weights $\{w_n : n \in N\}$ and is denoted by l^2_w. The symbol $B(l^2_w)$ denotes the Banach algebra of all bounded linear operators on l^2_w and the symbol f_0 denotes the Radon—Nikodym derivative of the measure λT^{-1} with respect to the measure λ.

Received November 30, 1982.

2. Normal composition operators

A bounded linear operator A on a Hilbert space is called normal if it commutes with A^*. The operator A is called quasinormal if it commutes with A^*A. In [7] SINGH, KUMAR and GUPTA made the study of these operators on a weighted sequence space l^2_w when $\sum_{n=1}^{\infty} w_n < \infty$. WHITLY [8] has studied the seoperators on $L^2(\lambda)$, when the underlying space is a finite measure space. He has proved that the class of unitary composition operators coincides with the class of normal composition operators. In our present note we have generalised this result to $L^2(\lambda)$, when the underlying measure space is a special type of σ-finite measure space. Some results on quasinormal, isometric and co-isometric composition operators are also reported.

We shall first give an example to show that a normal composition operator may not be unitary.

Example 2.1. Let $T\colon N \to N$ be the mapping defined by

$$T(n) = \begin{cases} 2 & \text{if } n = 1; \\ n+2 & \text{if } n \text{ is an even integer;} \\ n-2 & \text{if } n \text{ is an odd integer } > 1. \end{cases}$$

Let the sequence $\{w_n\}$ of weights be defined by

$$w_n = \begin{cases} 1 & \text{if } n = 1, \\ 1/2^n & \text{if } n \text{ is an ever integer,} \\ 2^{n-1} & \text{if } n \text{ is an odd integer } > 1. \end{cases}$$

Then $f_0(n) = 4$ for every $n \in N$. Hence in view of Theorem 1 of [5] C_T is a bounded operator. Clearly $f_0(n) = f_0(T(n))$ for every $n \in N$. Since T is an injection, $T^{-1}(\mathscr{S}) = \mathscr{S}$. Hence by Lemma 2 of [8] C_T is normal. Since $C_T^* C_T = M_{f_0} = 4I$, it is clear that C_T is not unitary.

If the sequence $\{w_n\}$ is a convergent sequence of positive real numbers, then every normal operator becomes unitary. It is given in the following theorem. We shall first give a definition.

Definition. Let $T\colon N \to N$ be a mapping. Then two integers m and n are said to be in the same orbit of T if each can be reached from the other by composing T and T^{-1} (T^{-1} means a multivalued function) sufficiently many times.

Theorem 2.2. *Let $C_T \in B(l^2_w)$ and let $w = \{w_n\}$ be a convergent sequence of positive real numbers. Then the following are equivalent:*

(i) *C_T is unitary,*

(ii) *C_T is normal.*

Proof. The implication (i)⇒(ii) is true for any bounded operator A. We show that (ii)⇒(i). Assume that C_T is normal. Then by Lemma 2 of WHITLEY [8], $f_0=f_0\circ T$ and $T^{-1}(\mathcal{S})=\mathcal{S}$. From Lemma 1 of WHITLEY [8], C_T has dense range and hence C_T is onto in view of the normality of C_T. Thus by Corollary 2.3 and Corollary 2.5 of SINGH and KUMAR [6] T is invertible. Let $n_i\in T^{-1}(\{n\})$. Then $f_0(n_i)=f_0(T(n_i))=$ $=f_0(n)$. Similarly, we can show that f_0 is constant on the orbit of n. Further let $n_0\in N$. Then $O(n_0)$, the orbit of n_0 is either a finite set or it is an infinite set. We first suppose that $O(n_0)$ is a finite set. Then there is an integer m in N such that $T^m(n_0)=n_0$. Now $f_0(T(n))=f_0(T^2(n_0))=\ldots=f_0(T^m(n_0))$. Equivalently,

$$\frac{w_{n_0}}{w_{n_1}}=\frac{w_{n_1}}{w_{n_2}}=\ldots=\frac{w_{n_{m-1}}}{w_{n_m}}=\beta \quad \text{(say)}$$

where $T^k(n_0)=n_k$ for $k\leqq m$, and $n_m=n_0$. From this we get $w_{n_k}=w_{n_0}/\beta^k$ for $k\leqq m$ and hence $\beta^m=1$ which implies that $\beta=1$. Next, if $O(n_0)$ is an infinite subset of N, then $T^k(n_0)\neq n_0$ for every $k\in N$. Let $(T^k)^{-1}(n_0)=n_{-k}$. Then f_0 is constant on the set $\{n_k\colon k\in Z\}$, where Z is the set of all integers. Thus as shown in the first case $w_{n_k}=w_{n_0}/\beta^k$ (i) and $w_{n_{-k}}=\beta^k w_{n_0}$ (ii). If $\beta\neq 1$, then at least one of the subsequences (i) and (ii) is divergent. This contradicts the fact that every subsequence of a convergent sequence is convergent. Hence $\beta=1$. Thus $f_0(n)=1$ for every $n\in N$. This implies that C_T is an isometry by [3]. Since C_T has dense range, we can conclude that C_T is unitary.

Theorem 2.3. *Let $C_T\in B(l_w^2)$. Then C_T^* is an isometry if and only if $w=w\circ T$ and T is an injection.*

Proof. Suppose C_T is a co-isometry. Then $C_TC_T^*e_n'=e_n'$, where $e_n'=X_{\{n\}}/w_n$, $X_{\{n\}}$ being the sequence each terms of which is 0, except for the nth one which equals 1. Using the definition of C_T^* [5], we get $C_Te_{T(n)}'=e_n'$. This implies that

$$X_{T^{-1}(\{T(n)\})}/w_{T(n)}=X_{\{n\}}/w_n.$$

Hence we can conclude that T is an injection and $w=w\circ T$.

Conversely, if $w=w\circ T$ i.e. $w_n=w_{T(n)}$ for every $n\in N$ and T is an injection then $C_TC_T^*e_n'=e_n'$. Let $f\in l_w^2$. Then

$$(C_TC_T^*f)(n)=\langle C_TC_T^*f, e_n'\rangle=\langle f, C_TC_T^*e_n'\rangle=\langle f, e_n'\rangle=f(n)$$

for every $n\in N$. Hence $C_TC_T^*f=f$ for every $f\in l_w^2$. This completes the proof of the theorem.

Theorem 2.4. *Let $T\colon N\to N$ be an injection such that $C_T\in B(l_w^2)$. Then the following are equivalent:*

(i) *C_T^* is an isometry,*

(ii) *C_T is a partial isometry,*

(iii) $w=w\circ T$.

Proof. (i) implies (ii): Suppose C_T^* is an isometry. Then C_T^* is a partial isometry. Hence C_T is a partial isometry [1, p. 96]. (ii) implies (iii): If C_T is a partial isometry, then from a corollary to Problem 98 of [2] $C_T C_T^* C_T = C_T$. Since $C_T^* C_T = M_{f_0}$, this implies that $M_{f_0 \circ T} C_T = C_T$. Thus $f_0 \circ T = 1$ on the range of C_T. Now T is an injection, therefore by Corollary 2.4 of [6] C_T has dense range. Hence $(f_0 \circ T)(n) = 1$ for every $n \in N$. Thus $T^{-1}(\{T(n)\})/T(n) = 1$ for every $n \in N$ which implies that $w_n = w_{T(n)}$ for every $n \in N$. Hence $w = w \circ T$. (iii) implies (i): This proof is given in Theorem 2.3.

Whitley [8] has given an example to show that not every quasinormal composition operator is normal. We show that if T is an injection, then every quasinormal composition operator becomes normal. It is given in the following theorem.

Theorem 2.5. *Let $T: N \to N$ be an injection such that $C_T \in B(l_w^2)$. Then the following are equivalent:*

(i) *C_T is normal,*

(ii) *C_T is quasinormal,*

(iii) *C_T is an isometry.*

Proof. (i)⇒(ii) is trivial; (ii)⇒(iii) follows from Theorem 2 of [8]. (iii)⇒(i): By the assumption of the theorem, Corollary 2.4 of [6] guarantees that C_T has dense range.

References

[1] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press Inc. (New York—London, 1972).

[2] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand (Princeton, N. J., 1967).

[3] R. K. Singh, Normal and Hermitian composition operators, *Proc. Amer. Math. Soc.*, **47** (1975), 348—350.

[4] R. K. Singh, Composition operators induced by rational functions, *Proc. Amer. Math. Soc.*, **59** (1976), 329—333.

[5] R. K. Singh and D. K. Gupta, Normal and invertible composition operators, *Bull. Austral. Math. Soc.*, **18** (1978), 45—54.

[6] R. K. Singh and A. Kumar, Characterization of invertible, unitary and normal composition operators, *Bull. Austral. Math. Soc.*, **19** (1978), 81—95.

[7] R. K. Singh, A. Kumar and D. K. Gupta, Quasinormal composition operators, *Indian J. Pure Appl. Math.*, **11** (1980), 904—907.

[8] R. Whitley, Normal and quasinormal composition operators, *Proc. Amer. Math. Soc.*, **70** (1978), 114—118.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF JAMMU
JAMMU-180001, INDIA