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On distances between unitary orbits of self-adjoint operators

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Dedicated to Béla Szőkefalvi-Nagy

1. Introduction

In this paper we study distances between unitary equivalence classes of self-adjoint operators. Our starting point is the following fact, observed by H. Weyl [10, Theorem 1].

Theorem 1.1. *Let A and B be self-adjoint operators acting on a finite-dimensional Hilbert space, and write $\alpha_1 \leqq \alpha_2 \leqq \ldots \leqq \alpha_n$ and $\beta_1 \leqq \beta_2 \leqq \ldots \leqq \beta_n$ for their eigenvalues, repeated according to multiplicity. Then*

$$\|A-B\| \geqq \max_j |\alpha_j - \beta_j|. \tag{1.1}$$

There are several alternate expressions for the number $\max|\alpha_j-\beta_j|$, but for now, we only want to emphasize the fact that it can be computed from the multiplicity functions α and β of A and B respectively, so we denote it by $\delta(\alpha, \beta)$. In particular, (1.1) persists if A and B are replaced by unitary transforms. In fact, if these transforms are chosen to have a common basis of eigenvectors corresponding to the ordered sets of eigenvalues in the Theorem, then equality will hold in (1.1). This leads to the following restatement of Theorem 1.1.

Theorem 1.2. *Let A and B be self-adjoint operators acting on a finite-dimensional Hilbert space, and write α, β for their multiplicity functions. Then $\delta(\alpha, \beta)$ measures the distance between the unitary equivalence classes $\mathcal{U}(A)$ and $\mathcal{U}(B)$. Moreover, there exist commuting representatives A', B' of $\mathcal{U}(A)$ and $\mathcal{U}(B)$ respectively such that $\|A'-B'\| = \delta(\alpha, \beta)$.*

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In seeking to generalize Theorem 1.2 to infinite-dimensional spaces, it is important to realize that unitary orbits may fail to be closed. This is both good and bad news. It is good because the distance between two unitary orbits is the same as the distance between their closures, so the invariant α which we associate with A does not have to be a complete invariant for $\mathcal{U}(A)$ but only for $\overline{\mathcal{U}(A)}$. Such an invariant already exists in the literature — it is the function which assigns to each open set of real numbers the rank of the corresponding spectral projection of A. We call this function the crude multiplicity function of A. Crude multiplicity functions have pleasant properties and it is easy to define a natural distance δ between them.

The bad news is that we can't expect unitary orbits on infinite-dimensional spaces to have closest representatives. Indeed, if B belongs to the closure of $\mathcal{U}(A)$, but not to $\mathcal{U}(A)$ itself, then the distance between $\mathcal{U}(A)$ and $\mathcal{U}(B)$ will be zero, so the representatives A' and B', mentioned in the last sentence of Theorem 1.2, cannot be found in $\mathcal{U}(A)$ and $\mathcal{U}(B)$. The main result of the paper thus reads as follows.

Theorem 1.3. *Let A and B be self-adjoint operators acting on a common Hilbert space, and write α, β for their crude multiplicity functions. Then $\delta(\alpha, \beta)$ measures the distance between $\mathcal{U}(A)$ and $\mathcal{U}(B)$. Moreover, there exist commuting operators A', B' in the closures of these orbits such that $\|A'-B'\| = \delta(\alpha, \beta)$.*

Crude multiplicity functions are studied in Section 2. Most relevant to Theorem 1.3 are definition of the distance δ between them, and the proof of the fact that the distance between $\mathcal{U}(A)$ and $\mathcal{U}(B)$ is at least $\delta(\alpha, \beta)$, but we also digress to show how crude multiplicity functions can be viewed as cardinal-valued functions and measures on **R**.

In Section 3, we study operators with finite spectra. These have closed unitary orbits, and a slight generalization of a combinatorial result known as the Marriage Theorem is used to show that they satisfy the conclusion of Theorem 1.2. A redistribution of spectral measures argument is then employed to establish the first assertion of Theorem 1.3 for arbitrary operators.

Section 4 opens by introducing the notion of a monotone pair of operators — the idea is to generalize the observation, implicit in inequality (1.1), that $\|A'-B'\|$ is minimized when eigenvectors corresponding to the smaller eigenvalues of A' are simultaneously eigenvectors for the smaller eigenvalues of B'. Monotone pairs of operators always commute, and can be simultaneously decomposed as 'monotone' direct sums of operators with smaller spectra. Such decompositions correspond to 'monotone' decompositions of crude multiplicity functions, and the technical heart of the paper, Proposition 4.5, amounts to carrying out the simultaneous decomposition of pairs of crude multiplicity functions in an efficient manner. The proof of Theorem 1.3 is completed by using Proposition 4.5 to construct A' and B'.

Section 5 shows that the operators A', B' of Theorem 1.3 can always be chosen to be diagonal. It also provides a more geometric interpretation of the earlier sections of the paper. Briefly, the idea is that the joint spectral measure of a commuting pair A', B' of operators gives rise to a crude multiplicity function ϱ on $\mathbf{R}^2$ whose 'marginals' are the crude multiplicity functions of the original operators. Whether (A', B') form a monotone pair can be read off from the support of ϱ; so can the value of $\|A'-B'\|$. The correspondence $(A', B')\to\varrho$ is many-to-one, and it is this latitude that allows the modification of the A' and B' of Theorem 1.3 to diagonal operators.

The final section of the paper discusses the prospects for generalizing Theorem 1.3 to normal operators.

It is important to note that the number

$$\max_j |\alpha_j-\beta_j| \tag{1.2}$$

appearing in Theorem 1.1 can alternatively be written

$$\min_\pi \max_j |\alpha_j-\beta_{\pi j}| \tag{1.3}$$

where π ranges over the permutations of $1, 2, \dots, n$. The equality of (1.2) and (1.3) can of course be established directly, but it also follows from Theorem 1.2 and the fact that (1.3) represents the minimal distance between commuting representatives of $\mathcal{U}(A)$ and $\mathcal{U}(B)$. Whereas Theorem 1.1 was formulated in a way altogether dependent on the order of $\mathbf{R}$, (1.3) escapes reliance on order.

Let us emphasize that the spectral distance treated in this paper is different from the Hausdorff distance between spectra; see the discussion after Proposition 2.3. Our problem, in that it concerns unitary equivalence, is also to be distinguished from the study of similarity orbits [8], with which however it has some points of contact.

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2. Crude multiplicity functions

Our first task is to assign invariants to self-adjoint operators which can be used as a basis for measuring the distance between their unitary equivalence classes. Theoretically, any complete unitary invariant would serve this purpose, but as mentioned in the Introduction, we do not need to distinguish between unitary equivalence classes, but only between their closures.

Definition 2.1. Let A be a self-adjoint Hilbert space operator with spectral measure E. The function which assigns the cardinal number $\operatorname{rank} E(V)$ to each open subset V of $\mathbf{R}$ is called the *crude multiplicity function* of A.

This concept (but not the terminology) was discovered independently by D. HADWIN [6] and by R. GELLAR and L. PAGE [5], and both of these references show that it is a complete invariant for closures of unitary equivalence classes. We will see this shortly, but one way to understand why it works on separable spaces is to recall Weyl's result that only the essential spectrum and the multiplicities of isolated eigenvalues are preserved under all the norm limits of the unitary transforms of a self-adjoint operator — this is precisely the information stored in the crude multiplicity function of the operator. To mention a specific example, all self-adjoint operators on separable spaces whose spectra are the unit interval share a common crude multiplicity function.

Spectral measures are countably subadditive in the sense that $E(\bigcup_{n=1}^{\infty} V_n) = \bigvee_{n=1}^{\infty} E(V_n)$ for every sequence of open sets. In particular, if the $\{V_n\}$ are monotone increasing, we have $\alpha(\bigcup_{n=1}^{\infty} V_n)=\sup_n \alpha(V_n)$. Thus α enjoys the regularity property $\alpha(V)=\sup\{\alpha(W)\,|\,W$ is compactly contained in $V\}$. This will prove useful later.

To motivate a notion of distance between crude multiplicity functions, consider the quantity $\max|\alpha_j-\beta_j|$ of (1.1). Suppose its value is r. Then if I is any open interval in $\mathbf{R}$, and I_r is obtained by extending it r units in each direction, then there must be at least as many β_j's in I_r as there are α_j's in I. In terms of the crude multiplicity functions α and β of A and B respectively, this means $\alpha(I)\leqq\beta(I_r)$, and of course by symmetry $\beta(I)\leqq\alpha(I_r)$. The argument is reversible in the sense that if $\alpha(I)\leqq\beta(I_r)$ and $\beta(I)\leqq\alpha(I_r)$ hold for every open interval I, then $\max|\alpha_j-\beta_j|\leqq r$.

Definition 2.2. Let α and β be crude multiplicity functions. Then the *distance* between them, denoted $\delta(\alpha,\beta)$, is the infimum of the numbers $r\geqq 0$ such that $\alpha(I)\leqq\beta(I_r)$ and $\beta(I)\leqq\alpha(I_r)$ hold for all open intervals I.

Several comments are in order here. First, for each $S\subseteq\mathbf{R}$ and $r\geqq 0$, the notation S_r refers to $\{x\in\mathbf{R}\,|\,|x-y|\leqq r$ for some $y\in S\}$. If S is open, or closed, or an interval, then S_r will be the same; all three parts of the converse statement fail.

The infimum in the Definition is attained. Indeed, if $\alpha(I)\leqq\beta(I_{r+1/n})$ for all open intervals I and positive integers n, then $\alpha(J)\leqq\beta(I_r)$ for each open interval J compactly contained in I. Since $\alpha(I)$ is the supremum of $\{\alpha(J)\}$ for such J, we conclude $\alpha(I)\leqq\beta(I_r)$ as desired.

The truth of the equation $\alpha(I)\leqq\beta(I_r)$ for all open intervals I implies its validity for all open sets. Indeed, given V open, then V_r is the disjoint union of open intervals of the form I_r: $V_r=\dot{\bigcup\limits_n} I_r^n$, so that $V\subseteq\dot{\bigcup\limits_n} I^n$ and $\alpha(V)\leqq\sum\alpha(I^n)\leqq\sum\beta(I_r^n)=\beta(V_r)$. This argument makes enough use of monotonicity to be specific to **R**, but the strong notion of monotonicity implicit in (1.1) is muted in Definition 2.2. This will be rectified to some extent in Section 4, and a definition of δ which is a direct analogue of the quantity $\max|\alpha_j-\beta_j|$ will be presented in Section 5.

Finally, note that if $\alpha(\mathbf{R})\neq\beta(\mathbf{R})$, then the distance between α and β is infinite. This is appropriate since if A and B act on spaces of different dimensions, there is no way to compare their unitary equivalence classes.

Proposition 2.3. *Let A and B be self-adjoint operators and write α and β for their crude multiplicity functions. Then the distance between (the closures of) the unitary equivalence classes $\mathcal{U}(A)$ and $\mathcal{U}(B)$ is at least $\delta(\alpha, \beta)$.*

Proof. Write E and F for the spectral measures of A and B respectively and suppose $r<\delta(\alpha, \beta)$. Then there is an interval I for which $\operatorname{rank} E(I)>\operatorname{rank} F(I_r)$ or $\operatorname{rank} F(I)>\operatorname{rank} E(I_r)$. Without loss of generality, assume the former, and also that $I=(-a, a)$ is centered at the origin. Choose a unit vector x in the range of $E(I)$, but orthogonal to the range of $F(I_r)$. Then $\|Ax\|<a$ while $\|Bx\|\geqq a+r$. This means $\|A-B\|>r$. Since r is arbitrary, we have $\|A-B\|\geqq\delta(\alpha, \beta)$. Since crude multiplicity is a unitary invariant, this inequality persists when A and B are replaced by unitary transforms, and the proof is complete.

Remark. Except for notation, the inequality $\|A-B\|\geqq\delta(\alpha, \beta)$ is essentially Theorem 7(i) of [3]*.

Remark. If S and T are compact subsets of **R** (or **C**), then the *Hausdorff* distance between them is given by $\theta(S, T)=\max\{\max\limits_{x\in S}\operatorname{dist}(x, T), \max\limits_{y\in T}\operatorname{dist}(S, y)\}$. It is known, even in the infinite-dimensional normal case, that $\|A-B\|\geqq\theta(\sigma(A), \sigma(B))$ and various further developments in this direction have recently been made [7], [2]. Although we will eventually show that $\operatorname{dist}(\mathcal{U}(A), \mathcal{U}(B))$ always equals $\delta(\alpha, \beta)$, equality with $\theta(\sigma(A), \sigma(B))$ rarely occurs. For example, operators $A=\begin{bmatrix}1 & & \\ & 1 & \\ & & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & & \\ & 0 & \\ & & 0\end{bmatrix}$ have the same spectrum, so $\theta(\sigma(A), \sigma(B))=0$, but $\delta(\alpha, \beta)=1$.

*) The second author takes this occasion to call attention to errors in his paper [3]. The statement of the elementary Lemma on page 402 is too general (the second conclusion requires the hypothesis $Q=Q^*=Q^2$); this, however, is without effect on the rest of the paper. More serious, the proof of Theorem 3 is fallacious (the construction given is correct, but it does not establish the asserted inequality). This error invalidates Theorem 4, Theorem 5 (ii), Theorem 6 (iii)—(iv), and Theorem 7 (ii).

Definition 2.4. If α is a crude multiplicity function and S an arbitrary subset of $\mathbf{R}$, then $\alpha(S)\equiv\inf\{\alpha(V)|V$ an open set containing $S\}$.

This extension of the domain of α is basically a matter of convenience, but it has some surprising consequences, which will be explored after Proposition 2.5. In the meantime, two observations should be made.

(1) If $\alpha(S)\leqq\beta(S_r)$ holds for all open intervals, we have already noted that it remains valid for all open sets, and thus it holds for all subsets of $\mathbf{R}$.

(2) If E is the spectral measure of A, then rank $E(S)$ does *not* in general coincide with $\alpha(S)$ unless S is open; for example, $\alpha\{\lambda\}$ is non-zero for any λ in the spectrum of A, but $E\{\lambda\}=0$ unless λ is an eigenvalue of A.

We now prove, as promised earlier, that α is a complete invariant for the closure of $\mathcal{U}(A)$.

Proposition 2.5. *Let A and B be self-adjoint operators with crude multiplicity functions α and β respectively. Then the following are equivalent:*

(1) *the closures of $\mathcal{U}(A)$ and $\mathcal{U}(B)$ coincide;*

(2) *the distance between $\mathcal{U}(A)$ and $\mathcal{U}(B)$ is zero;*

(3) $\alpha=\beta$;

(4) $\delta(\alpha,\beta)=0$.

Proof. The implications (1)$\Leftrightarrow$(2) and (3)$\Rightarrow$(4) are clear. If $\delta(\alpha,\beta)=0$, then $\alpha(I)\leqq\beta(I)\leqq\alpha(I)$ for all intervals I since the infimum in Definition 2.2 is attained. This establishes (4)$\Rightarrow$(3).

That (2)$\Rightarrow$(4) follows from Proposition 2.3.

Suppose finally that $\alpha=\beta$. Call $\lambda\in\mathbf{R}$ *dispensable* for α if there is some open interval I containing λ with $\alpha(\lambda)=\inf_{\mu\in I}\alpha(\mu)$. Every open interval contains such points. Let $\lambda_0<\lambda_1<\ldots<\lambda_n$ be a partition of an interval containing $\sigma(A)=\sigma(B)$ and consisting of dispensable points. Then rank $E(\lambda_{i-1},\lambda_i]=$rank $F(\lambda_{i-1},\lambda_i]=$ $=\alpha(\lambda_{i-1},\lambda_i)$ for $i=1,\ldots,n$. In particular $\sum_{i=1}^{n}\lambda_i E(\lambda_{i-1},\lambda_i]$ and $\sum_{i=1}^{n}\lambda_i F(\lambda_{i-1},\lambda_i]$ are unitarily equivalent. Since these sums can be taken arbitrarily close to A, B respectively, we have established (3)$\Rightarrow$(2).

Let α be a crude multiplicity function. By the well ordering of the cardinal numbers the infimum in Definition 2.4 is always attained. Thus if S and T are disjoint compact sets in $\mathbf{R}$, there are disjoint open sets V and W containing them with $\alpha(S\cup T)=\alpha(V\cup W)=\alpha(V)+\alpha(W)=\alpha(S)+\alpha(T)$. It follows that $\alpha(S)=\sum_{x\in S}\alpha(x)$ for every finite set S. Outer regularity is built into Definition 2.4. The next result shows that α also enjoys a strong form of inner regularity. It implies that α can be reconstructed from its restriction to the collection of singleton sets, and in the sequel we will often regard α as a function on $\mathbf{R}$.

Proposition 2.6. *For any set S, we have $\alpha(S)=\sup\{\alpha(T)\,|\,T$ a finite subset of $S\}$.*

Proof. For each $x\in S$, choose an open set V containing x with $\alpha(x)=\alpha(V)$. These open sets cover S and thus admit a countable subcover $\{V_n\}$. Writing $\{x_n\}$ for the associated points in S, we have $\alpha(S)\leqq\alpha(\sum_{n=1}^{\infty}V_n)\leqq\sum_{n=1}^{\infty}\alpha(V_n)=\sum_{n=1}^{\infty}\alpha(x_n)$. This shows $\alpha(S)\leqq\sup\{\alpha(T)\,|\,T$ a finite subset of $S\}$. The reverse inequality is obvious.

Corollary 2.7. *α is countably additive.*

Proof. $\alpha(S)=\sup\{\alpha(T)|\ T$ a countable subset of $S\}$.

We close this section with an abstract characterization of crude multiplicity functions. Recall that a cardinal-valued function α is *upper semi-continuous* if $\{\lambda|\alpha(\lambda)<c\}$ is open for each cardinal number c.

Proposition 2.8. *A cardinal-valued function α defined on $\mathbf{R}$ is a crude multiplicity function if and only if*

(1) *α is compactly supported,*

(2) *α is upper semi-continuous, and*

(3) *the points at which α takes on finite non-zero values are isolated.*

Proof. The necessity of (1) is obvious, while (2) and (3) follow from the outer regularity built into Definition 2.4, and the inner regularity proved in Proposition 2.6.

Conversely, suppose α satisfies (1), (2) and (3). For each cardinal c in the range of α, choose a countable dense subset S_c of $\alpha^{-1}(c)$. There is a diagonal operator B with the nullity of $B-\lambda I$ being c iff $\lambda\in S_c$. The crude multiplicity function β of B is defined on open sets by $\beta(V)=\sum_{c}\sum_{\lambda\in S_c\cap V}\alpha(\lambda)$. We complete the proof by showing $\alpha=\beta$. Fix $\lambda_0\in\mathbf{R}$. Since every open set V containing λ_0 contains points in $S_{\alpha(\lambda_0)}$, we have $\beta(V)\geqq\alpha(\lambda_0)$ and hence $\beta(\lambda_0)\geqq\alpha(\lambda_0)$. If $\alpha(\lambda_0)$ is finite, (3) and (2) give $\beta(\lambda_0)=\alpha(\lambda_0)$. If on the other hand, $\alpha(\lambda_0)$ is infinite, use (2) to choose a neighborhood V_0 of λ_0 with $\alpha(\lambda)\leqq\alpha(\lambda_0)$ for all $\lambda\in V_0$. Then $\beta(\lambda_0)\leqq\beta(V_0)$, where $\beta(V_0)$ is a sum of cardinal numbers, each of which appears at most countably often, and all of which are $\leqq\alpha(\lambda_0)$. Thus we have $\beta(\lambda_0)\leqq\beta(V_0)\leqq\alpha(\lambda_0)$ and so $\alpha=\beta$ is a crude multiplicity function.

A totally different proof of this proposition will be outlined in Section 4, and will play an important role in establishing Theorem 1.3. The present simpler proof will be mimicked when we prove Proposition 5.5.

3. Operators with finite spectra

The separate treatment of operators with finite spectra presented in this section is not logically necessary for the sequel but the ideas involved are sufficiently different (and simpler!) to deserve exposition.

Proposition 3.1. *The unitary orbit of every self-adjoint operator with finite spectrum is closed.*

Proof. If the spectrum of A is finite and B belongs to the closure of $\mathcal{U}(A)$, then A and B have the same crude multiplicity function. This means $\sigma(A)=\sigma(B)$, and the corresponding eigenspaces have equal dimensions. This forces B to be unitarily equivalent to A.

The following combinatorial result was referred to in the Introduction. When X is finite (so that (1) is redundant) it is the classical result known as the Marriage Theorem and variously attributed to H. Weyl, J. Egerváry, P. Hall, and G. Pólya; see [11, Thm. 25A] or [9, Lemma 3.2].

Proposition 3.2. *Let $R\subseteq X\times Y$ be a relation with domain X satisfying:*

(1) *Only finitely many subsets of Y are of the form $R(x)$ for some $x\in X$, and*

(2) *For each subset S of X, the cardinality of $R(S)$ is at least as great as the cardinality of S.*

Then there is a one-to-one function $f\colon X\to Y$ whose graph is contained in R.

Proof. We use $|...|$ to denote cardinality.

Case 1: X is finite. We argue inductively on $|X|$. The result is clear if $|X|=1$. To effect the inductive step, note that if $|R(S)|=|S|$ for some proper subset of X, then $R\cap(S\times Y)$ and $R\cap[(X\setminus S)\times Y\setminus R(S)]$ again satisfy the hypothesis of the Proposition; on the other hand, if $|R(S)|>|S|$ for all proper subsets of X, then we could fix $x_0\in X$, $y_0\in R(x_0)$, and apply the inductive hypothesis to $R\cap[(X\setminus\{x_0\})\times\times Y\setminus\{y_0\}]$.

Case 2: The set $R(x)$ is infinite for each $x\in X$. Write $T_1, \dots, T_n$ for the various subsets of Y of the form $R(x)$ for some $x\in X$; and set $S_i=\{x\in X|R(x)=T_i\}$. Let $\mathcal{V}$ denote the collection of infinite subsets of Y which are obtained by intersecting some of the T_j's with the complements of the remaining T_j's. Express each $V\in\mathcal{V}$ as the disjoint union $V=\bigcup_{i=1}^{n} V_i$ of n sets of equal cardinality, and set $Y_i==\bigcup_{V\in\mathcal{V}} V_i$. Then $|T_i\cap Y_i|=|T_i|$ for each i; so there is a one-to-one map $f_i\colon S_i\to\to T_i\cap Y_i$. Take f to be the union of the $\{f_i\}$; this is injecitve since the $\{Y_i\}_{i=1}^n$ are disjoint.

Case 3: R is arbitrary. Let $S_1=\{x\in X \mid R(x)$ is finite$\}$. Then S_1 is finite since $R(S_1)$ must be the finite union of sets of the form $R(x)$ with $x\in S_1$, and $|R(S_1)|\geqq|S_1|$. Use Case 1 to define $f_1\colon S_1\to Y$ and apply Case 2 to the relation $R\cap[(X\setminus S_1)\times \times\{Y\setminus f_1(S_1)\}]$ to obtain a one-to-one f_2 on $X\setminus S_1$. Take $f=f_1\cup f_2$.

Remark. Let $X=Y$ be the positive integers and set $R=\{(x,y)\in X\times Y|(x=1$ and $y>1)$ or $x=y>1\}$. Although $|R(S)|\geqq|S|$ for every $S\subseteq X$, this R does not contain the graph of a one-to-one function. This example, which illustrates the necessity of hypothesis (1) in Proposition 3.2, was pointed out by Randy Tuler.

We can now extend Theorem 1.2 to operators with finite spectra.

Proposition 3.3. *Let A and B be self-adjoint operators with finite spectra which act on a common Hilbert space, and write α and β for their crude multiplicity functions. Then $\delta(\alpha,\beta)$ measures the distance between $\mathcal{U}(A)$ and $\mathcal{U}(B)$. Moreover there are commuting representatives A' and B' of $\mathcal{U}(A)$ and $\mathcal{U}(B)$ respectively such that $\|A'-B'\|=\delta(\alpha,\beta)$.*

Proof. Let X and Y be orthonormal bases of eigenvectors for A and B respectively and define a relation $R\subseteq X\times Y$ by $R\equiv\{(x,y)\in X\times Y|$ the eigenvalues corresponding to x and y differ by no more than $\delta(\alpha,\beta)\}$. Then R and R^{-1} satisfy the hypotheses of Proposition 3.2, so the Schroeder—Bernstein Theorem provides a bijection $\tau\colon X\to Y$ whose graph is contained in R. Let U be the unitary operator induced by (i.e. containing) τ. Set $A'=A$ and $B'=U^{-1}BU$. Then A' and B' commute and $\operatorname{dist}(\mathcal{U}(A),\mathcal{U}(B))\leqq\|A'-B'\|\leqq\delta(\alpha,\beta)$. Since we already know $\operatorname{dist}(\mathcal{U}(A),\mathcal{U}(B))\geqq\delta(\alpha,\beta)$, the proof is complete.

Proposition 3.3 leads to a quick proof of the first assertion of Theorem 1.3.

Proposition 3.4. *Let A and B be self-adjoint operators acting on a common Hilbert space, and write α,β for their crude multiplicity functions. Then $\delta(\alpha,\beta)$ measures the distance between $\mathcal{U}(A)$ and $\mathcal{U}(B)$.*

Proof. We already know $\operatorname{dist}(\mathcal{U}(A),\mathcal{U}(B))\geqq\delta(\alpha,\beta)$. Let $\varepsilon>0$ be given. By redistribution of spectral measures, we obtain self-adjoint operators A' and B' with finite spectra which are ε-perturbations of A and B respectively. Write α',β' for the crude multiplicity functions of A', B'. Then $\operatorname{dist}(\mathcal{U}(A),\mathcal{U}(B))<\operatorname{dist}(\mathcal{U}(A'),\mathcal{U}(B'))+2\varepsilon$ and $\delta(\alpha',\beta')<\delta(\alpha,\beta)+2\varepsilon$. Since ε was arbitrary and $\operatorname{dist}(\mathcal{U}(A'),\mathcal{U}(B'))= =\delta(\alpha',\beta')$ by Proposition 3.3, we conclude that $\operatorname{dist}(\mathcal{U}(A),\mathcal{U}(B))\leqq\delta(\alpha,\beta)$, and the proof is complete.

For the sake of completeness, we close the section by characterizing the self-adjoint operators whose unitary orbits are closed.

Proposition 3.5. *Let* A *be self-adjoint with crude multiplicity function* α. *Then the following are equivalent:*

(1) *The unitary orbit of* A *is closed;*

(2) *The spectrum of* A *is countable, and each* $\lambda\in\sigma(A)$ *has a neighborhood* U *with* $\alpha(\{\lambda\})>\alpha(U\setminus\{\lambda\})$.

Proof. (1)⇒(2). Suppose first that $\lambda_0\in\sigma(A)$, but that the condition does not hold at λ_0. Then for all sufficiently small neighborhoods U of λ_0 we have $\alpha(U\setminus\setminus\{\lambda_0\})\geqq\alpha(\{\lambda_0\})$. If λ_0 is an eigenvalue of A, take B to be the restriction of A to the orthogonal complement of $\operatorname{Ker}(A-\lambda_0 I)$. If λ_0 is not an eigenvalue of A, set $B==A\oplus\lambda_0 I$ where I acts on a one-dimensional space. In either case, A and B have the same crude multiplicity function, but are not unitarily equivalent. This shows that (1) implies the second part of (2).

Suppose now that α satisfies the second part of condition (2). In this case each λ in $\sigma(A)$ is an eigenvalue of A. If A is not diagonal, let B be the restriction of A to $\vee\{\operatorname{Ker}(A-\lambda I)|\lambda\in\sigma(A)\}$. So A and B share a common crude multiplicity function, but they are not unitarily equivalent. If, on the other hand, A is diagonal and $\sigma(A)$ is uncountable, then let μ be a non-atomic measure supported on $\sigma(A)$, and take B to be the direct sum of A with the position operator on $L^2(\mu)$. Here too, α is the crude multiplicity function of the non-unitarily-equivalent operators A and B.

(2)⇒(1). If α satisfies (2), then every operator having α as its crude multiplicity function must be diagonal; the dimensions of the various eigenspaces are completely determined by α. All such operators are unitarily equivalent.

On separable spaces, condition (2) means $\sigma(A)$ is finite. On non-separable spaces, $\sigma(A)$ may have limit points, even infinitely many limit points.

The authors thank K.R. Davidson for correcting their faulty version of this Proposition.

4. Monotonicity and commuting representatives

The following definition will enable us to adapt the notion of monotonicity implicit in Theorem 1.1 to general pairs of self-adjoint operators.

Definition 4.1. Let A, B be self-adjoint operators on a common Hilbert space with spectral measures E, F respectively. We say the pair (A, B) is *monotone* if for each pair (a, b) of real numbers, either $E(-\infty, a)\leqq F(-\infty, b)$ of $F(-\infty, b)\leqq\leqq E(-\infty, a)$.

Proposition 4.2. *Let* (A, B) *be a monotone pair. Then there is a non-decreasing function* $\tau\colon\mathbf{R}\to\mathbf{R}$ *so that* $F(-\infty, \tau(a))\leqq E(-\infty, a)\leqq F(-\infty, \tau(a)]$ *for all* $a\in\mathbf{R}$.

Proof. For each $a \in \mathbf{R}$, set $\tau(a) = \inf\{b \geqq -\|B\| \mid E(-\infty, a) \leqq F(-\infty, b)\}$. For $b < \tau(a)$, we have $F(-\infty, b) \leqq E(-\infty, a)$ so the double inequality follows.

Corollary 4.3. *Every monotone pair of self-adjoint operators commutes.*

Proof. Let A, B, E, and F be as in Proposition 4.2. The conclusion of that result shows that $E(-\infty, a)$ commutes with every spectral projection of B. It follows that all the spectral projections of A and B commute with each other, and hence, so do A and B.

If the diagonal entries in two diagonal matrices are simultaneously non-decreasing, then the corresponding operators form a monotone pair. The operators A' and B' of Theorem 1.2, i.e., those which make equality hold in relation (1.1), can be taken to be a monotone pair, and we will use monotone pairs to establish the final assertion of Theorem 1.3.

Definition 4.4. The equation $\alpha = \alpha_1 + \alpha_2$ represents a *monotone decomposition* of the crude multiplicity function α if α_1 and α_2 are also crude multiplicity functions and there is a real number a, called a *break-point* of the decomposition, such that $\alpha_1(x) = 0$ for $x > a$ while $\alpha_2(x) = 0$ for $x < a$.

It is easy to construct monotone decompositions — simply start with any number a, and choose appropriate values for $\alpha_i(a)$. (Beside the obvious restriction $\alpha_1(a) + \alpha_2(a) = \alpha(a)$, we must also have $\alpha_1(a) \geqq \limsup_{x \to a^-} \alpha(x)$ and $\alpha_2(a) \geqq \limsup_{x \to a^+} \alpha(x)$ to insure that the $\{\alpha_i\}$ are crude multiplicity functions — cf. Proposition 2.8 (2)). If A_1 and A_2 are operators with crude multiplicity functions α_1 and α_2 respectively, then α is the crude multiplicity function of the direct sum $A' \equiv A_1 \oplus A_2$.

In fact, repeated monotone decomposition of α could be used to construct the implementing operator A' in the first place, thereby providing a (more technically complicated) proof of Proposition 2.8. To prove Theorem 1.3, we basically need to carry out this program on the crude multiplicity functions α and β simultaneously. The following proposition tells us how to get started, and Theorem 4.13 applies it to construct a monotone pair (A', B') which will satisfy Theorem 1.3.

Proposition 4.5. *Let $\beta_1 + \beta_2$ be a monotone decomposition of a crude multiplicity function β, and suppose α is another crude multiplicity function with $\delta(\alpha, \beta) = r < \infty$. Then there is a monotone decomposition $\alpha_1 + \alpha_2$ of α such that $\delta(\alpha_1, \beta_1)$ and $\delta(\alpha_2, \beta_2)$ are both less than or equal to r.*

Before embarking on the proof of this result, we illustrate its usefulness by establishing a special case of Theorem 1.3. It improves on Proposition 3.3 by only requiring A to have finite spectrum.

Corollary 4.6. *Let A and B be self-adjoint operators acting on a common Hilbert space, and write α, β for their crude multiplicity functions. Suppose A has finite spectrum. Then there is an operator $B' \in \overline{\mathcal{U}(B)}$ such that (A, B') is a monotone pair and $\|A-B'\|=\delta(\alpha, \beta)=\operatorname{dist}(\mathcal{U}(A), \mathcal{U}(B))$.*

Proof. We argue inductively on the cardinality of $\sigma(A)$. If $A=\lambda I$ is a scalar multiple of the identity, then (A, B) is itself a monotone pair, and $\|A-B\|= =\operatorname{dist}(\mathcal{U}(A). \mathcal{U}(B))$ since $\mathcal{U}(A)=\{A\}$.

To establish the inductive step, write $A=A_1\oplus A_2$ by splitting off the eigenspace corresponding to the smallest eigenvalue of A. Let $\alpha=\alpha_1+\alpha_2$ be the corresponding (monotone) decomposition of α, and decompose $\beta=\beta_1+\beta_2$ via Proposition 4.5. Choose operators B_1 and B_2 having these crude multiplicity functions. By the inductive hypothesis, it is possible to have $\|A_i-B_i'\|=\delta(\alpha_i, \beta_i)$ with (A_i, B_i') monotone pairs. Then $B'=B_1'\oplus B_2'$ satisfies the conclusion of the corollary.

We now work toward a proof of Proposition 4.5. Until this is completed, we will fix the notation of that proposition, i.e., α and β are crude multiplicity functions with $\delta(\alpha, \beta)=r<\infty$ and $\beta=\beta_1+\beta_2$ is a monotone decomposition of β. We seek a monotone decomposition $\alpha=\alpha_1+\alpha_2$ with both $\delta(\alpha_1, \beta_1)\leqq r$ and $\delta(\alpha_2, \beta_2)\leqq r$.

Consider first the problem of constructing α_1 — this must be a left restriction of α in the sense of the following definition.

Definition 4.7. Let γ_1 and γ be crude multiplicity functions, and write a for the largest x satisfying $\gamma_1(x)\neq 0$. We say γ_1 is a *left restriction* of γ and write $\gamma_1\leqq\gamma$ if $\gamma_1(a)\leqq\gamma(a)$ and $\gamma_1(x)=\gamma(x)$ for $x<a$. The ordered pair $(a, \gamma_1(a))$ is called the *boundary point* of γ_1. *Right restrictions* are defined similarly.

If γ is understood, then γ_1 is completely determined by its boundary point. Note that $\leqq$ is a total order on the collection of left restrictions on γ; thought of in terms of boundary points, it is the usual dictionary order. Thus $\leqq$ has the least upper bound and greatest lower bound properties.

Returning to α_1, the requirement $\delta(\alpha_1, \beta_1)\leqq r$ means that α_1 must belong to the sets

$$\mathcal{S}^+ \equiv \{\gamma \leqq \alpha | \gamma(I) \leqq \beta_1(I_r) \text{ for all open intervals } I\},$$

and

$$\mathcal{S}^- \equiv \{\gamma \leqq \alpha | \beta_1(I) \leqq \gamma(I_r) \text{ for all open intervals } I\}.$$

Write α_1^+ for the supremum of $\mathcal{S}^+$. Since $\alpha_1^+(I)=\sup\{\gamma(I)|\gamma\in\mathcal{S}^+\}$ for every interval I, we see that α_1^+ belongs to $\mathcal{S}^+$. Similarly, $\alpha_1^-\equiv\inf\mathcal{S}^-$ belongs to $\mathcal{S}^-$. Thus $\mathcal{S}^+\cap\mathcal{S}^-=\{\gamma|\alpha_1^-\leqq\gamma\leqq\alpha_1^+\}$ constitute our candidates for α_1. Lemma 4.9 shows that this set is nonempty.

Lemma 4.8. *Suppose* γ *is a left restriction of* α. *If* $\gamma(I)>\beta_1(I_r)$ *holds for* $I=(c,d)$, *then it holds for* $I=(c,\infty)$. *The same is true for the inequality* $\beta_1(I)>\gamma(I_r)$.

Proof. If $\gamma(c,d)>\beta_1(c-r,d+r)$, then β must have a break point below $d+r$, since otherwise $\gamma(c,d)\leqq\alpha(c,d)\leqq\beta(c-r,d+r)=\beta_1(c-r,d+r)$, the second inequality following from the assumption $\delta(\alpha,\beta)=r$. Thus replacing d by ∞ can only enlarge $\gamma(c,d)$ but will not change $\beta_1(c-r,d+r)$.

Similarly, the inequality $\beta_1(c,d)>\gamma(c-r,d+r)$ means that the boundary point $(a,\gamma(a))$ of γ satisfies $a<d+r$ so replacing d by ∞ leaves this intact as well.

Lemma 4.9. $\alpha_1^-\leqq\alpha_1^+$.

Proof. We argue by contradiction, assuming that $\alpha_1^->\alpha_1^+$. Then either there is a γ satisfying $\alpha_1^->\gamma>\alpha_1^+$ or α_1^- is an immediate successor of α_1^+. In the former case, set $\theta^+=\theta^-=\gamma$; in the latter, take $\theta^+=\alpha_1^-$ and $\theta^-=\alpha_1^+$. There are intervals $I=(c,\infty)$ and $J=(d,\infty)$ satisfying

$$\beta_1(I_r)<\theta^+(I) \tag{4.1}$$

and

$$\theta^-(J_r)<\beta_1(J). \tag{4.2}$$

If $|c-d|\leqq r$, we would have $I\subseteqq J_r$ and $J\subseteqq I_r$, so $\theta^-(I)\leqq\theta^-(J_r)<\beta_1(J)\leqq$ $\leqq\beta_1(I_r)<\theta^+(I)$, a contradiction since θ^+ is at most an immediate successor of θ^-. Thus, if we assume for definiteness that $c\leqq d$, then we actually have $c<d-r$. By (4.2), there is a break point for β greater than d, so

$$\theta^+(c,d-r]\leqq\alpha(c,d-r]\leqq\beta(c-r,d]=\beta_1(c-r,d]. \tag{4.3}$$

Since θ^+ is at most an immediate successor of θ^-, we conclude from (4.2) that

$$\theta^+(d-r,\infty)\leqq\beta_1(d,\infty). \tag{4.4}$$

Adding (4.3) and (4.4), we contradict (4.1), and the proof is complete.

Of course, right restrictions of α are handled analogously to left restrictions. (The dictionary order on boundary points uses the order on $\mathbf{R}$ opposite to the usual one.) In particular, we take α_2^+ to be the maximal right restriction of α satisfying $\alpha_2^+(I)\leqq\beta_2(I_r)$ for all I and α_2^- to be the minimal right restriction of α satisfying $\beta_2(I)\leqq\alpha_2^-(I_r)$ for all I. The following analogue of Lemma 4.9 shows there are candidates for α_2.

Lemma 4.10. $\alpha_2^-\leqq\alpha_2^+$.

Proof. For each crude multiplicity function θ, write $\tilde\theta$ for its opposite, defined by $\tilde\theta(x)=\theta(-x)$. The operation $\sim$ converts right restrictions to left restrictions, so the present result is a corollary of Lemma 4.9.

We now have plenty of candidates for α_1 and α_2, but we must still choose carefully if $\alpha=\alpha_1+\alpha_2$ is to represent a monotone decomposition. Lemma 4.11 says that α_1^- and α_2^- are 'too small' to do the job; Lemma 4.12 says that α_1^+ and α_2^+ are 'too big'. We then complete the proof of Proposition 4.5 by 'interpolation'.

Lemma 4.11. *There is at most one number a such that $\alpha_1^-(a)$ and $\alpha_2^-(a)$ are simultaneously non-zero, and $\alpha_1^-(x)+\alpha_2^-(x)\leqq\alpha(x)$ for all x.*

Proof. We first show that if $\theta_i<\alpha_i^-$, then $\theta_1(x)+\theta_2(x)<\alpha(x)$ for some x. Indeed, by Lemma 4.8 (and its analogue for right restrictions), there are intervals satisfying

$$\theta_1(c-r,\infty)<\beta_1(c,\infty) \tag{4.5}$$

and

$$\theta_2(-\infty,d+r)<\beta_2(-\infty,d). \tag{4.6}$$

These inequalities force β to have a break-point between c and d. Adding them, we get

$$\theta_1(c-r,\infty)+\theta_2(-\infty,d+r)<\beta(c,d)\leqq\alpha(c-r,d+r). \tag{4.7}$$

This forces $\theta_1(x)+\theta_2(x)<\alpha(x)$ for some x, as desired.

Suppose there are three (or more) distinct numbers $a_1<a_2<a_3$ at which α_1^- and α_2^- are simultaneously non-zero. Let

$$\theta_1(x)=\begin{cases}\alpha(x) & \text{if } x\leqq a_2\\ 0 & \text{if } x>a_2\end{cases} \quad\text{and}\quad \theta_2(x)=\begin{cases}0 & \text{if } x<a_2\\ \alpha(x) & \text{if } x\geqq a_2.\end{cases}$$

Then $\theta_1<\alpha_1^-$ and $\theta_2<\alpha_2^-$ and $\theta_1(x)+\theta_2(x)\geqq\alpha(x)$ for all x. In view of the preceding paragraph, this case cannot occur.

The assumption that there are precisely two numbers $a_1<a_2$ at which α_1^- and α_2^- are both non-zero leads to the same contradiction by consideration of

$$\theta_1(x)=\begin{cases}\alpha(x) & \text{if } x\leqq a_1\\ 0 & \text{if } x>a_1,\end{cases} \quad \theta_2(x)=\begin{cases}0 & \text{if } x<a_2\\ \alpha(x) & \text{if } x\geqq a_2.\end{cases}$$

We conclude there is at most one number at which α_1^- and α_2^- are both non-zero. If there are no such numbers, or if the number, a, satisfies $\alpha(a)$ infinite, the proof is complete. In the remaining case, i.e. $\alpha_1^-(a)$ and $\alpha_2^-(a)$ both finite, but non-zero, choose θ_1 and θ_2 to be immediate predecessors of α_1^- and α_2^- respectively. Reviewing the first paragraph of the proof, we note that the strict inequalities in (4.5), (4.6) and (4.7) all become equalities when θ_i is replaced by α_i^-. In particular all the numbers involved are finite and a must lie between $c-r$ and $d+r$. The revised (4.7) reads

$$\alpha_1^-(c-r,\infty)+\alpha_2^-(-\infty,d+r)\leqq\alpha(c-r,d+r), \tag{4.8}$$

or alternatively

(4.9) $\alpha(c-r, a)+\alpha_1^-(a)+\alpha_2^-(a)+\alpha(a, d+r) \leqq \alpha(c-r, a)+\alpha(a)+\alpha(a, d+r)$.

All numbers in this inequality are finite, and we conclude $\alpha_1^-(a)+\alpha_2^-(a) \leqq \alpha(a)$ as desired.

Lemma 4.12. $\alpha_1^+(x)+\alpha_2^+(x) \geqq \alpha(x)$ *for all* $x \in \mathbf{R}$.

Proof. We closely parallel the proof of Lemma 4.11. First, observe that if $\theta_i > \alpha_i^+$, then $\theta_1(x)+\theta_2(x) > \alpha(x)$ for some x. The relevant inequalities, replacing (4.5), (4.6) and (4.7), are:

(4.10) $$\beta_1(c-r, \infty) < \theta_1(c, \infty),$$

(4.11) $$\beta_2(-\infty, d+r) < \theta_2(-\infty, d),$$

and

(4.12) $$\alpha(c, d) \leqq \beta(c-r, d+r) < \theta_1(c, \infty)+\theta_2(-\infty, d).$$

Suppose now that $\alpha_1^+(a)+\alpha_2^+(a) < \alpha(a)$. If $\alpha(a)$ is infinite, set

$$\theta_1(x) = \begin{cases} \alpha(x) & \text{if } x \leqq a \\ 0 & \text{if } x > a, \end{cases} \qquad \theta_2(x) = \begin{cases} 0 & \text{if } x < a \\ \alpha(x) & \text{if } x \geqq a \end{cases}$$

to obtain a contradiction with the preceding paragraph. On the other hand, if $\alpha(a)$ is finite, choose θ_i to be an immediate successor of α_i^+. Review of the first paragraph of the proof shows that if θ_i is replaced by α_i^+ in (4.12), we get

(4.13) $$\alpha(c, d) \leqq \alpha_1^+(c, \infty)+\alpha_2^+(-\infty, d).$$

Since a is between c and d, and the numbers in (4.13) are finite, this means $\alpha(a) \leqq \alpha_1^+(a)+\alpha_2^+(a)$.

Proof of Proposition 4.5. Lemmas 4.9 and 4.10 tell us $\alpha_i^- \leqq \alpha_i^+$. We will construct α_i such that $\alpha_i^- \leqq \alpha_i \leqq \alpha_i^+$ with $\alpha=\alpha_1+\alpha_2$ a monotone decomposition. The double inequalities force $\delta(\alpha_i, \beta_i) \leqq r$, so this will complete the proof.

We begin by choosing a break point a for our decomposition. Write $a_1 = \sup\{x \mid \alpha_1^-(x) \neq 0\}$ and $a_2 = \inf\{x \mid \alpha_2^-(x) \neq 0\}$. Lemma 4.12 shows that $a_1 \leqq a_2$. We distinguish several (overlapping) cases:

Case 1: $\alpha_2^+(a_1) \neq 0$. Take $a=a_1$.

Case 2: $\alpha_1^+(a_2) \neq 0$. Take $a=a_2$.

Case 3: There is a number a between a_1 and a_2 such that both $\alpha_1^+(a)$ and $\alpha_2^+(a)$ are non-zero.

In all these cases, set:

$$\alpha_1(x) = \begin{cases} \alpha(x) & \text{if} \quad x < a, \\ 0 & \text{if} \quad x > a, \end{cases} \qquad \alpha_2(x) = \begin{cases} 0 & \text{if} \quad x < a \\ \alpha(x) & \text{if} \quad x > a, \end{cases}$$

and use the following recipe to define $\alpha_i(a)$:

Case A: $\alpha(a)$ is infinite. Set $\alpha_i(a)=\alpha_i^+(a)$.

Case B: $\alpha(a)$ is finite. Choose $\alpha_i(a)$ to satisfy $\alpha_i^-(a)\leqq\alpha_i(a)\leqq\alpha_i^+(a)$ and $\alpha_1(a)+$ $+\alpha_2(a)=\alpha(a)$. This is possible since $\alpha_1^-(a)+\alpha_2^-(a)\leqq\alpha(a)\leqq\alpha_1^+(a)+\alpha_2^+(a)$.

It is easy to check that in all these cases we have $\alpha_i^-\leqq\alpha_i\leqq\alpha_i^+$, the equation $\alpha_1+\alpha_2=\alpha$ is true, and α_1, α_2 are crude multiplicity functions by construction. There is one additional possibility not covered by Cases 1—3 above, namely when $\alpha_1^+(x)$ and $\alpha_2^+(x)$ are never simultaneously positive — but then $\alpha=\alpha_1^++\alpha_2^+$ by Lemma 4.12, so we may take $\alpha_i=\alpha_i^+$.

We are now in a position to prove the last assertion of Theorem 1.3. As mentioned earlier in the section, we will use a (necessarily commuting) monotone pair for (A', B'). In following the proof, the reader may want to keep the special cases $\alpha=\beta$ (Proposition 1.8) and α of finite support (Corollary 4.6) in mind.

Theorem 4.13. *Let α, β be crude multiplicity functions with $\delta(\alpha, \beta)<\infty$. Then there exists a monotone pair (A', B') of operators having α, β as their respective crude multiplicity functions and satisfying $\|A'-B'\|=\delta(\alpha, \beta)$.*

Proof. We first construct two families of crude multiplicity functions $\{\alpha_k\}$ and $\{\beta_k\}$ where k ranges over all finite sequences of 1's and 2's. We use the standard notations $k*j$ for the sequence k concatenated with (or followed by) j, and $|k|$ for the length of k, i.e., its number of terms. It is convenient to allow the empty sequence $k=\emptyset$ (of length zero) and to begin our construction by setting $\alpha_\emptyset=\alpha$ and $\beta_\emptyset=\beta$. We will also use the notations I_k and J_k for the support intervals of α_k and β_k respectively. (These are closed intervals whose endpoints are the smallest and largest points where α_k and β_k fail to vanish.)

Suppose α_k and β_k have been defined and $|k|$ is even. Then we choose a monotone decomposition $\alpha_k=\alpha_{k*1}+\alpha_{k*2}$ with the support intervals of α_{k*1} and α_{k*2} being at most half as long as I_k. Then we use Proposition 4.5 to construct a corresponding decomposition $\beta_n=\beta_{k*1}+\beta_{k*2}$. We proceed similarly if $|k|$ is odd, except that we first decompose β_k, controlling the lengths of J_{k*1} and J_{k*2}; and then apply Proposition 4.5 to decompose α_k.

If $\alpha_k=\beta_k=0$, take $a_k=b_k=0$; otherwise, fix points a_k and b_k in I_k and J_k respectively. For each integer n, write ε_n for the maximal length of the intervals I_k and J_k with $|k|=n$. By construction $\varepsilon_n\to 0$ as $n\to\infty$, and our application of Proposition 4.5 guarantees that $\delta(\alpha_k, \beta_k)\leqq\delta(\alpha, \beta)$ for all k. In particular $|a_k-b_k|\leqq$ $\leqq\delta(\alpha, \beta)+2\varepsilon_{|k|}$.

Now fix a Hilbert space of dimension $\alpha(\mathbf{R})$, and construct a family $\{P_k\}$ of projections on it satisfying $\operatorname{rank} P_k=\alpha_k(\mathbf{R})$ and $P_k=P_{k*1}+P_{k*2}$ for each multi-index k. For each integer n, set

$$A_n=\sum_{|k|=n} a_k P_k \quad \text{and} \quad B_n=\sum_{|k|=n} b_k P_k .$$

Each pair (A_n, B_n) is monotone and we have $\|A_n-B_n\|\leqq\delta(\alpha,\beta)+2\varepsilon_n$ for each n. Since $a_{k*j}\in I_k$ for all j, we also have $\|A_n-A_m\|\leqq\varepsilon_n$ for $m\geqq n$. This means the sequences $\{A_n\}$ and $\{B_n\}$ converge (in norm) to operators A' and B' respectively. We have that (A', B') is a monotone pair and $\|A'-B'\|\leqq\delta(\alpha,\beta)$.

Write α^n for the crude multiplicity function of A_n. Then α^n is a 'redistribution' of α which concentrates all of $\alpha(I_k)$ at a_k whenever $|k|=n$. Thus $\delta(\alpha^n,\alpha)\leqq\varepsilon_n$. We conclude that α and β are the crude multiplicity functions of A' and B' respectively, and the proof is complete.

Remark. The construction in the proof is sufficiently general to produce all pairs (A', B') satisfying the conclusion of the Theorem, but it is difficult to predict a priori what these will be. We will see in the next section that they can always be chosen to be diagonal.

Proof of Theorem 1.3. Choose A' and B' as in Theorem 4.13. That they belong to the closures of $\mathcal{U}(A)$ and $\mathcal{U}(B)$ respectively follows from Proposition 2.5, that they commute from Corollary 4.3. Finally, $\operatorname{dist}(\mathcal{U}(A),\mathcal{U}(B))\geqq\delta(\alpha,\beta)$ by Proposition 2.3 while $\delta(\alpha,\beta)=\|A'-B'\|\geqq\operatorname{dist}(\mathcal{U}(A,\mathcal{U}(B))$ by definition of distance.

5. Diagonal representatives

In this section we introduce an additional characterization of the distance between crude multiplicity functions which is closer in spirit to the quantity $\max|\alpha_j-\beta_j|$ of Theorem 1.1. This characterization provides a geometric interpretation of monotonicity and leads to a proof of the fact that the representatives in Theorem 4.13 can be chosen to be diagonal.

Definiton 5.1. Let G be a spectral measure on $\mathbf{R}^2$. The *crude multiplicity function* of G is the function ϱ which assigns the cardinal number $\operatorname{rank} G(V)$ to each open subset V of $\mathbf{R}^2$.

As in Section 2, we extend the domain of ϱ by setting $\varrho(S)=\inf\{\varrho(V)|\ V \text{ open}, V\supseteqq S\}$ for every subset S of $\mathbf{R}^2$. The extended ϱ is countably additive and inner regular in the sense that $\varrho(S)=\sup\{\varrho(F)|\ F \text{ finite}, F\subseteqq S\}$ for $S\subseteqq\mathbf{R}^2$.

Definition 5.2. Let ϱ be a crude multiplicity function on $\mathbf{R}^2$: The *marginals* α and β of ϱ are defined by $\alpha(S)=\varrho(S\times\mathbf{R})$ and $\beta(S)=\varrho(\mathbf{R}\times S)$ for every $S\subseteq\mathbf{R}^1$.

Marginals are crude multiplicity functions (on $\mathbf{R}^1$).

Proposition 5.3. *Let A and B be commuting self-adjoint operators with spectral measures E, F, and crude multiplicity functions α, β respectively. Write G for their joint spectral measure on $\mathbf{R}^2$, and ϱ for the crude multiplicity function of G.*

(1) *The marginals of ϱ are α and β.*

(2) $\|A-B\|=\sup\{|x-y| \mid \varrho(x,y)\neq 0\}$.

(3) *The pair (A,B) is monotone iff $x_1<x_2$ and $y_1>y_2$ implies at least one of $\varrho(x_1,y_1)$, $\varrho(x_2,y_2)$ is zero.*

Proof. (1) Follows immediately from the definition.

(2) If $A=\sum_{i,j} a_i P_{ij}$ and $B=\sum_{i,j} b_j P_{ij}$ are diagonal operators, then $\|A-B\|=$ $=\sup\{|a_i-b_j| \mid P_{ij}\neq 0\} = \sup\{|x-y| \mid \varrho(x,y)\neq 0\}$. The case of general A and B follows by redistribution of spectral measures.

(3) Suppose (A,B) is monotone, and $x_1<c<x_2$, $y_1>d>y_2$. If $E(-\infty,c)\leqq$ $\leqq F(-\infty,d)$, then $\varrho((-\infty,c)\times(d,\infty))=0$ so $\varrho(x_1,y_1)=0$, while if $F(-\infty,d)\leqq$ $\leqq E(-\infty,c)$, then $\varrho(x_2,y_2)=0$.

Suppose conversely ϱ is as stated in (3) and fix a, b. Then either $\varrho(x,y)=0$ for all $x<a$, $y>b$, or $\varrho(x,y)=0$ for all $x>a$, $y<b$. In the former case, we have $E(-\infty,a)\leqq F(-\infty,b)$; in the latter $F(-\infty,b)\leqq E(-\infty,a)$.

It is natural to call ϱ *monotone* if (3) of the Proposition holds — this means that the support of ϱ is a monotone relation in $\mathbf{R}^2$ in the usual sense. The number $\sup\{|x-y| \mid \varrho(x,y)\neq 0\}$ will be called the *departure* of ϱ — the smaller it is, the closer the support of ϱ is to the diagonal $x=y$.

Corollary 5.4. *Let α and β be crude multiplicity functions. The following numbers are equal:*

(1) *the distance $\delta(\alpha,\beta)$ between α and β,*

(2) *the minimum departure of all crude multiplicity functions on $\mathbf{R}^2$ having α and β as marginals,*

(3) *the minimum departure of all monotone crude multiplicity functions on $\mathbf{R}^2$ having α and β as marginals.*

Proof. By Propositions 2.3 and 2.5, we know that $\|A-B\|\geqq\delta(\alpha,\beta)$ for any operators A, B with crude multiplicity functions α, β respectively, and Theorem 4.13 tells us there is a monotone pair (A',B') with $\|A'-B'\|=\delta(\alpha,\beta)$. Application of Proposition 5.3 (2) completes the proof.

The numbers described in (2) and (3) of Corollary 5.4 are appropriate analogues of the expressions (1.3) and (1.2) of the Introduction. Indeed, let A and B be as in Theorem 1.1, and assume for simplicity that none of their eigenvalues $\alpha_1<\ldots<\alpha_n$ or $\beta_1<\ldots<\beta_n$ is repeated. Then the (crude) multiplicity functions α and β only take on the values 0 and 1. Every multiplicity function ϱ on $\mathbf{R}^2$ with these marginals must 'pair' the α_j's with the β_j's, i.e., there must be a permutation π so that ϱ takes on the value 1 at the points $(\alpha_j, \beta_{\pi j})$ and vanishes elsewhere. The number (2) of the Corollary is thus $\min_\pi \max_j |\alpha_j-\beta_{\pi j}|$, in agreement with (1.3). Since ϱ can only be monotone when π is the identity permutation, we also see that the expression in (3) of the Corollary reduces to $\max_j |\alpha_j-\beta_j|$.

The geometric appeal of Corollary 5.4 is somewhat offset by Definition 5.1, in which crude multiplicity functions on $\mathbf{R}^2$ are defined in terms of the somewhat elusive spectral measures on $\mathbf{R}^2$. The following analogue of Proposition 1.8 is intended to circumvent this problem.

Proposition 5.5. *Every crude multiplicity function on* $\mathbf{R}^2$ *is* (1) *compactly supported,* (2) *upper semi-continuous, and* (3) *vanishes in a deleted neighborhood of each point at which its value is finite. Conversely if* ϱ *is a cardinal-valued function on* $\mathbf{R}^2$ *having these properties, then there is a commuting pair* (A', B') *of diagonal operators such that* ϱ *is the crude multiplicity function of their joint spectral measure.*

Proof. The first assertion is a consequence of regularity. For the converse, suppose ϱ is a cardinal-valued function on $\mathbf{R}^2$ satisfying (1), (2) and (3). For each cardinal c, choose a countable dense subset S_c of $\varrho^{-1}(c)$. Let H be a Hilbert space of dimension $\varrho(\mathbf{R}^2)$, and choose an orthogonal supplementary family $\{P_p\}_{p\in\mathbf{R}^2}$ of projections on H such that $\operatorname{rank} P_p=c$ iff $p\in S_c$. Define the (discrete) spectral measure G on $\mathbf{R}^2$ by $G(S)=\bigvee_{p\in S} P_p$. Then G is the joint spectral measure of the operators $A'\equiv\Sigma^{\oplus} xP_{xy}$ and $B'\equiv\Sigma^{\oplus} yP_{xy}$. Since $\operatorname{rank} G(V)=\sum_{p\in V}\operatorname{rank} P_p=$ $=\sum_c \sum_{\lambda\in S_c\cap V}\varrho(\lambda)=\varrho(V)$, we see ϱ is the crude multiplicity function of G, and the proof is complete.

Corollary 5.6. *The operators* (A', B') *of Theorem* 4.13 *can be chosen to be diagonal.*

Proof. Let G be the joint spectral measure for any pair of operators satisfying the conclusion of Theorem 4.13, and write ϱ for the crude multiplicity function of G. Take (A', B') to be the pair of operators associated with ϱ by the final statement of Proposition 5.5.

6. Normal operators

It is a long-standing question whether the analogue of (1.1), i.e.,

$$\|A-B\| \geqq \min_{\pi} \max_{j} |\alpha_j - \beta_{\pi j}| \tag{6.1}$$

is valid for (finite-dimensional) normal operators, and the present paper has nothing to add to the subject. For a history of the problem and a summary of known partial results, the reader should consult [1], [4].

Of course, if (6.1) turns out to be false, none of the Theorems stated in § 1 would generalize to the normal case. Even if (6.1) is valid, it is hard to imagine a normal analogue for the monotonicity notions of § 4, but it is possible to formulate a plan for generalizing the balance of the paper.

So assume (6.1) is true. There is little trouble in adapting §§ 2—3 to the normal case — it is only necessary to allow the sets V and I of Definitions 2.1 and 2.2 respectively to range over the open subsets of the plane. The proof of Proposition 2.3 would have to be changed, but it seems reasonable to assume that (6.1) would at least carry over to operators with finite spectra, and then one could apply the redistribution of spectral measures technique. The real challenge would be in proving a substitute for Proposition 4.5. The truth of the following conjecture would imply the normal analogues of Theorems 4.13 and 1.3.

Conjecture. Let $\beta=\beta_1+\beta_2$ be crude multiplicity functions on $\mathbf{C}$, and assume $\beta_1(z)=0$ for $\operatorname{Re} z>0$ while $\beta_2(z)=0$ for $\operatorname{Re} z<0$. Then every α satisfying $\delta(\alpha, \beta)=r<\infty$ admits a decomposition $\alpha=\alpha_1+\alpha_2$ with $\delta(\alpha_i, \beta_i)\leqq r$ for $i=1, 2$.

This could perhaps be attacked via an 'exhaustion argument' similar to that used in the proof of the Hahn Decomposition Theorem for signed measures.

Bibliographical note. After our work was completed, we learned from E. C. Milner that a necessary and sufficient condition is now known for a relation between infinite sets to satisfy the conclusion of the Marriage Theorem. See R. Aharoni, C. St. J. A. Nash-Williams, S. Shelah, A general criterion for the existence of transversals, *Proc. London Mat. Soc.*, (3)**47** (1983), 43—68. However, this theorem does not seem to help in obtaining the conclusion we need in this paper (Proposition 3.2).

Note added in proof: For striking subsequent progress, see the forthcoming papers by K.R. Davidson, The distance between unitary orbits of normal operators, and The distance between unitary orbits of normal operators in the Calkin algebra.

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