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Classification and construction of complete hypersurfaces satisfying $R(X, Y)\cdot R=0$

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In one of his papers K. NOMIZU [3] examined the immersed hypersurfaces in $\mathbf{R}^{n+1}$ satisfying $R(X, Y)\cdot R=0$ for all tangent vectors X; Y, where the curvature endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of the manifold. The main theorem of Nomizu's paper is the following.

Theorem (K. Nomizu). *Let M be an n-dimensional, connected, complete Riemannian manifold, which is isometrically immersed in $\mathbf{R}^{n+1}$ so that the type number is greater than 2 at least at one point. If M satisfies the condition $R(X, Y)\cdot R=0$ then it is of the form $M=S^k\times\mathbf{R}^{n-k}$, where S^k is a hypersphere in a euclidean subspace $\mathbf{R}^{k+1}$ of $\mathbf{R}^{n+1}$ and $\mathbf{R}^{n-k}$ is a euclidean subspace orthogonal to $\mathbf{R}^{k+1}$.*

This theorem inspired the so called Nomizu conjecture: Every irreducible complete space with $\dim\geqq 3$ and $R(X, Y)\cdot R=0$ is locally symmetric.

But the answer for this conjecture was negative as H. TAKAGI [6] constructed a 3-dimensional counterexample. This counterexample is a connected complete immersed hypersurface in $\mathbf{R}^4$. Thus the problem is to determine all the connected complete n-dimensional immersed hypersurfaces in $\mathbf{R}^{n+1}$ satisfying $R(X, Y)\cdot R=0$, the description of which completes Nomizu's theorem. The main purpose of this paper is to give a complete description and classification of these hypersurfaces.

1. Basic formulas

A C^∞ Riemannian manifold*) (M^n, g) with the property $R(X, Y)\cdot R=0$ is called a semisymmetric manifold. Let us assume that the semisymmetric manifold (M^n, g) is an immersed hypersurface in $\mathbf{R}^{n+1}$. Let $\mathbf{n}$ be a normal unit vector field on a connected orientable neighbourhood U of M^n. If D resp. ∇ denotes the Riemannian

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*) The notion differentiable is used in the meaning C^∞.

covariant derivative in $\mathbf{R}^{n+1}$ resp. in M^n, then

$$
(1.1)\qquad \begin{aligned} D_X Y &= \nabla_X Y + H(X, Y)\mathbf{n}, \\ D_X \mathbf{n} &= A(X), \quad H(X, Y) = -g(A(X), Y) \end{aligned}
$$

holds, for all differentiable vector fields X, Y on U tangent to M^n. $H(X, Y)$ is the so-called second fundamental form of the hypersurface, and $A(X)$ is the so-called Weingarten field. The $A(X)$ is a symmetric endomorphism's field on the manifold. The rank of A at a point $p \in M^n$ is called the type number at p and it is denoted by $k(p)$.

The curvature tensor field $R(X, Y)Z$ of M^n is of the form

$$
(1.2)\qquad R(X, Y)Z = -g(A(X), Z)A(Y) + g(A(Y), Z)A(X)
$$

by the Gauss' equation.

The nullspace of the cuvature operator at a point p consists of vetors $Z \in T_p(M)$ for which $R(X, Y)Z = 0$ holds for all vectors $X; Y \in T_p(M)$. The dimension of the nullspace at p is called the index of nullity, and it is denoted by $i(p)$. If $k(p)$ is 0 or 1, then $R_p = 0$ holds, and $i(p) = n$ in this case. But if $k(p) > 1$ holds, then $k(p) = n - i(p)$ (see in [2], p. 42).

It is not hard to see, that all the hypersurfaces with $k(p) \leqq 2$ (or equivalently $i(p) \geqq n-2$) are semisymmetric. By Nomizu's theorem every connected, complete immersed semisymmetric hypersurface M^n in $\mathbf{R}^{n+1}$ is a cylinder, if at least at one point p, $k(p) > 2$ holds, so in what follows we examine only the hypersurfaces for which $k(p) \leqq 2$ holds at every point $p \in M^n$.

If at a point $k(p) = 2$ holds, then $i(p) = n-2$. Let λ_1 and λ_2 be the two non-trivial eigenvalues of A_p, and let $\mathbf{x}_1, \mathbf{x}_2$ be the corresponding orthogonal unit eigenvectors. If V_p^1 denotes the 2-dimensional subspace spanned by $\mathbf{x}_1$ and $\mathbf{x}_2$, then the orthogonal complement V_p^0 of V_p^1 is just the nullspace of the curvature operator, and also

$$
T_p(M) = V_p^0 + V_p^1
$$

holds. This direct sum is called the *V-decomposition* of the tangent space $T_p(M)$. Since $k(p) \leqq 2$ holds everywhere, and the eigenvalue functions $\lambda_1(q) \leqq \lambda_2(q)$ are continuous, so $k(q) = 2$ holds in a neighbourhood of p. I.e. the set, where $k(q) = 2$ holds, is an open set U in M^n. If we consider the above V-decomposition on U, then the distributions V^i, $i = 0; 1$, are differentiable, since V^1 is spanned by the vector fields of the form $R(X, Y)Z$.

The V-decomposition is defined at the points p with $k(p) < 2$ by the trivial decomposition $T_p(M) = V_p^0$.

Further on we examine the hypersurface on the open set U, where $k(q) = 2$ holds.

The following relations are simple consequences of the Bianchi identity

$\sigma(\nabla_X R)(Y, Z)=0$:

$$\nabla_{V^0}V^1 \subseteqq V^1, \quad \nabla_{V^0}V^0 \subseteqq V^0, \quad \nabla_{V^1}V^1 \subseteqq V^0+V^1 = T(M), \tag{1.3}$$

where the formula $\nabla_{V^l}V^j \subseteqq V^k$ means that for the differentiable vector fields X_l, tangent to V^l, the vector field $\nabla_{X_l}X_j$ is tangent to V^k.

We mention that the distribution V^1 is in general not integrable, but by the second relation in (1.3) it follows, that the distribution V^0 on U is always integrable and the integral manifolds are totally geodesic and locally euclidean submanifolds. From the first formula in (1.3) we can see too, that the distribution V^1 is parallel along the curves which are going in the above totalgeodesic integral manifolds of V^0.

Now let us consider a local system $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_{n-2}$ of differentiable unit vector fields tangent to V^0 which are paarwise orthogonal, furthermore, also $\nabla_{\mathbf{m}_\alpha}\mathbf{m}_\beta=0$ hold. From the above considerations it follows, that such a vector field system exists around every point of U.

Next we introduce some basic formulas w.r.t. the system $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_{n-2}$. For the differentiable vector fields X; Y tangent to V^1 we can write

$$\nabla_X \mathbf{m}_\alpha = B_\alpha(X)+\sum_\beta M_\alpha^\beta(X)\mathbf{m}_\beta, \quad \text{where} \quad B_\alpha(X)_{/p}\in V_p^1, \tag{1.4}$$

$$\nabla_X Y = \tilde{\nabla}_X Y+\sum_\alpha M^\alpha(X, Y)\mathbf{m}_\alpha, \quad \text{where} \quad \tilde{\nabla}_X Y_{/p}\in V_p^1. \tag{1.5}$$

Using these formulas we define the tensor fields B_α, M^α, M_α^β and the covariant derivative $\tilde{\nabla}$ only on the distribution V^1.

But let us extend these tensor fields and this covariant derivative over the whole tangent bundle in such a way that $B_\alpha(\mathbf{m}_p)=0$, $M_\alpha^\beta(\mathbf{m}_\gamma)=0$, $M^\alpha(\mathbf{m}_\beta, X)= =M^\alpha(\mathbf{m}_\beta, \mathbf{m}_\gamma)=0$ and $\tilde{\nabla}_{\mathbf{m}_\alpha}X=\nabla_{\mathbf{m}_\alpha}X, \tilde{\nabla}_{\mathbf{m}_\alpha}\mathbf{m}_\beta=0$ hold. Then the fields B_α, M^α, M_α^β are differentiable tensor fields indeed, furthermore, $\tilde{\nabla}$ is a metrical covariant derivative, i.e: $\tilde{\nabla}g=0$ holds. The following formulas are also obvious:

$$M^\alpha(X, Y) =-g\big(B_\alpha(X), Y\big), \quad M_\beta^\alpha(X) =-M_\alpha^\beta(X). \tag{1.6}$$

We leave the proof of these facts to the reader. Let $\tilde{R}(X, Y)Z$ be the curvature tensor of $\tilde{\nabla}$.

Proposition 1.1. *For differentiable vector fields X, Y, Z tangent to V^1 the tensor fields B_α, M^α, M_β^α, $\tilde{R}$ satisfy the following basic formulas:*

$$R(X, Y)Z = \tilde{R}(X, Y)Z+\sum_\alpha \{M^\alpha(Y, Z)B_\alpha(X)-M^\alpha(X, Z)B_\alpha(Y)\}, \tag{1.7}$$

$$(\tilde{\nabla}_X B_\alpha)(Y)-(\tilde{\nabla}_Y B_\alpha)(X) = \sum_\beta \{M_\alpha^\beta(X)B_\beta(Y)-M_\alpha^\beta(Y)B_\beta(X)\}, \tag{1.8}$$

$$\begin{aligned} &dM_\alpha^\beta(X, Y) = \\ &= \sum_\gamma M_\gamma^\beta(X)\wedge M_\alpha^\gamma(Y)-(1/2)\big\{M^\beta(X, B_\alpha(Y))-M^\beta(Y, B_\alpha(X))\big\}, \end{aligned} \tag{1.9}$$

$$(1.10)\qquad (\nabla_{\mathbf{m}_\alpha} B_\beta)(X) = -B_\beta \circ B_\alpha(X),$$

$$(1.11)\qquad (\nabla_{\mathbf{m}_\alpha} M_\beta^\gamma)(X) = -M_\beta^\gamma(B_\alpha(X)),$$

$$(1.12)\qquad \begin{gathered} \tilde{R}(\mathbf{m}_\alpha, X)Y = 0, \\ \text{i.e.}\quad \nabla_{\mathbf{m}_\alpha}\tilde{\nabla}_X Y = \tilde{\nabla}_X \nabla_{\mathbf{m}_\alpha} Y + \tilde{\nabla}_{\nabla_{\mathbf{m}_\alpha} X} Y - \tilde{\nabla}_{B_\alpha(X)} Y - \sum_\beta M_\alpha^\beta(X) \nabla_{\mathbf{m}_\beta} Y, \end{gathered}$$

$$(1.13)\qquad (\nabla_{\mathbf{m}_\alpha} R)(X, Y) = R(Y, B_\alpha(X)) + R(B_\alpha(Y), X),$$

where d is the exterior derivative and the symbol $\wedge$ denotes the skew-product.

The complete proof of these formulas is contained in [4]. But we mention, that (1.7) follows by (1.4) and (1.5) from the formula $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$, the formulas (1.8)—(1.12) are equivalent to the identities $R(X, Y)\mathbf{m}_\alpha = 0$, $R(\mathbf{m}_\alpha, X)Y = 0$, $R(\mathbf{m}_\alpha, X)\mathbf{m}_\beta = 0$, and formula (1.13) follows from the Bianchi identity and from (1.4) in the following manner:

$$(\nabla_{\mathbf{m}_\alpha} R)(X, Y) = -(\nabla_X R)(Y, \mathbf{m}_\alpha) - (\nabla_Y R)(\mathbf{m}_\alpha, X) = R(Y, B_\alpha(X)) + R(B_\alpha(Y), X).$$

Here the details are also left to the reader.

2. Reduction of the basic formulas

Further on let us examine the complete connected semisymmetric hypersurface M^n in $\mathbf{R}^{n+1}$ with $k(p) \leqq 2$ on the open set U, where $k(p) = 2$ and thus $R_p(X, Y)Z \neq 0$ holds. Let us consider also the V-decomposition $T(M) = V^0 + V^1$ on U and for a point $p \in U$ let us consider the maximal connected integral manifold N of V^0 through a point p. If $c(s)$ is a differentiable curve in N, parametrized by arc-length and if $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_{n-2}$ is a vector field system around $c(s)$ defined in the previous chapter, then for the tangent vector $\dot{c}(s) = \sum_\alpha a^\alpha(s)\mathbf{m}_\alpha$ the tensor, defined by

$$(2.1)\qquad B_{\dot{c}(s)} := \sum_\alpha a^\alpha(s) B_{\alpha/c(s)},$$

is uniquely determined, and it is independent from the choice of the system $\mathbf{m}_1, \ldots$ $\ldots, \mathbf{m}_{n-2}$ around $c(s)$. Indeed if $\tilde{\mathbf{m}}_1, \ldots, \tilde{\mathbf{m}}_{n-2}$ is another system around $c(s)$ with $\tilde{\mathbf{m}}_\alpha = \sum_\beta b_\alpha^\beta \mathbf{m}_\beta$, and the corresponding tensors w.r.t. this system are denoted by $\tilde{B}_\alpha$, then from

$$\tilde{B}_\alpha = \sum_\beta b_\alpha^\beta B_\beta, \quad \mathbf{m}_\beta = \sum_\alpha (b^{-1})_\beta^\alpha \tilde{\mathbf{m}}_\alpha, \quad \dot{c}(s) = \sum_\beta a^\beta(s)\mathbf{m}_\beta = \sum_{\alpha,\beta} a^\beta (b^{-1})_\beta^\alpha \mathbf{m}_\alpha$$

we get

$$\sum_{\alpha,\beta} a^\beta (b^{-1})_\beta^\alpha \tilde{B}_\alpha = \sum_{\alpha,\beta,\gamma} a^\beta (b^{-1})_\beta^\alpha b_\alpha^\gamma B_\gamma = \sum_\alpha a^\alpha B_\alpha,$$

which proves the above statement.

Let us notice too, that the curvature tensor $R(X, Y)Z$ is of the form

$$R(X,Y)Z = K\big(g(Y,Z)X - g(X,Z)Y\big), \quad X; Y; Z \in V^1, \tag{2.2}$$

on V^1, where $K(p)$ is the sectional curvature w.r.t. the section V_p^1 at p. From (1.13) it follows that the function $K(s)=K(c(s))$ satisfies the differential equation

$$\frac{dK}{ds} = -(\operatorname{Tr} B_{\dot c})K, \tag{2.3}$$

and thus we have

$$K(s) = K(0)e^{-\int_0^s \operatorname{Tr} B_{\dot c}\, ds}. \tag{2.4}$$

From this formula we get, that K is zero neither on N nor on the boundary of N, and thus the boundary of N is inside of U. But N is maximal, thus N cannot have boundary points. As the space is complete, N is a complete, connected, locally euclidean and totally geodesic submanifold in the maifold M. On the other hand the second fundamental form A vanishes on the tangent spaces of N, further V^1 is totally parallel along N, thus N is an open subset in an $(n-2)$-dimensional euclidean subspace $\mathbf{R}^{n-2}$ of $\mathbf{R}^{n+1}$. But because of the completeness of N it must be equal to the whole euclidean subspace $\mathbf{R}^{n-2}$, and thus we have

Proposition 2.1. *Every maximal integral manifold N of V^0, through a point p, where $R_p \neq 0$ holds, is complete, totally geodesic and isometric with $\mathbf{R}^{n-2}$. In addition N is an $(n-2)$-dimensional euclidean subspace in $\mathbf{R}^{n+1}$. The curvature tensor R_p of the space M^n never vanishes at the points of such a submanifold N.*

Now let $c(s)$, $-\infty < s < \infty$, be a complete geodesic in a subspace N, considered in the above proposition and parametrised by arc-length s. Let us consider also $B_{\dot c}$ along $c(s)$ defined in (2.1). Then

$$\nabla_{\dot c} B_{\dot c} = -B_{\dot c}^2 \tag{2.5}$$

holds. From this equation it follows, that $B_{\dot c}$ never vanishes along $c(s)$ if it is non-zero at a point $c(s_0)$, and so it is a zero-field, if it is zero at a point. Let us remember too, that V^1 is invariant under the action of $B_{\dot c}$, and that also $B_{\dot c}(V^0)=0$ holds.

Next we solve the differential equtiaon (2.5). We can distinguish two cases.

Accordingly let $\dot c$ and $B_{\dot c}$ be as above in a connected and complete semisymmetric hypersurface M^n with $k(p) \leqq 2$.

Proposition 2.2. *If the endomorphism $B_{\dot c}$ degenerates at a point $c(s_0)$ in $V^1_{c(s_0)}$, then $B_{\dot c}^2=0$ holds along the whole $c(s)$ and $B_{\dot c}$ is parallel along $c(s)$.*

Proposition 2.3. *If the endomorphism $B_{\dot c}$ is non-singular at one point $c(s_0)$ in $V^1_{c(s_0)}$, then it is non-singular along $c(s)$ in $V^1_{c(s_0)}$, and at every point $c(s)$ the eigenvalues of $B_{\dot c}$ are non-real complex numbers in V^1_c:*

As a consequence we get, that in a complete semisymmetric hypersurface with $k(p)\leqq 2$ the endomorphisms $B_{\dot c}$ cannot have real non-zero eigenvalues.

In the following proofs the completeness of the manifold is important.

Proof of Proposition 2.2. Let $\mathbf{x}_1(s_0)$ be the unit vector in $V^1_{c(s_0)}$ belonging to the image set of $B_{\dot c(s_0)}$ and let $\mathbf{x}_2(s_0)$ be the orthogonal unit vector in $V^1_{c(s_0)}$. Let us extend these vectors into parallel vector fields $\mathbf{x}_1(s)$, $\mathbf{x}_2(s)$ along $c(s)$. Then these are tangent to $V^1_{c(s)}$.

The restriction of $B_{\dot c(s_0)}$ onto $V^1_{c(s_0)}$ has the matrix in $\{\mathbf{x}_1(s_0), \mathbf{x}_2(s_0)\}$ of the form

$$\begin{bmatrix}\lambda(s_0), & \gamma(s_0)\\ 0, & 0\end{bmatrix}, \tag{2.6}$$

where $\lambda(s_0)=0$ holds iff $B^2_{\dot c(s_0)}=0$ is satisfied. The solutions of (2.5) are uniquely determined by the initial value (2.6), so if $\lambda(s_0)=0$ holds, then the solution of (2.5) has the matrix of the form

$$\begin{bmatrix}0, & \gamma(s)=\gamma(s_0)\\ 0, & 0\end{bmatrix}, \tag{2.7}$$

w.r.t. the basis $\{\mathbf{x}_1(s), \mathbf{x}_2(s)\}$ in $V^1_{c(s)}$, since (2.7) is a solution of (2.5) with the above initial conditions.

Now if $\lambda(s_0)\neq 0$ holds, then the solution of (2.5) has the matrix of the form

$$\begin{bmatrix}\dfrac{1}{s+c_1}, & \gamma(s_0)e^{-\int_{s_0}^{s} dt/(t+c_1)}\\ 0, & 0\end{bmatrix}, \tag{2.8}$$

w.r.t. $\{\mathbf{x}_1(s), \mathbf{x}_2(s)\}$ in $V^1_{c(s)}$, where $c_1=(1-s_0\lambda(s_0))/\lambda(s_0)$ is constant. But in this case the functions $\lambda(s)$, $\gamma(s)$, $K(s)$ have infinity value at $-c_1$ which contradicts the completeness of the manifold. Thus this case doesn't occur and $\lambda(s_0)=0$ holds, which proves the proposition.

Proof of Proposition 2.3. Let $\{\mathbf{x}_1(s_0), \mathbf{x}_2(s_0)\}$ be an orthonormed basis in $V^1_{c(s_0)}$ such that the vectors $\mathbf{x}_i(s_0)$ are the eigenvectors of the symmetric part of $B_{\dot c(s_0)}$. The matrix of $B_{\dot c(s_0)}$ restricted onto $V^1_{c(s_0)}$ is of the form

$$\begin{bmatrix}\alpha_1(s_0), & -\beta(s_0)\\ \beta(s_0), & \alpha_2(s_0)\end{bmatrix}, \tag{2.9}$$

w.r.t. this basis. Let $\{\mathbf{x}_1(s), \mathbf{x}_2(s)\}$ be the extension of $\{\mathbf{x}_1(s_0), \mathbf{x}_2(s_0)\}$ onto $c(s)$ by parallel displacement. If we consider $B_{\dot c(s)}$ only in $V^1_{c(s)}$, then from (2.5) we get the following:

$$B_{\dot c}^{-1}\nabla_{\dot c}B_{\dot c} = -B_{\dot c}, \quad \nabla_{\dot c}B_{\dot c}^{-1} = I.$$

Thus the matrix of the solution $B_{\dot{c}}$ of (2.5) with initial condition (2.9) is

$$\tag{2.10} \begin{bmatrix} \dfrac{s+c_1}{(s+c_1)(s+c_2)+c_3^2}, & \dfrac{-c_3}{(s+c_1)(s+c_2)+c_3^2} \\ \dfrac{c_3}{(s+c_1)(s+c_2)+c_3^2}, & \dfrac{s+c_2}{(s+c_1)(s+c_2)+c_3^2} \end{bmatrix},$$

w.r.t. $\{\mathbf{x}_1(s), \mathbf{x}_2(s)\}$, where

$$c_1 = \big((\alpha_1(0)/\det B_{\dot{c}(s_0)}\big)-s_0, \quad c_2 = \big(\alpha_2(0)/\det B_{\dot{c}(s_0)}\big)-s_0, \quad c_3 = \big(\beta(0)/\det B_{\dot{c}(s_0)}\big)-s_0.$$

Because of the completeness of the hypersurfaces the equation

$$(s+c_1)(s+c_2)+c_3^2 = 0$$

of second order can't have real solution, i.e. for it's discriminant Δ

$$\Delta = (c_1-c_2)^2-4c_3^2 < 0$$

holds. It is easy to see from (2.10) that by this conditon the eigenvalues of the restricted $B_{\dot{c}(s)}$ are non-real along $c(s)$ which proves the proposition.

After these propositions we examine the orthogonal projection of vector fields $\nabla_X Y$ onto V^0, where X and Y are tangent to V^1. We denote this projected vector field by $v(\nabla_X Y)$.

Proposition 2.4. *Let M^n be a connected complete semisymmetric hypersurface with $k(p)\leqq 2$. Then the vectors $v(\nabla_X Y)$ span an at most 1-dimensional subspace S_p in V_p^0 for every point p.*

Proof. We start with the indirect assumption $\dim S_p \geqq 2$ for a point p. By the assumption the V-decomposition is of the form $T_q(M)=V_q^0+V_q^1$ around p, where $\dim V_q^0=n-2$. Let $\{\mathbf{x}_1, \mathbf{x}_2\}$ be an orthonormed differentiable basic field around p in V^1. Let us denote the vector $v(\nabla_{\mathbf{x}_i}\mathbf{x}_j)_{/p}$ by $\mathbf{x}_{ij}$. Then for arbitrary unit vector $\mathbf{m}$, tangent to V_p^0, the matrix of $B_{\mathbf{m}}$ w.r.t. $(\mathbf{x}_1, \mathbf{x}_2)$ is the following:

$$\begin{bmatrix} -g(\mathbf{x}_{11}, \mathbf{m}), & -g(\mathbf{x}_{21}, \mathbf{m}) \\ -g(\mathbf{x}_{12}, \mathbf{m}), & -g(\mathbf{x}_{22}, \mathbf{m}) \end{bmatrix}.$$

The characteristic equation of this matrix is

$$\lambda^2+\{g(\mathbf{x}_{11}, \mathbf{m})+g(\mathbf{x}_{22}, \mathbf{m})\}\lambda+\{g(\mathbf{x}_{11}, \mathbf{m})g(\mathbf{x}_{22}, \mathbf{m})-g(\mathbf{x}_{12}, \mathbf{m})g(\mathbf{x}_{21}, \mathbf{m})\} = 0,$$

which has the discriminant

$$\Delta = \{g(\mathbf{x}_{11}, \mathbf{m})-g(\mathbf{x}_{22}, \mathbf{m})\}^2+4g(\mathbf{x}_{12}, \mathbf{m})g(\mathbf{x}_{21}, \mathbf{m}).$$

If $\mathbf{x}_{11}\neq 0$ or $\mathbf{x}_{22}\neq 0$ holds and $\mathbf{m}$ is orthogonal to $\mathbf{x}_{12}$ or to $\mathbf{x}_{21}$, then the eigenvalues are $-g(\mathbf{x}_{11}, \mathbf{m})$, $-g(\mathbf{x}_{22}, \mathbf{m})$. And if $\mathbf{x}_{11}=\mathbf{x}_{22}=0$ holds, furthermore $\mathbf{m}$

halves the angle of $\mathbf{x}_{12}$ and $\mathbf{x}_{21}$, then the eigenvalues are $\pm\sqrt{g(\mathbf{x}_{12},\mathbf{m})g(\mathbf{x}_{21},\mathbf{m})}\neq\neq 0$, and these eigenvalues are also reals. Consequently we can choose such a vector $\mathbf{m}$ for which $B_{\mathbf{m}}$ has real, non-zero eigenvalue. This contradicts the previous proposition and the proof is complete.

Let p be a point for which $\dim S_p=1$ holds. Then $\dim S_p=1$ holds in a neighbourhood of p. Let M^2 be such a 2-dimensional submanifold through p in the points of which

$$T_q(M^n) = T_q(M^2)+V_q^0, \quad \dim S_q = 1$$

hold. Let us choose such a system $\mathbf{m}_1, \ldots, \mathbf{m}_{n-2}$ around p for which the vectors $\mathbf{m}_1(q)$, $q\in M^2$, are pointing in the direction of S_q. Then in the points $q\in M^2$

$$B_1(q)\neq 0, \quad B_2(q)=\ldots=B_{n-2}(q)=0$$

holds. Since the differential equation (1.10) is of first order, so

$$B_1\neq 0, \quad B_2=\ldots=B_{n-2}=0$$

hold everywhere, and $\mathbf{m}_1$ is pointing in the direction of S.

A system $\mathbf{m}_1, \ldots, \mathbf{m}_{n-2}$ constructed in this way is called a *reduced system.* For such a system only the first tensor B_1 is non-trivial, which we denote by B. Also the basic formulas (1.8) and (1.9) are more simple w.r.t. such a system, and we get for them:

$$(\tilde{\nabla}_X B)(Y)-(\tilde{\nabla}_Y B)(X) = 0, \tag{2.12}$$

$$M_\alpha^1(X)B(Y)-M_\alpha^1(Y)B(X) = 0, \tag{2.13}$$

$$dM_\alpha^\beta-\sum_\gamma M_\alpha^\gamma\wedge M_\gamma^\beta = 0. \tag{2.14}$$

The other basic formulas are unchanged.

At the end we give some definitions.

Let M^n be a connected complete immersed hypersurface in $\mathbf{R}^{n+1}$ with $k(p)\leqq 2$ everywhere. Let $\mathscr{V}_1$ be the open set, where $k(p)=2$, i.e. $K(p)\neq 0$ holds for the Riemannian curvature scalar K. Then in the interior $\mathscr{V}_0$ of $M^n\setminus\mathscr{V}_1$ the Riemann curvature $R(X, Y)Z$ vanishes. Let $\mathscr{V}_2\subseteqq\mathscr{V}_1$ be the open set where the subspace S_p (defined in Proposition 2.4) is 1-dimensional. Then the tensor B vanishes in the interior $\mathscr{V}_t$ of $\mathscr{V}_1\setminus\mathscr{V}_2$. The open set $\mathscr{V}_t$ is called the pure trivial part of M^n. At the end let $\mathscr{V}_h\subseteqq\mathscr{V}_2$ be the open set where B has two non-real eigenvalues. Then in the interior $\mathscr{V}_p$ of $\mathscr{V}_2\setminus\mathscr{V}_h$ B doesn't vanish and it has only zero eigenvalues on $\mathscr{V}_p$. The open sets $\mathscr{V}_p$ resp. $\mathscr{V}_h$ are called the pure parabolic resp. pure hyperbolic part of M^n.

It is rather trivial that the open set

$$\mathscr{V}_0\cup\mathscr{V}_t\cup\mathscr{V}_p\cup\mathscr{V}_h \tag{2.15}$$

is everywhere dense in M^n. Furthermore the open sets $\mathscr{V}_t, \mathscr{V}_p$ resp. $\mathscr{V}_h$ always contain the complete integral manifolds of V^0, i.e. the type of the hypersurface is uniquely determined along a maximal integralmanifold of V^0, where $\dim V_q^0=n-2$ holds.

Now let M^n be a general (not necessarily complete) immersed hypersurface, with $k(p)\leqq 2$ everywhere. The V-decomposition is defined for it in the same way as in § 1. This decomposition is of the form

$$T_p(M^n) = V_p^0+V_p^1, \quad \dim V_p^0 = n-2,$$

iff the Riemannian curvature scalar $K(p)$ doesn't vanish. The maximal integral manifold of V^0 through such a point p is always an open set in an euclidean subspace $\mathbf{R}^{n-2}$ of $\mathbf{R}^n$. The M^n is called *vertically complete* iff all these integral manifolds are complete euclidean suspaces $\mathbf{R}^{n-2}$ in $\mathbf{R}^n$.

We can define the open sets $\mathscr{V}_0, \mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_t, \mathscr{V}_p, \mathscr{V}_h$ for vertically complete hypersurfaces with $k(p)\leqq 2$ in some way as before, since propositions (2.2), (2.3) and (2.4) hold for such hypersurfaces also. The type of hypersurfaces along an integral manifold of V^0 (where $\dim V_q^0=n-2$) is also uniquely determined.

Definition. A vertically complete immersed hypersurface M^n with $k(p)\leqq 2$ is said to be of

1) trivial type if $\mathscr{V}_2=\emptyset$ holds, i.e. M^n contains only $\mathscr{V}_0$ resp. pure trivial parts,

2) parabolic type if $\mathscr{V}_t=\mathscr{V}_h=\emptyset$, $\mathscr{V}_p\neq\emptyset$, hold, i.e. M^n contains only $\mathscr{V}_0$ and nonempty pure parabolic part,

3) hyperbolic type if $M^n=\mathscr{V}_h$, i.e. M^n contains only pure hyperbolic part.

By formula (2.15) all complete hypersurfaces with $k(p)\leqq 2$ can be built up from vertically complete hypersurfaces of the above types. In the next sections we give general procedures for the construction of vertically complete immersed hypersurfaces of the above types.

3. Hypersurfaces of trivial type

Strong theorems are known — local or global — which describe all the hypersurfaces with zero Riemannian curvature. For example a complete connected hypersurface M^n with zero Riemannian curvature is a cylinder of the form $M^n=c\times\mathbf{R}^{n-1}$ where c is a curve in an euclidean plane $\mathbf{R}^2$ and $\mathbf{R}^{n-1}$ is the orthogonal complement of $\mathbf{R}^2$ [1]. So by the description of hypersurfaces of trivial type we assume that the open set $\mathscr{V}_t$ is nonempty.

Proposition 3.1. *Let U be a connected component of $\mathscr{V}_t$ in a hypersurface of trivial type. Then U is a cylinder of the form $U=M^2\times\mathbf{R}^{n-2}$, where M^2 is a hypersurface in a euclidean subspace $\mathbf{R}^3$ and $\mathbf{R}^{n-2}$ is the orthogonal complement to $\mathbf{R}^3$.*

Proof. The tensor fields B_α are zero in the considered case, so $\nabla_{V^1}V^1 \subseteqq V^1$ holds. So the distribution V^1 is integrable and the integral manifolds are totally geodesic. Let M^2 be an integral manifold of V^1. From $B_\alpha=0$ and $A(V^0)=0$,

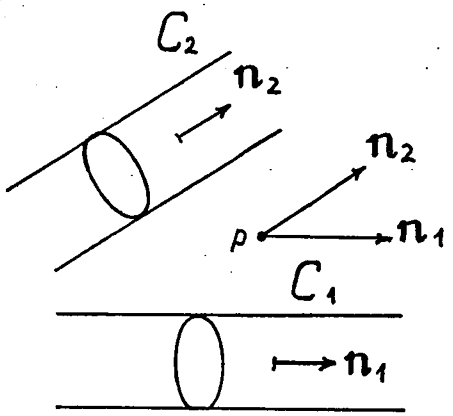
$$D_{V^1}V^0 \subseteqq V^0$$

follows, where D is the covariant derivative of $\mathbf{R}^{n+1}$. Thus the integral manifolds of V^0 are parallel euclidean subspaces, and M^2 is contained in the orthogonal complement $\mathbf{R}^3$ of these parallel subspaces. It is rather trivial, that U is of the form $U=M^2\times\mathbf{R}^{n-2}$ indeed.

The following theorem is obvious.

Theorem 3.1. *For a hypersurface of trivial type there exists an everywhere dense open subset, on the connected component of which the space is of zero Riemannian curvature or it is a cylinder described in the above proposition.*

Generally a hypersurface of trivial type doesn't split into a global direct product of the form $M^2\times\mathbf{R}^{n-2}$. To show this fact we construct a 3-dimensional irreducible hypersurface of trivial type.



Let C_1 and C_2 be two infinite closed circle-cylindrical domains without common points in $\mathbf{R}^3$, which are pointing in different directions $\mathbf{n}_1$ resp. $\mathbf{n}_2$. Furthermore let $f(x, y, z)$ be such a differentiable real function on $\mathbf{R}^3$ which has zero value on $\mathbf{R}^3\setminus(C_1\cup C_2)$ and f is positive inside of C_i, $i=1, 2$, such that it is constant along the lines parallel to $\mathbf{n}_i$. Such functions obviously exist.

Proposition 3.2. *The hypersurface M^3 represented by $(x, y, z, f(x, y, z))$ in $\mathbf{R}^4$ is a complete irreducible hypersurface of trivial type, diffeomorphic to $\mathbf{R}^3$.*

Proof. The open sets $\mathscr{V}^i\subset M^3$, $i=1; 2$, represented by $(x, y, z, f(x, y, z))$, $(x, y, z)\in C_i$, are cylindrical of the form $\mathscr{V}^i=M_i^2\times\mathbf{R}$, furthermore the Riemannian curvature vanishes on $M^3\setminus(\mathscr{V}^1\cup\mathscr{V}^2)$. Thus M^3 is of trivial type.

Let p be arbitrary point of $\mathbf{R}^3\setminus(C_1\cup C_2)$. Then p is a point of M^3. It is easy to show, that the holonomy group H_p of M^3 is generated by the rotation groups $SO(2)_1$, $SO(2)_2$, where $SO(2)_i$, $i=1; 2$, acts around the axis through p pointing in the direction of $\mathbf{n}_i$. Thus $H_p\cong SO(3)$ holds, and M^3 is irreducible. The other statement in the proposition is obvious.

Since the above example is not locally symmetric, so it is also a counterexample to Nomizu's conjecture.

With the above method one can construct n-dimensional complete irreducible hypersurfaces of trivial type for any dimension n.

4. Hypersurfaces of parabolic type

Let us consider the hypersurface M^n on the open set $\mathscr{V}_p$, where $R\neq 0$, $B\neq 0$ with $B^2=0$. The system $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_{n-2}$ is by assumption a reduced system. Let $\{\partial_0, \partial_1\}$ be an orthonormed basis in V^1 such that ∂_1 is tangent to the image space of B.

By $\nabla_{\mathbf{m}_\alpha}B=0$ we get that ∂_0 and ∂_1 are parallel vector fields along any integral manifold of V^0, i.e. $\nabla_{\mathbf{m}_\alpha}\partial_i=0$ holds. Furthermore from $B^2=0$ we have that the matrix of the restricted B (onto V^1) is of the form

$$\begin{bmatrix} 0, & 0 \\ b, & 0 \end{bmatrix}, \tag{4.1}$$

w.r.t. $\{\partial_0, \partial_1\}$.

Let us introduce also the functions λ, λ_1 by

$$\tilde{\nabla}_{\partial_0}\partial_0=\lambda\partial_1,\quad \tilde{\nabla}_{\partial_1}\partial_1=\lambda_1\partial_0,\quad \tilde{\nabla}_{\partial_0}\partial_1=-\lambda\partial_0,\quad \tilde{\nabla}_{\partial_1}\partial_0=-\lambda_1\partial_1. \tag{4.2}$$

Proposition 4.1. *The above functions satisfy the following equations:*

$$\lambda_1=0,\quad \partial_1(b)=\lambda b, \tag{4.3}$$

$$\nabla_{\partial_1}\partial_0=\nabla_{\mathbf{m}_\alpha}\partial_i=0. \tag{4.4}$$

Proof. From (2.12), (4.1) and (4.2) we have

$$(\tilde{\nabla}_{\partial_1}B)(\partial_0)=\partial_1(b)\partial_1+b\lambda_1\partial_0=(\tilde{\nabla}_{\partial_0}B)(\partial_1)=B(\lambda\partial_0)=\lambda b\partial_1$$

so we get (4.3). (4.4) is obvious by $\lambda_1=0$ and by the above considerations.

Now let us examine the Weingarten field A of the hypersurface. As for it $A(V^0)=0$, $A(V^1)=V^1$ hold, so let $\tilde{A}$ be the restriction of A onto V^1. The matrix of $\tilde{A}$ w.r.t. $\{\partial_0, \partial_1\}$ is of the form

$$\begin{bmatrix} \gamma_0, & \delta \\ \delta, & \gamma_1 \end{bmatrix}. \tag{4.5}$$

Proposition 4.2. *The Weingarten field A satisfies the following relations:*

$$\nabla_{\mathbf{m}_1} A = -A\circ B, \quad \nabla_{\mathbf{m}_\alpha} A = 0 \quad \textit{for} \quad \alpha \geqq 2, \tag{4.6}$$

$$A\circ B \quad \textit{is symmetric,} \quad (\tilde{\nabla}_{\partial_0} A)(\partial_1) = (\tilde{\nabla}_{\partial_1} A)(\partial_0), \tag{4.7}$$

$$\gamma_1 = 0, \quad \mathbf{m}_1(\gamma_0) + \delta b = 0, \quad \mathbf{m}_\alpha(\gamma_0) = 0 \quad \text{for} \quad \alpha \geqq 2, \tag{4.8}$$

$$\mathbf{m}_\alpha(\delta) = 0 \quad \textit{if} \quad \alpha \geqq 1, \tag{4.9}$$

thus δ is constant along the integral manifolds of V^0,

$$\partial_1(\gamma_0) = \partial_0(\delta) + \lambda\gamma_0, \quad \partial_1(\delta) = 2\lambda\delta. \tag{4.10}$$

Proof. Equations (4.6) and (4.7) come from the Codazzi—Mainardi equation

$$(\nabla_X A)(Y) = (\nabla_Y A)(X),$$

using the vector fields $\partial_0, \partial_1, \mathbf{m}_1, \ldots, \mathbf{m}_{n-2}$. The equation $\gamma_1 = 0$ comes from symmetry of $A\circ B$, and the others are equivalent to (4.6) and (4.7) using the formulas (4.1)—(4.5).

By (4.8) and (4.5) the curvature scalar K of M^n is

$$K = \det \tilde{A} = -\delta^2 < 0 \tag{4.11}$$

on $\mathcal{V}_p$, so the matrix of $\tilde{A}$ in $\{\partial_0, \partial_1\}$ is of the form

$$\tilde{A}^i_j = \begin{bmatrix} \gamma_0, & \sqrt{-K} \\ \sqrt{-K}, & 0 \end{bmatrix}. \tag{4.12}$$

By the second equation of (4.10) also the equation

$$\partial_1(K) = 4\lambda K \tag{4.13}$$

holds.

Let us notice too, that the sectional curvature K_σ is non-positive in a hypersurface of parabolic type so from the Hadamard—Cartan theorem we get:

Proposition 4.3. *The sectional curvature K_σ of a hypersurface M^n of parabolic type is non-positive. Thus if M^n is complete and simply connected then it is diffeomorphic to $\mathbf{R}^n$.*

Proposition 4.4. *The distribution W^0, spanned by ∂_1 and V^0, is involutive, and the integral manifolds of W^0 are open sets in $(n-1)$-dimensional euclidean subspaces of $\mathbf{R}^{n+1}$. In addition if the hypersurface is complete, then the maximal integral manifolds of W^0 are complete $(n-1)$-dimensional euclidean subspaces in $\mathbf{R}^{n+1}$.*

Proof. For the Lie derivative $[\partial_1, \mathbf{m}_\alpha]$ resp. $[\mathbf{m}_\alpha, \mathbf{m}_\beta]$ we have

$$[\partial_1, \mathbf{m}_\alpha] = \nabla_{\partial_1}\mathbf{m}_\alpha - \nabla_{\mathbf{m}_\alpha}\partial_1 = \nabla_{\partial_1}\mathbf{m}_\alpha = B_\alpha(\partial_1) + \sum_\gamma M^\gamma_\alpha(\partial_1)\mathbf{m}_\gamma = \sum_\gamma M^\gamma_\alpha(\partial_1)\mathbf{m}_\gamma,$$

$$[\mathbf{m}_\alpha, \mathbf{m}_\beta] = 0,$$

thus W^0 is involutive.

Let H be an integral manifold of W^0. Then H is a hypersurface in M^n with normal vector field ∂_0. H is by (4.4) a totally goedesic hypersurface in M^n with zero Riemannian curvature as well.

Let D be the covariant derivative in $\mathbf{R}^{n+1}$. By (4.1) and (4.5) we have

$$D_{\partial_1}\mathbf{n} = \delta\partial_0, \quad D_{\partial_1}\partial_0 = -\delta\mathbf{n}, \quad D_{\mathbf{m}_\alpha}\mathbf{n} = 0, \quad D_{\mathbf{m}_\alpha}\partial_0 = 0.$$

Thus the planes spanned by $\mathbf{n}$ and ∂_0 (along H) are parallel, and so H is an open set in the euclidean subspace which is orthogonal to the above parallel planes.

Now let M^n be a complete hypersurface of parabolic type and let H be a maximal integral manifold of W^0. From the second equation of (4.3) and from (4.13) we get, that K resp. B vanishes neither on H nor on the boundary of H. Thus H is without boundary points and so it is a complete $(n-1)$-dimensional euclidean subspace in $\mathbf{R}^{n+1}$.

By the above proposition every connected component $\mathscr{V}_p^i$ of $\mathscr{V}_p$ in a complete M^n can be considered as a fibred space $\Pi\colon \mathscr{V}_p^i\to\mathbf{R}$, where the fibres $\Pi^{-1}(q)$, $q\in\mathbf{R}$, are $(n-1)$-dimensional euclidean spaces. In the following proposition we make this fibration into a global fibration.

Theorem 4.1. *Let M^n be a simply connected and complete immersed hypersurface of parabolic type in $\mathbf{R}^{n+1}$. Then M^n is in a natural manner a fibred space $\Pi\colon M^n\to\mathbf{R}$, where the fibres $\Pi^{-1}(q)$, $q\in\mathbf{R}$, are $(n-1)$-dimensional euclidean subspaces in $\mathbf{R}^{n+1}$.*

Proof. Let us examine M^n on the open set $\mathscr{V}_0$. The rank of the Weingarten field A on $\mathscr{V}_0$ is 1 or 0. Let $\mathscr{V}_0^1\subseteqq\mathscr{V}_0$ be the open set, where $\operatorname{rank} A=1$ holds, and let $\mathscr{V}_0^0$ be the interior of $\mathscr{V}_0\setminus\mathscr{V}_0^1$. If ∂_0 is the unit vector field on $\mathscr{V}_0^1$, tangent to the image-space of A, then

$$A(\partial_0) = \gamma_0\partial_0 \quad\text{with}\quad \gamma_0\neq 0$$

holds. Let $\overset{*}{W}{}_q^0\subset T_q(\mathscr{V}_0^1)$, $q\in\mathscr{V}_0^1$, be the subspace orthogonal to $\partial_0(q)$. It is well known that the distribution $\overset{*}{W}{}^0$ is involutive and the integral manifolds of it are open sets in the $(n-1)$-dimensional euclidean subspaces of $\mathbf{R}^{n+1}$. In the following we prove the completeness of these integral manifolds.

First of all let us notice, that the fibration described in Proposition 4.4. can be extended continuously onto the boundary of $\mathscr{V}_p$. In fact, in the opposite case two sequences $p_i, q_i\in\mathscr{V}_p$ could be chosen such that $p=\lim p_i=\lim q_i=q$ is on the boundary of $\mathscr{V}_p$, the integral manifolds H_{p_i} resp. H_{q_i} of W^0 through p_i resp. q_i converge to H_p resp. H_q, but $H_p\neq H_q$ holds. As the spaces $H_{p_i}, H_{q_i}, H_p, H_q$ are hypersurfaces in M^n thus

$$\dim(H_{p_i}\cap H_{q_i}) = n-2.$$

would hold for large numbers i, which is a contradiction. Thus the proof of the statement is complete.

Let us return to the investigation of $\overset{*}{W}{}^0$'s integral manifolds. Let H be a maximal integral mainfold. For a vector field X tangent to H we have

$$(\nabla_{\partial_0}A)(X) = (\nabla_X A)(\partial_0),$$

from which we get

$$\nabla_X\partial_0 = 0, \quad X(\gamma_0) = \gamma_0 g(X, \nabla_{\partial_0}\partial_0). \tag{4.14}$$

So if $x(t)$ denotes an integral curve of X, then along it

$$\gamma_0(t) = \gamma_0(0)e^{\int_0^t g(\dot{x}, \nabla_{\partial_0}\partial_0)}$$

holds. From this we have, that A vanishes neither on H nor on the boundary of H. So every boundary point of H is a boundary point of $\mathscr{V}_p$, too. We prove, that such a boundary point doesn't exist for H.

We start with the indirect assumption. If q would be such a boundary point, then let H_q be the subspace through q which we get by the extension of the fibration, described in Proposition 4.4, onto the boundary of $\mathscr{V}_p$. Then $\dim(H\cap H_q)=n-2$ holds obviously. Let $\bar{\partial}_0$ be the normal vector of H_q in $T_q(M^n)$. Since $K(q)=0$, $A(q)\neq 0$ hold, so by (4.12) we get, that $\bar{\partial}_0$ is the unique non-trivial eigenvector of $A(q)$. But by (4.14) the non-trivial eigenvector ∂_0 is parallel along H, so the vector $\partial_0(q)$ is also a non-trivial eigenvector of $A(q)$. This is contradiction, because $\partial_0(q)\neq \neq\bar{\partial}_0$ holds.

So we get, that the maximal integral manifolds of $\overset{*}{W}{}^0$ are also complete $(n-1)$-dimensional euclidean subspaces in $\mathbf{R}^{n+1}$. Now let us consider a connected component $\mathscr{V}_0^{0i}$ of $\mathscr{V}_0^0$. From the above considerations it follows, that $\mathscr{V}_0^{0i}$ is an open set in an n-dimensional euclidean hyperspace, such that the boundary of $\mathscr{V}_0^{0i}$ is either an $(n-1)$-dimensional euclidean subspace, or two parallel $(n-1)$-dimensional subspaces. Thus the extension of the fibration onto $\mathscr{V}_0^0$ is trivial, which proves the proposition.

The above statements suggest a simple constructional method for hypersurfaces of parabolic type.

Proposition 4.5. *Let $c(s)$ be an immersed curve in $\mathbf{R}^{n+1}$, parametrised by arc-length. Furthermore let $H_{c(s)}$ be a differentiable field of $(n-1)$-dimensional euclidean subspaces along $c(s)$ such that $H_{c(s)}$ is orthogonal to $\dot{c}(s)$. Then the subspaces $H_{c(s)}$ cover an immersed hypersurface with $k(p)\leqq 2$ around $c(s)$.*

Proof. It is trivial, that the subspaces $H_{c(s)}$ cover an immersed hypersurface M^n in a neighbourhood of $c(s)$. Let $\mathbf{n}$ be the normal vector field of this hypersurface

M^n, and let ∂_0 be the unit vector field in M^n, orthogonal to the subspaces $H_{c(s)}$. Since the vector $D_X\mathbf{n}$, where X is tangent to $H_{c(s)}$, is pointing always in the direction of ∂_0, so the image-space of Weingarten map A is spanned by the vectors ∂_0 and $D_{\partial_0}\mathbf{n}$. Thus rank $A\leqq 2$ holds, and the proof is finished.

The spaces constructed in the previous proposition are in general not complete. But in many cases a field $H_{c(s)}$ described above covers globally a complete immersed hypersurface M^n. This is the case, if we consider an arbitrary differentiable field $H_{c(s)}$ of orthogonal $(n-1)$-dimensional euclidean subspaces along a line $c(s)$ of $\mathbf{R}^{n+1}$. Of course there can be given more complicated cases. Since such a hypersurface is in general not of the form

$$c\times H_c,$$

where c is a plane curve in a euclidean subplane $\mathbf{R}^2$ and H_c is orthogonal to $\mathbf{R}^2$, so these hypersurfaces have non-zero curvature in general.

Theorem 4.2. *Let $c(s)$, $-\infty<s<\infty$, be an immersed curve in $\mathbf{R}^{n+1}$ and let $H_{c(s)}$ be such a differentiable field of orthogonal (to $\dot{c}(s)$), $(n-1)$-dimensional euclidean subspaces along $c(s)$, which cover a complete hypersurface M^n. Then for M^n we have $k(p)\leqq 2$, $B^2=0$ and*

$$K=-(D_{\partial_0}\mathbf{n}, D_{\partial_0}\mathbf{n})+(D_{\partial_0}\mathbf{n}, \partial_0)^2\leqq 0. \tag{4.15}$$

Furthermore if $K(p)<0$ holds in a point $p\in H_p$, then $K<0$ is satisfied along H_p.

Proof. By Proposition 4.5 $k(p)\leqq 2$ holds for M^n, and if $K(p)\neq 0$ (i.e. $k(p)=2$) is satisfied, then the image space of the Weingarten field A_p is spanned by ∂_0 and $D_{\partial_0}\mathbf{n}$, where $D_{\partial_0}\mathbf{n}$ has non-zero projection onto the fibre H_p. Let ∂_1 be the unit vector pointing in the direction of this projected vector. Then the non-trivial subspace of A_p is spanned by ∂_0 and ∂_1. Since for $D_{\partial_1}\mathbf{n}$ the relation $D_{\partial_1}\mathbf{n}=\delta\,\partial_0=A(\partial_1)$ holds, so the matrix of A_p w.r.t. $\{\partial_0, \partial_1\}$ is of the form

$$\begin{bmatrix}\gamma_0, & \delta\\ \delta, & 0\end{bmatrix}$$

with $\delta\neq 0$. Since $D_{\partial_0}\mathbf{n}=\gamma_0\partial_0+\delta\partial_1$ holds, so by $K=-\delta^2$ we get the relation (4.15). Of course (4.15) holds also in the case $K(p)=0$, as in this case $D_{\partial_0}\mathbf{n}$ is pointing in the direction of ∂_0.

The subspaces $H_{c(s)}$ are totally geodesic so $\nabla_{\partial_1}\partial_0=0$ follows. From this we get $g(B(\partial_1), \partial_0)=0$ i.e. ∂_1 is an eigenvector of B. But the space is complete so B has only zero real eigenvalue. Thus $B(\partial_1)=0$ and $B^2=0$ follows.

The integral manifolds of V^0 are parallel hyperspaces in the fibres $H_{c(s)}$, and so the integral curves of ∂_1 are lines in $H_{c(s)}$. From (2.4) and (4.13) we get, that $K<0$ holds along $H_{c(s)}$ if in a point $p\in H_{c(s)}$, $K(p)<0$ is satisfied.

We are going to investigate the irreducibility of the previously described spaces. Let M^n be a complete simple connected immersed hypersurface as in Theorem 4.2 with $K<0$, and let $c(s)$, $-\infty<s<\infty$, be an arbitrary fixed integral curve of ∂_0. The subspaces $H_{c(s)}$ can be described uniquely by the normal vector field $\mathbf{n}(s)$ along $c(s)$.

Theorem 4.3*). *The hypersurface M^n with $K<0$ is reducible iff a euclidean subspace $\mathbf{R}^k$ with $k<n+1$ exists, which contains $c(s)$ with the vector field $\mathbf{n}(s)$ as well. If $\mathbf{R}^k$ is the smallest such subspace, then M^n is of the form*

$$M^n = M^{k-1}\times\mathbf{R}^{n-k+1}, \tag{4.16}$$

where M^{k-1} is an irreducible complete hypersurface in $\mathbf{R}^k$ covered by a one-parametrized family $H^*_{c(s)}$ of $(k-1)$-dimensional euclidean subspaces, furthermore $\mathbf{R}^{n-k+1}$ is euclidean subspace in $\mathbf{R}^{n+1}$ orthogonal to $\mathbf{R}^k$.

Proof. If $c(s)$ with $\mathbf{n}(s)$ is contained in a subspace $\mathbf{R}^k$, $k<n+1$, then M^n is obviously of the form (4.16). Thus we examine the other direction, and let us assume that M^n is reducible, and it is of the form

$$M^n = Q^{k-1}\times Q^{n-k+1} \tag{4.17}$$

with $k<n$.

First we prove that (4.17) is a cylindrical decomposition. Let T^1 resp. T^2 be the tangent space of Q^{k-1} resp. Q^{n-k+1}. Since for the curvature tensor R the equation $R(T^1, T^2)X=0$ holds, so by the Gauss equation we get

$$g(X, A(T^1))A(T^2) = g(X, A(T^2))A(T^1), \tag{4.18}$$

for every tangent vector $X\in T(M)$. We show, that A vanishes on one of the tangent spaces T^i.

In fact, if there were tangent vectors $X^i\in T^i_p$, $i=1;2$ for which $A(X^i)\neq 0$ holded, then by (4.18) the vectors $A(X^i)$ would point in the same direction, and so A would be of rank 1. But this is imposible, because $K<0$ holds.

So we get, that one of the spaces Q^{k-1}, Q^{n-k+1} has negative scalar curvature, and the other is of zero curvature. Let Q^{k-1} be the space with $K<0$. Since $A(T^2)=0$ holds, so $T^2\subseteqq V^0$ and the integral manifolds of T^2 are complete $(n-k+1)$-dimensional euclidean subspaces. Because of the decomposition (4.17) these euclidean subspaces must be parallel subspaces in $\mathbf{R}^{n+1}$. So (4.17) is a cylindrical decomposition of the form

$$M^n = Q^{k-1}\times\mathbf{R}^{n-k+1},$$

where Q^{k-1} is a hypersurface in $\mathbf{R}^k$ orthogonal to $\mathbf{R}^{n-k+1}$. Since $\mathbf{R}^{n-k+1}$ is orthogonal to $c(s)$ and $\mathbf{n}(s)$ as well, so $c(s)$ and $\mathbf{n}(s)$ are contained in $\mathbf{R}^k$.

*) The theorem is true also in case $K\leqq 0$.

The last statement in the theorem is obvious.

We mention, that the above theorem is true also in the case, when we consider M^n only for an open interval $a<s<b$.

By Theorem 4.2 the hypersurfaces described in the theorem can contain also pure trivial part $\mathscr{V}_t$, i.e. on which $K<0$, $B=0$ hold. It is clear by the above remark, that $\mathscr{V}_t$ is non-empty iff an open interval $a<s<b$ exists, for which $c(s)$ with $\mathbf{n}(s)$ is contained in a 3-dimensional subspace $\mathbf{R}^3$, but a smaller subspace doesn't contain the system $\{c(s), \mathbf{n}(s)\}$. So excluding this possibility the other hypersurfaces described in Theorem 4.2 are of parabolic type.

It is very easy to construct such complete, irreducible hypersurfaces which contain pure parabolic part only.

For example let us consider a differentiable field of unit vectors $\mathbf{n}(s)$ along a line $c(s)$, $-\infty<s<\infty$, in $\mathbf{R}^{n+1}$ for which

1. the vector $D_{\dot c}\mathbf{n}$ is non-zero along $c(s)$,
2. the system $\{c(s), \mathbf{n}(s)\}$, $-\infty<s<\infty$, is not contained in a subspace $\mathbf{R}^k$ with $k<(n+1)$.
3. There is no interval $a<s<b$, for which $\{c(s), \mathbf{n}(s)\}$ is in a subspace $\mathbf{R}^3$.

Then the euclidean subspaces $H_{c(s)}$, orthogonal to $c(s)$ and $\mathbf{n}(s)$, inscribe in $\mathbf{R}^{n+1}$ an irreducible complete hypersurface with pure parabolic part only.

It is very easy to contruct also such hypersurfaces which contain only pure trivial and pure parabolic parts.

5. Hypersurfaces of hyperbolic type

Theorem 5.1. *Every connected and simply connected immersed hypersurface M^n of hyperbolic type is of the form $M^n=M^3\times\mathbf{R}^{n-3}$, where M^3 is an immersed hypersurface of hyperbolic type in a euclidean subspace $\mathbf{R}^4$ and $\mathbf{R}^{n-3}$ is euclidean subspace orthogonal to $\mathbf{R}^4$.*

Proof. By (2.13)

$$M_\alpha^1(X)B_1(Y)-M_\alpha^1(Y)B_1(Y)=0$$

holds. Since B_1 is non-degenerate thus $M_\alpha^1=-M_1^\alpha=0$ holds for $\alpha\geqq 2$. This means that $\nabla_X\mathbf{m}_1$ is contained in V_p^1 for every vector $X\in V_p^1$. By formulas (1.3) and Proposition 2.4 the distribution V_p^*, spanned by V_p^1 and $\mathbf{m}_1/p$, is involutive and the integral manifolds of this distribution are totally geodesic. It is also trivial, that the orthogonal complement V_p^{**} of V_p^* is also involutive, and the maximal integral manifolds of it are $(n-3)$-dimensional euclidean subspaces in $\mathbf{R}^{n+1}$. Let M^3 be a maximal integral manifold of V^*. Then for every vector field Y tangent to V^{**} and for every vector field X tangent to V^* the vector field $D_X Y$ is also tangent to V^{**}, where D is the covariant derivative in $\mathbf{R}^{n+1}$. This means, that the integral manifolds of V^{**} are parallel euclidean subspaces in $\mathbf{R}^{n+1}$ and that M^3 is an immersed hypersurface of hyperbolic type in an orthogonal complement $\mathbf{R}^4$ of the above parallel euclidean spaces. From the basic formulas it is rather trivial, that the metric of M^n is of the form $M^n = M^3 \times \mathbf{R}^{n-3}$ indeed.

From the above theorem we can see, that for the construction of hyperbolic hypersurfaces we must construct only the 3-dimensional cases. In the following we describe a general construction for such hypersurfaces.

At first let us consider a one-fold covering of a simply connected open set U of $\mathbf{R}^3$ with complete lines such that the unit vector field $\mathbf{u}$ tangent to these lines is differentiable. We call such a covering a *line-fibration of U*. For a point $p \in U$ let $\overset{*}{V}{}_p^1$ be the orthogonal complement of $\mathbf{u}_p$ and let $\overset{*}{V}{}_p^0$ be the 1-dimensional subspace in $T_p(U)$ spanned by $\mathbf{u}_p$. The following relations are obvious for the covariant derivative D of $\mathbf{R}^3$:

$$D_{\overset{*}{V}{}^0} \overset{*}{V}{}^1 \subseteqq \overset{*}{V}{}^1, \quad D_{\overset{*}{V}{}^0} \overset{*}{V}{}^0 \subseteqq \overset{*}{V}{}^0, \quad D_{\overset{*}{V}{}^1} \overset{*}{V}{}^1 \subseteqq \overset{*}{V}{}^0 + \overset{*}{V}{}^1. \tag{5.1}$$

Furthermore let $\overset{*}{B}(X) := D_X \mathbf{u}$ be the derived tensor field of $\mathbf{u}$ and let $\overset{*}{\nabla}$ be the covariant derivative defined by

$$\begin{gathered}\overset{*}{\nabla}_X Y := D_X Y - (D_X Y, \mathbf{u})\mathbf{u} = D_X Y + (\overset{*}{B}(X), Y)\mathbf{u}, \quad X_p;\ Y_p \in \overset{*}{V}{}_p^1, \\ \overset{*}{\nabla}_X \mathbf{u} := 0 \quad \text{for every vector field} \quad X, \quad \text{and} \quad \overset{*}{\nabla}_{\mathbf{u}} X := D_{\mathbf{u}} X \ \text{ if } \ X_p \in \overset{*}{V}{}_p^1\end{gathered} \tag{5.2}$$

on U, where (X, Y) denotes the inner product in $\mathbf{R}^3$. It is rather trivial that $\overset{*}{\nabla}$ is metrical w.r.t. (X, Y). If $\overset{*}{R}$ denotes the curvature tensor of $\overset{*}{\nabla}$, then the following basic formulas hold for the given line fibration:

$$\begin{gathered}\overset{*}{R}(X, Y)Z = (\overset{*}{B}(Y), Z)\overset{*}{B}(X) - (\overset{*}{B}(X), Z)\overset{*}{B}(Y), \\ (\overset{*}{\nabla}_X \overset{*}{B})(Y) - (\overset{*}{\nabla}_Y \overset{*}{B})(X) = 0 \quad \text{if} \quad X_p;\ Y_p \in \overset{*}{V}{}_p^1, \\ \nabla_{\mathbf{u}} B^* = -B^* \circ B^*, \\ \overset{*}{R}(X, \mathbf{u})Y = \overset{*}{R}(X, Y)\mathbf{u} = 0.\end{gathered} \tag{5.3}$$

These formulas can be proved in a similar way as the formulas of Proposition 1.1. Since the lines in the fibration are complete lines so it can be proved (similarly to Proposition 2.2 and 2.3) that along a line either $B^{*2} = 0$ holds or B^* is non-degenerated on $\overset{*}{V}{}^1$ and it has two non-real eigenvalues.

Now let $U_1 \subseteqq U$ be the maximal open set where $B^{*2}=0$ holds and let $U_2 \subseteqq U$ be the open set where B^* is non-degenerated in $\overset{*}{V}{}^1$. Then the open set $U_1 \cup U_2$ is everywhere dense in U, and both open sets are line-fibred open sets. Thus for the line fibrations we can give the following local classification. One class of such fibrations contains the fibrations for which $B^{*2}=0$ holds, and the other class contains the fibrations for which B^* is non-degenerated in $\overset{*}{V}{}^1$. We describe this classification form a more geometric point of view.

First let us consider the case $B^{*2}=0$. If $\overset{*}{B}=0$ holds on an open set, then this open set is fibred with parallel lines. And if $B^* \neq 0$ holds, then let $\overset{*}{\partial}_0$, $\overset{*}{\partial}_1$ be the orthogonal unit vector fields tangent to $\overset{*}{V}{}^1$, such that $\overset{*}{\partial}_1$ is tangent to the kernel of $\overset{*}{B}$. The following statement can be proved in the same way as Proposition 4.2.

Proposition 5.1. *The distribution $\overset{*}{W}{}^0$ spanned by* $\mathbf{u}$ *and $\overset{*}{\partial}_1$ is involutive.*

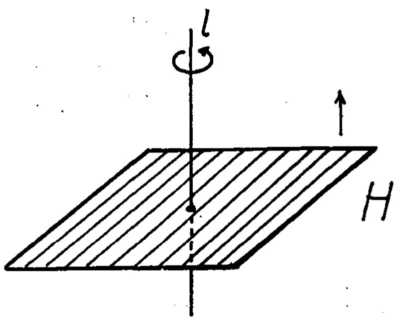
A maximal integral manifold $\overset{*}{H}$ of $\overset{*}{W}{}^0$ is an open set in a euclidean hyperplane of $\mathbf{R}^3$ such that the lines of fibration, which have common point with $\overset{*}{H}$, are parallel lines in this hyperplane and the integral curves of $\overset{*}{\partial}_1$ in $\overset{*}{H}$ are parallel line segments in the plane.

Conversely, if through every line l of a line-fibration there exists a euclidean hyperplane H such that H covers parallel lines from the fibration around l then the equation $\overset{*}{B}{}^2=0$ holds for the line-fibration.

The last statement of the above proposition is also obvious.

Thus the above local classification of line-fibrations is the following. One class contains the line-fibrations which can be covered with one parametric family of hyperplanes in the sense of Proposition 5.1 and the elements of other class cannot be covered in such a way. So we call the elements of the first class *plane-coverable line-fibrations* and the elements of the second class *plane-uncoverable line-fibrations.*

It is easy to give plane-coverable line-fibrations. For example let us consider a family of parallel lines in a hyperplane H of $\mathbf{R}^3$.



Let us move H along a line l (perpendicular to H) in such a way that H also turns around l. In this way we get a plane-coverable line-fibration of the whole $\mathbf{R}^3$. In order to show the existence of fibrations belonging to the second class we also give an example of a plane-uncoverable line-fibration of whole $\mathbf{R}^3$.

Let us consider the unit vector field

(5.4)
$$\mathbf{u} = (z^2+1)^{-1/2}(x^2+y^2+z^2+1)^{-1/2}\left\{(xz-y)\frac{\partial}{\partial x}+(yz+x)\frac{\partial}{\partial y}+(z^2+1)\frac{\partial}{\partial z}\right\}$$

defined in a Cartesian coordinate neighbourhood (x, y, z) of $\mathbf{R}^3$. A simple computation shows the equation $D_{\mathbf{u}}\mathbf{u}=0$, thus the maximal integral curves of u are lines and these lines define a line-fibration of $\mathbf{R}^3$. Every line intersects the (x, y)-plane $(z=0)$ just in one point. It can be simply computed that the eigenvalues of $\overset{*}{B}(X)=D_X\mathbf{u}$ at the point of the (x, y)-plane are

$$0, \quad (x^2+y^2+1)^{-1/2}\mathbf{i}, \quad -(x^2+y^2+1)^{-1/2}\mathbf{i}, \tag{5.5}$$

where $\mathbf{i}$ is the imaginary number. Thus $\overset{*}{B}$ has two non-real eigenvalues at every point of $\mathbf{R}^3$ and the fibration is a plane-uncoverable line-fibration.

Now let us consider a 3-dimensional hypersurface M^3 of hyperbolic type in $\mathbf{R}^4$. The integral curves of the vector field $\mathbf{m}$ in M^3 are lines in $\mathbf{R}^4$ and the tangent hyperspaces $T_p(M^3)$ coincide along such an integral curve l. Let us denote this constant hyperspace by $T_l(M^3)$. If S is such a euclidean hyperspace in $\mathbf{R}^4$, which is not orthogonal to l, then the orthogonal projection $\Pi\colon M^3 \to S$ maps an open neighbourhood U of l diffeomorphically onto an open set U^* of S such that the image of $\mathbf{m}$'-s integral curves form a line-fibration of U^*. This line-fibration is called the projected line-fibration of U^*.

Proposition 5.2. *The projected line-fibration of U^* is plane-uncoverable if M^3 is of hyperbolic type.*

Proof. Let α be the angle between the line l and the projected line l'. Then α can be cosidered as a differentiable function on U^* which is constant along the projected lines l'. If $\lambda_i(p)$, $p\in U^*$, $i=1, 2, 3$ denotes the eigenvalues of $B(X)=\nabla_X\mathbf{m}$ at the point $\Pi^{-1}(p)\in U$ then by a simple computation we get, that the eigenvalues of $\overset{*}{B}(X)=D_X\mathbf{u}$ are $\lambda_i^*=\cos\alpha\lambda_i$, $i=1, 2, 3$, which proves the proposition.

By the above considerations every hypersurface M^3 of hyperbolic type can be represented locally as the position of the points

$$(x, y, z, f(x, y, z)), \tag{5.6}$$

where $f(x, y, z)$ is a differentiable function on an open set $U^*\subsetneqq\mathbf{R}^3$, where U^* is an open set, line-fibred in a plane-uncoverable way.

We mention, that the unit normal vector field $\mathbf{n}$ of M^3 is represented by

$$\mathbf{n} = \frac{1}{h}(-f_x, -f_y, -f_z, 1), \tag{5.7}$$

where $h=(1+f_x^2+f_y^2+f_z^2)^{1/2}$, furthermore the second fundamental form is represented by

$$H = \frac{1}{h}\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}. \tag{5.8}$$

(For details see [6].) Let $\mathbf{u}$ be the unit vector field referring to the line-fibration of U^*. Then the covariant vector field df is parallel along the integral curves of $\mathbf{u}$, i.e. $D_{\mathbf{u}}df=0$ holds, furthermore rank $H=2$ holds at every point $p\in U^*$, and the nullspace of H is spanned by $\mathbf{u}$.

Now we turn to the reversed problem, and we give a general construction for hyper-surfaces M^3 of hyperbolic type.

Theorem 5.2. *Let $U^*\subseteqq\mathbf{R}^3$ be an open set which is line-fibred in a plane-uncoverable way. Then around every line of the fibration there exist differentiable functions $f(x, y, z)$ such that the points*

$$\big(x, y, z, f(x, y, z)\big)$$

represent hypersurfaces of hyperbolic type.

Proof. Let $\mathbf{u}$ be the vector field referring to the fibration of U^*.

Lemma 5.2.1. *The hypersurface $\big(x, y, z, f(x, y, z)\big)$ is of hyperbolic type referring to the fibration of U^* iff*

$$D_{\mathbf{u}}df = 0, \quad \operatorname{rank} D^2f = 2 \tag{5.9}$$

hold.

The proof is obvious by Proposition 5.2 and formula (5.8).

Let $M^2\subset U^*$ be such a hypersurface in $\mathbf{R}^3$ for which the tangent spaces $T_p(M^2)$ are complements of $\mathbf{u}_p$, i.e. $T_p(M^2)+S_p=T_p(\mathbf{R}^3)$ holds, where S_p is the 1-dimensional subspace spanned by $\mathbf{u}_p$. Thus M^2 can be considered as a cross-section of U^*'s fibration. If (x^1, x^2) is a coordinate neighbourhood of M^2, then it can be extended uniquely onto a coordinate neighbourhood (x^1, x^2, t) of U^* such that $\partial/\partial t=\mathbf{u}$ holds, and $(x^1, x^2, 0)$ is just (x^1, x^2) on M^2. The vector fields $\partial/\partial x^i$ can be written in the form

$$\frac{\partial}{\partial x^i} = E_i+\Phi_i\mathbf{u}, \tag{5.10}$$

where E_i is orthogonal to $\mathbf{u}$ and thus also

$$\Phi_i = \left(\frac{\partial}{\partial x^i}, \mathbf{u}\right) \tag{5.11}$$

holds. For the tensor field $\overset{*}{B}$ the following holds:

$$\overset{*}{B}\left(\frac{\partial}{\partial x^i}\right) = \overset{*}{B}(E_i+\Phi_i\mathbf{u}) = \overset{*}{B}(E_i) = \overset{*}{B}{}_i^r E_r = \overset{*}{B}{}_i^r \frac{\partial}{\partial x^r} - \overset{*}{B}{}_i^r \Phi_r \mathbf{u}. \tag{5.12}$$

Lemma 5.2.2. *The fields E_i, Φ_i, $\overset{*}{B}{}_i^r$ fulfill the following formulas:*

$$\begin{gathered} \frac{\partial \Phi_i}{\partial t} = 0, \quad D_{\mathbf{u}} E_i = \overset{*}{B}(E_i) = \overset{*}{B}{}_i^r E_r, \quad \frac{\partial \overset{*}{B}{}_i^j}{\partial t} = -\overset{*}{B}{}_r^j \overset{*}{B}{}_i^r, \\ (\overset{*}{B}(E_j), E_i) - (\overset{*}{B}(E_i), E_j) = E_j(\Phi_i) - E_i(\Phi_j) = \partial\Phi_i/\partial x^j - \partial\Phi_j/\partial x^i. \end{gathered} \tag{5.13}$$

Proof. From $[\partial/\partial x^i, \mathbf{u}] = [\partial/\partial x^i, \partial/\partial t] = 0$ we get

$$0 = \left[\frac{\partial}{\partial x^i}, \mathbf{u}\right] = [E_i + \Phi_i \mathbf{u}, \mathbf{u}] = [E_i, \mathbf{u}] - \frac{\partial \Phi_i}{\partial t}\mathbf{u}.$$

On the other hand

$$[E_i, \mathbf{u}] = D_{E_i} u - D_{\mathbf{u}} E_i = \overset{*}{B}(E_i) - D_{\mathbf{u}} E_i.$$

Since both components of these equations are orthogonal to $\mathbf{u}$, so we get the first two equations in (5.13). We get the third equation form $D_{\mathbf{u}}\overset{*}{B} = -\overset{*}{B}{}^2$ and from the second equation. We get the last equation in the following way:

$$0 = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = [E_i + \Phi_i \mathbf{u}, E_j + \Phi_j \mathbf{u}] =$$

$$= [E_i, E_j]^{\sim} + \{(\overset{*}{B}(E_j), E_i) - (\overset{*}{B}(E_i), E_j) - E_j(\Phi_i) + E_i(\Phi_j)\}\mathbf{u},$$

thus the last equation is also satisfied.

Every solution f of $D_{\mathbf{u}} df = 0$ satisfies $\mathbf{u}\cdot\mathbf{u}(f) = 0$, thus f must be of the form $f = \varrho(x^1, x^2)t + \lambda(x^1, x^2)$ in the above coordinate neighbourhood (x^1, x^2, t), where the functions ϱ, λ are the functions of the variables (x^1, x^2) only.

Lemma 5.2.3. *A function $f = \varrho(x^1, x^2)t + \lambda(x^1, x^2)$ is the solution of $D_{\mathbf{u}} df = 0$ iff for ϱ and λ the differential equation*

$$\frac{\partial \varrho}{\partial x^i} - \overset{*}{B}{}_i^r \left(\frac{\partial \varrho}{\partial x^r} t + \frac{\partial \lambda}{\partial x^r} - \Phi_r \varrho\right) = 0, \quad i = 1;\ 2, \tag{5.14}$$

holds.

Proof. This equation comes from (5.13) by

$$(D_{\mathbf{u}}df)\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial^2 f}{\partial t\partial x^i} - df\left(D_{\mathbf{u}}\frac{\partial}{\partial x^i}\right) = \frac{\partial^2 f}{\partial t\partial x^i} - df(D_{\mathbf{u}}E_i + \Phi_i\mathbf{u}) =$$

$$= \frac{\partial^2 f}{\partial t\partial x^i} - df(\overset{*}{B}(E_i)) = \frac{\partial \varrho}{\partial x^i} - \overset{*}{B}{}_i^r\left(\frac{\partial \varrho}{\partial x^r}t + \frac{\partial \lambda}{\partial x^r} - \varrho\Phi_r\right).$$

For every solution f the restrictions of ϱ and λ onto M^2 satisfy the differential equation

$$\frac{\partial \varrho}{\partial x^i} - \overset{*}{B}{}_i^r\left(\frac{\partial \lambda}{\partial x^r} - \varrho\Phi_r\right) = 0. \tag{5.15}$$

Lemma 5.2.4. *Let $\varrho(x^1, x^2)$ and $\lambda(x^1, x^2)$ be the solutions of* (5.15) *on M^2. Then the function $f=\varrho t+\lambda$ defined on (x^1, x^2, t) is a solution of $D_{\mathbf{u}}df=0$.*

Proof. Let $\omega_i(t)$ be the functions defined by the left side of (5.14) along a line of the fibration. Since $\overset{*}{B}{}_i^r$ is of the form (2.10) along a line thus $\omega_i(t)$ are analytical functions with $\omega_i(t)=0$. A simple computation shows the equation

$$\frac{d^n\omega_i}{dt^n} = (-1)^n\overset{*}{B}{}_i^{l_1}\overset{*}{B}{}_{l_1}^{l_2}\dots\overset{*}{B}{}_{l_{n-1}}^{l_n}\omega_{l_n},$$

so $d^n\omega_i/dt^n_{/0}=0$, i.e. $\omega_i=0$ everywhere. This proves the statement.

Now let us assume that M^2 is a hyperplane in $\mathbf{R}^3$ and that (x^1, x^2) is a Descartesian coordinate system on it.

Lemma 5.2.5. *The covariant vector field $p_i=\overset{*}{B}{}_i^r\Phi_r$ is a closed form on a hyperplane M^2.*

Proof. It can be seen from (5.3) that the equation

$$(D_X\overset{*}{B})(Y) = (D_Y\overset{*}{B})(X) \tag{5.16}$$

holds for every vector field X, Y in $\mathbf{R}^3$. By this formula we get

$$0 = D_{\partial/\partial x^i}\overset{*}{B}\left(\frac{\partial}{\partial x^j}\right) - D_{\partial/\partial x^j}\overset{*}{B}\left(\frac{\partial}{\partial x^i}\right) =$$

$$= D_{\partial/\partial x^i}\left(\overset{*}{B}{}_j^r\frac{\partial}{\partial x^r} - \overset{*}{B}{}_j^r\Phi_r\mathbf{u}\right) - D_{\partial/\partial x^j}\left(\overset{*}{B}{}_i^r\frac{\partial}{\partial x^r} - B_i^r\Phi_r\mathbf{u}\right) =$$

$$= \left\{\frac{\partial \overset{*}{B}{}_j^r}{\partial x^i} - \frac{\partial \overset{*}{B}{}_i^r}{\partial x^j} - \overset{*}{B}{}_j^q\Phi_q\overset{*}{B}{}_i^r + \overset{*}{B}{}_i^q\Phi_q\overset{*}{B}{}_j^r\right\}\frac{\partial}{\partial x^r} + \left\{\frac{\partial \overset{*}{B}{}_i^r\Phi_r}{\partial x^j} - \frac{\partial \overset{*}{B}{}_j^r\Phi_r}{\partial x^i}\right\}\mathbf{u},$$

and so

$$(5.17)\qquad \begin{gathered} \frac{\partial \overset{*}{B}{}^r_j}{\partial x^i} - \frac{\partial \overset{*}{B}{}^r_i}{\partial x^j} = \overset{*}{B}{}^q_j \Phi_q \overset{*}{B}{}^r_i - \overset{*}{B}{}^q_i \Phi_q \overset{*}{B}{}^r_j, \\ \frac{\partial \overset{*}{B}{}^r_i \Phi_r}{\partial x^j} - \frac{\partial \overset{*}{B}{}^r_j \Phi_r}{\partial x^i} = 0. \end{gathered}$$

By the last formula the proof is complete.

Let us define the matrix field

$$(5.18)\qquad a^{ij} := \begin{bmatrix} -\overset{*}{B}{}^1_2, & (1/2)(\overset{*}{B}{}^1_1 - \overset{*}{B}{}^2_2) \\ (1/2)(\overset{*}{B}{}^1_1 - \overset{*}{B}{}^2_2), & \overset{*}{B}{}^2_1 \end{bmatrix}$$

on M^2. This matrix field is positive definite as by the plane-uncoverable fibration

$$(5.19)\qquad \det(a^{ij}) = -\overset{*}{B}{}^2_1 \overset{*}{B}{}^1_2 - (1/4)(\overset{*}{B}{}^1_1 - \overset{*}{B}{}^2_2)^2 > 0$$

holds, since the discriminant $\Delta(= -\det(a^{ij}))$ of the characteristic equation

$$\lambda^2 - \operatorname{Tr} \overset{*}{B} \lambda + \det \overset{*}{B} = 0$$

is negative.

Lemma 5.2.6. *In a hyperplane M^2 the differential equation* (5.15) *is equivalent to the equations*

$$(5.20)\qquad a^{ij} \frac{\partial^2 \lambda}{\partial x^i \partial x^j} = 0, \quad \det(a^{ij}) > 0,$$

$$(5.21)\qquad \frac{\partial \varrho}{\partial x^i} + \overset{*}{B}{}^r_i \Phi_r \varrho = \overset{*}{B}{}^r_i \frac{\partial \lambda}{\partial x^r}.$$

Furthermore for a fixed solution λ of (5.20) *the differential equation* (5.21) *is completely integrable w.r.t. ϱ.*

Proof. We can write the equation (5.15) also in the following invariant form

$$(5.22)\qquad d\varrho + \varrho\delta - \omega = 0,$$

where δ resp. ω are the covariant vector fields $\overset{*}{B}{}^r_i \Phi_r$ resp. $\overset{*}{B}{}^r_i \partial\lambda/\partial x^r$. As the operator d acts on the left side of this equation so we get by Lemma 5.2.5:

$$(5.23)\qquad d\omega = d\varrho \wedge \delta = \omega \wedge \delta.$$

We show, that this equation is equivalent to (5.20). Indeed, the equation (5.23) is just the following:

$$\frac{\partial \overset{*}{B}{}^r_i \lambda_r}{\partial x^j} - \frac{\partial \overset{*}{B}{}^r_j \lambda_r}{\partial x^i} = \overset{*}{B}{}^r_i \Phi_r \overset{*}{B}{}^p_j \lambda_p - \overset{*}{B}{}^r_j \Phi_r \overset{*}{B}{}^p_i \lambda_p,$$

where $\lambda_r := \lambda/\partial x^r$. By the first equation of (5.23) we get

$$\overset{*}{B}{}_i^r \frac{\partial^2\lambda}{\partial x^r \partial x^j} - \overset{*}{B}{}_j^r \frac{\partial^2\lambda}{\partial x^r \partial x^i} = 0, \tag{5.24}$$

which is equivalent to (5.20) indeed. Since (5.23) is the condition of integrability for (5.21) thus the last statement is in the lemma also obvious.

Now let l be a line from the line-fibration of U^*. For a point $p \in l$ let M^2 be a hyperplane such that l is not belonging to M^2. Then there exists a neighbourhood V of p in M^2 such that the lines going through points of V are not belonging to M^2. Let (x^1, x^2) be a Descartesian coordinate neighbourhood on M^2 and let λ be a non-linear solution of (5.20) around p. Then λ is non-linear in a neighbourhood V^* of p, i.e. the matrix field $\partial^2\lambda/\partial x^i \partial x^j$ is non-trivial on V^*. Let ϱ be a solution of (5.21) w.r.t. the fixed λ. Then ϱ is uniquely determined by the initial value $\varrho(p)$. By the above considerations the function $f(x^1, x^2, t) = \varrho(x^1, x^2)t + \lambda(x^1, x^2)$ satisfies the differential equation $D_u df = 0$. On the other hand the rank of $D^2 f$ is 2 in a neighbourhood of l. To prove this statement we only have to show that the matrix field $\partial^2\lambda/\partial x^i \partial x^j$ is non-singular on V^*. Indeed, by (5.24) the field $\partial^2\lambda/\lambda x^i \partial x^j$ cannot be of rank 1, on V^*, because in the opposite case the null-space would be an eigen direction of $\overset{*}{B}{}_j^i$ by (5.24). This is impossible, because the two eigenvalues of $\overset{*}{B}{}_j^i$ are non-real. So for a neighbourhood of l the points $(x, y, z, f(x, y, z))$ represent a hypersurface of hyperbolic type and the proof of Theorem is complete.

Now we turn to Takagi's counterexample. Let us consider the line-fibration (5.4). Then every line of the fibration intersects the (x, y)-plane only in one point. Let us denote this canonical coordinate neighbourhood on this plane by (x^1, x^2). A simple computation shows, that the matrix field $\overset{*}{B}{}_i^j$ is of the form

$$\overset{*}{B}{}_i^j = ((x^1)^2 + (x^2)^2 + 1)\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

on this plane and so the function $\lambda(x^1, x^2) := -x^1 x^2$ satisfies the differential equation (5.20) with $\det(\partial^2\lambda/\partial x^i \partial x^j) = -1$. From (5.21) we get the solution

$$\varrho = (1/2)((x^2)^2 - (x^1)^2)((x^1)^2 + (x^2)^2 + 1)^{-1/2}.$$

If we compute the function $f(x^1, x^2, t) = \varrho t + \lambda$ in the Descartesian coordinate neighbourhood (x, y, z) of $\mathbf{R}^3$, we have

$$f(x, y, z) = \frac{x^2 z - y^2 z - 2xy}{2(z^2+1)},$$

and so the points $(x, y, z, f(x, y, z))$ represent a complete irreducible hypersurface of hyperbolic type which is of course irreducible and non-symmetric. But this is just Takagi's counterexample, so we have:

Proposition 5.3. *Takagi's counterexample is a complete hypersurface of hyperbolic type.*

Proposition 5.4. *The sectional curvature K_σ is non-positive for every plane σ in a hypersurfaces of hyperbolic type. So every complete and simple connected immersed hypersurface M^n of hyperbolic type is diffeomorphic to $\mathbf{R}^n$.*

Proof. It is enough to prove, that the sectional curvature w.r.t. $\sigma = V_p^1$ is negative. If (A_j^i), $i; j = 1; 2$, is the Weingarten field, restricted onto $\sigma = V_p^1$, then $K_\sigma = \det(A_j^i)$ holds. On the other hand $\nabla_m A = -A \circ B$ holds, thus we get

$$B_i^r A_{rj} = B_j^r A_{ri}.$$

If A_{ij} were positive definite, then B would have two non-zero real eigenvalues. So the signature of A_{ij} is 1, and thus $K_\sigma = \det(A_j^i) < 0$ holds.

6. Classification of complete semisymmetric hypersurfaces

At the end we can summarize the results of the paper in the following manner.

Theorem 6.1. *Let M^n be a complete semisymmetric immersed hypersurface in $\mathbf{R}^{n+1}$. Then M^n is one of the following types.*

1. *M^n is of zero curvature, and it is of the form $M^n = c \times \mathbf{R}^{n-1}$, where c is a curve in a hyperplane $\mathbf{R}^2$ and $\mathbf{R}^{n-1}$ is orthogonal to $\mathbf{R}^2$.*
2. *M^n is a straight cylinder of the form $M^n = S^k \times \mathbf{R}^{n-k}$ described in Nomizu's theorem.*
3. *M^n is pure trivial of the form $M^n = M^2 \times \mathbf{R}^{n-2}$, where M^2 is a hypersurface in a 3-dimensional euclidean subspace $\mathbf{R}^3$ and $\mathbf{R}^{n-2}$ is orthogonal to $\mathbf{R}^3$.*
4. *M^n is pure parabolic of the form $M^n = M^k \times \mathbf{R}^{n-k}$, where M^k is an irreducible pure parabolic hypersurface in a euclidean subspace $\mathbf{R}^{k+1}$ and $\mathbf{R}^{n-k}$ is orthogonal to $\mathbf{R}^{k+1}$.*
5. *M^n is pure hyperbolic of the form $M^n = M^3 \times \mathbf{R}^{n-3}$, where M^3 is a pure hyperbolic irreducible hypersurface in a 4-dimensional euclidean subspace $\mathbf{R}^4$ and $\mathbf{R}^{n-3}$ is orthogonal to $\mathbf{R}^4$.*
6. *M^n satisfies the relation $k(p) \leqq 2$ and it is mixed with $\mathcal{V}_0$, $\mathcal{V}_t$, $\mathcal{V}_p$, $\mathcal{V}_h$ parts.*

Theorem 6.2. *A complete semisymmetric immersed hypersurface with $K > 0$ is one of the following types.*

1. *M^n is a cylinder $M^n=S^{k-1}\times\mathbf{R}^{n-k}$ described in Nomizu's theorem.*
2. *M^n is pure trivial of the form $M^n=M^2\times\mathbf{R}^{n-2}$ described above in point 3.*

Theorem 6.3. *Let M^n be a complete immersed semisymmetric hypersurface with $|K|\geqq\varepsilon>0$ for a constant ε. Then M^n is also one of the types described in the above theorem.*

Proof. Let M^n have the property $k(p)\leqq 2$. Then M^n can't have hyperbolic part, because on an integral line of $\mathbf{m}_1$ on this part the function $K(s)$ is of the form

$$K(s)=\frac{Q}{(s+c_1)(s+c_2)+c_3^2},\quad Q=\text{constant},$$

by (2.4) and (2.10).

But M^n can't have pure parabolic part either. Indeed, on this part the integral manifolds of W^0 would be complete $(n-1)$-dimensional euclidean subspaces in $\mathbf{R}^{n+1}$ by (2.4), (2.7), (4.3) and (4.13), and the maximal integral curves of ∂_1 would be complete lines in these subspaces.

On the other hand B degenerates on this part, so by (1.7) $R(\partial_1, \partial_0)\partial_0=\tilde{R}(\partial_1, \partial_0)\partial_0$ holds. From this relation we get

$$\partial_1(\lambda)=K+\lambda^2,$$

so along an integral curve of ∂_1

$$\frac{dK}{ds}=4\lambda K,\quad \frac{\partial\lambda}{ds}=K+\lambda^2$$

hold. The general solutions of this system with $K<0$ are the following:

$$K(t)=\frac{Q_1}{(Q_1-(t+Q_2)^2)},\quad \lambda(t)=\frac{t+Q_2}{-(t+Q_2)^2+Q_1},$$

where Q_1 and Q_2 are constants with $Q_1<0$. So this case is also impossible and M^n contains only pure trivial part. By Proposition 3.1 the proof is finished.

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