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Term functions and subalgebras

EMIL W. KISS

Answering a question of A. F. Pixley this note shows that the class of para primal algebras cannot be characterised by the preservation properties of the term functions. Moreover all classes are described which can be characterised in such a way.

1. The characterisability result

Let $\mathfrak{A}$ be a finite algebra with underlying set A and Q a collection of finitary relations on A. A $\varrho \in Q$ is called compatible on $\mathfrak{A}$ if each term function f of $\mathfrak{A}$ preserves ϱ. Now if each finitary function f that preserves all the compatible elements of Q is a term function of $\mathfrak{A}$ then Q is said to characterise the term functions of $\mathfrak{A}$. The class of the clones of all such $\mathfrak{A}$ is denoted by Q^* and we say that *a class $\mathcal{K}$ of algebras on A can be characterised by the preservation properties of the term functions* if the set of clones of all the algebras in $\mathcal{K}$ is of the form Q^* for an appropriate collection Q.

This complicated definition can lead to very useful characterisations of classes $\mathcal{K}$ when Q is a concrete collection. The most important example is the class of quasi primal algebras where Q consists of the partial bijections on A (cf. WERNER [7] also for other examples).

In order to give an internal description of characterisible classes let us call a clone F *cocyclic* if $F = \mathrm{Pol}\,\varrho$ for some (finitary) relation ϱ on A (for notation and elementary results concerning the Pol—Inv connection see Pöschel—Kalužnin [5]).

Theorem. *A class $\mathcal{C}$ of clones on a finite set A is of the form Q^* iff*

(i) *$\mathcal{C}$ is closed under intersection (in particular the clone of all operations is in $\mathcal{C}$);*

(ii) *Each element of $\mathcal{C}$ is the intersection of cocyclic elements of $\mathcal{C}$.*

$\mathcal{C}$ is of the form Q^* for some finite Q iff (i);

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(iii) *Each element of* $\mathscr{C}$ *is cocyclic;*
(iv) $\mathscr{C}$ *is finite.*

Proof. The key observation is the following:

$(*)$ $F \in Q^*$ iff $F = \mathrm{Pol}\, Q'$ for some $Q' \subseteq Q$.

Indeed, suppose $F \in Q^*$ and let Q' be the set of all compatible elements of Q. Then $F = \mathrm{Pol}\, Q'$ by the definition of Q^*. Conversely, suppose $F = \mathrm{Pol}\, Q'$ for some $Q' \subseteq Q$. If an operation f preserves all the elements of Q that are compatible with F then, in particular, f preserves those of Q' (by $Q' \subseteq Q$ and $F \subseteq \mathrm{Pol}\, Q'$) so by $F \supseteq \mathrm{Pol}\, Q'$ we have $f \in F$ as desired.

Now the Theorem is obvious by using the rule

$$\mathrm{Pol}\,\{\cup Q_i\} = \cap \mathrm{Pol}\, Q_i$$

and the following observation which gives also an intrinsic characterisation of cocyclic clones (see e.g. Jablonskiĭ [3]):

Proposition. Pol $\{\varrho_1, \ldots, \varrho_k\}$ *is always cocyclic. A clone F is cocyclic iff there is an integer n such that* $f \in F$ *if and only if every at most n-ary function resulting from f by identifying certain (maybe no) variables is contained in F.*

Proof. Let $F = \mathrm{Pol}\,\{\varrho_1, \ldots, \varrho_k\}$ and choose n to be the maximum of the cardinalities of the ϱ_i-s. Then F satisfies the property in the second assertion. Conversely, if F is such and $\varrho \subseteq A^{A^n}$ is the set of all n-ary elements of F then clearly $F = \mathrm{Pol}\, \varrho$.

2. Para primal algebras

We prove

Corollary. *Suppose A is a finite set of at least two elements. Then the class of all para primal algebras on A can be defined by the preservation properties of the term functions iff A has two elements. In this case this class can be defined by finitely many relations.*

Proof. In the case $|A| \geqq 3$ let F_c be the clone of a cyclic group on A and F_t be the clone generated by the ternary discriminator. As the elements of F_t preserve all subsets of A and the elements of F_c are of the form

$$f(x_1, \ldots, x_n) = k_1 x_1 + \ldots + k_n x_n$$

where the k_i-s are integers, an easy calculation shows that $F_c \cap F_t$ consists of the projections. Thus the class of para primal clones does not satisfy (i) of the Theorem.

The case $|A| = 2$ could be settled by an elementary argument: all para primal clones are either quasi primal or affine by McKenzie [4], such clones are always cocyclic by the Proposition and easy calculations. However, for the sake of better visibility of the situation we derive the poset (in fact the lattice) of para primal clones on the set $\{0, 1\}$ from Post's classification ([6], for a considerably shorter proof see [1]). The clones D_1 (generated by the discriminator), D_3, C_1, C_2, C_3, C_4 defined below are quasi primal and L_1, L_2, L_3, L_4, L_5 are affine. These are eleven clones but C_2 and C_3 as well as L_2 and L_3 give cryptomorphic algebras by $0 \leftrightarrow 1$, so one can obtain the list of two element para primal algebras found in CLARK—KRAUSS [2].

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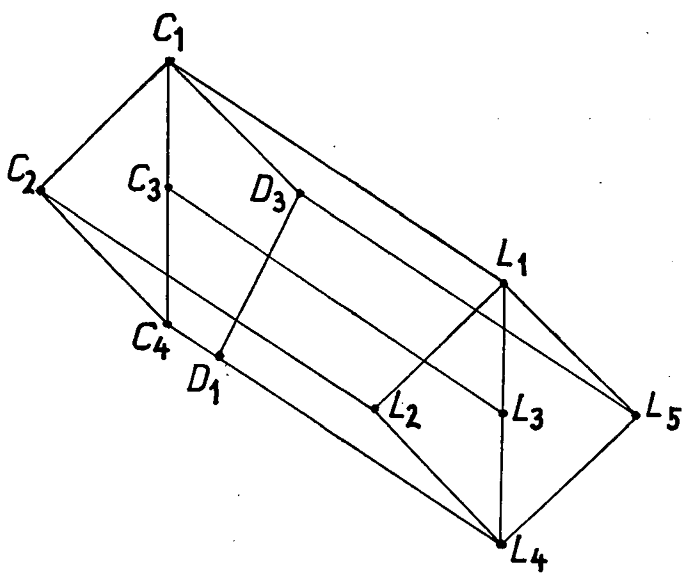

$C_1 = \{\text{all finitary functions on } \{0, 1\}\}$;

+	0	1
0	0	1
1	1	0

$\mathbf{1}\colon \{0, 1\} \to \{1\}, \quad \bar{0} = 1,$

$\mathbf{0}\colon \{0, 1\} \to \{0\}, \quad \bar{1} = 0,$

$C_4 = \{f \in C_1 | f(x, \dots, x) = x\}$,

$C_3 = \{f \in C_1 | f(0, \dots, 0) = 0\}$,

$C_2 = \{f \in C_1 | f(1, \dots, 1) = 1\}$,

$D_1 = \{f \in C_4 | f(\bar{x}_1, \dots, \bar{x}_n) = \overline{f(x_1, \dots, x_n)}\}$,

$D_3 = \{f \in C_1 | f(\bar{x}_1, \dots, \bar{x}_n) = \overline{f(x_1, \dots, x_n)}\}$,

$L_1 = [x+y, \mathbf{1}]$ (that is, the clone generated by these operations),

$L_2 = [x+y+1], \quad L_3 = [x+y], \quad L_4 = [x+y+z], \quad L_5 = [x+y+z+1]$.

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MATHEMATICAL INSTITUTE OF THE
HUNGARIAN ACADEMY OF SCIENCES
REÁLTANODA U. 13—15
1053 BUDAPEST, HUNGARY