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On approximation of the solutions of quasi-linear elliptic equations in $\mathbf{R}^n$

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Introduction

Let $P=P(D)$ be an elliptic differential operator of order $2m$ with constant coefficients $\left(D=\left(-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n}\right)\right)$ and $Q=Q(x, D)$ a differential operator of order $2m$ with smooth coefficients which vanish for $|x|>a$.

For any domain $\Omega\subset\mathbf{R}^n$ and any integer $k\geqq 0$ denote by $H^k(\Omega)$ the Hilbert space of functions u with the norm

$$\|u\|_{H^k(\Omega)}=\Big\{\sum_{|\alpha|\leqq k}\int_\Omega |D^\alpha u|^2\,dx\Big\}^{1/2}$$

(Sobolev space); $L^2(\Omega)=H^0(\Omega)$. Further denote by $H^k_{\mathrm{loc}}(\Omega)$ the set of functions u satisfying the condition: $\varphi u\in H^k(\Omega)$ for arbitrary infinitely differentiable function φ which is equal to zero out of a compact subset of Ω.

In [1] the elliptic equation

$$Au \equiv (P+Q)u = f \quad \text{in} \quad \mathbf{R}^n \tag{0.1}$$

has been considered when $P(\xi)\neq 0$ for all $\xi\in\mathbf{R}^n$. It has been proved that if for any $f\in L^2_a(\mathbf{R}^n)$ $\big($i.e. $f\in L^2(\mathbf{R}^n), f(x)=0$ if $|x|>a\big)$ there exists a solution u of the equation (0.1) which tends to zero at infinity then the solution is unique. Furthermore, by use of methods of [2] it is easy to show that for this solution the estimation

$$\|u\|_{H^{2m}(\mathbf{R}^n)} \leqq c_1\|f\|_{L^2_a(\mathbf{R}^n)} \tag{0.2}$$

holds. (c_1 is a constant which does not depend on f.) In [1] there have been formulated conditions on the differential operators $B_j(x, D)$ which guarantee that for sufficiently

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large $\varrho>0$ the boundary value problem in $B_\varrho=\{x\in\mathbf{R}^n: |x|<\varrho\}$

$$Au_\varrho=f \quad \text{in} \quad B_\varrho \tag{0.3}$$

$$B_j(x, D)u_\varrho=0 \quad \text{on} \quad S_\varrho, \quad j=1, \ldots, m \tag{0.4}$$

($S_\varrho=\{x\in\mathbf{R}^n: |x|=\varrho\}$) has a unique solution u_ϱ in the Sobolev space $H^{2m}(B_\varrho)$ and an estimation of the form

$$\|u-u_\varrho\|_{H^{2m}(B_\varrho)} \leqq c_2\|f\|_{L^2_a(\mathbf{R}^n)}e^{-c_3\varrho} \tag{0.5}$$

holds, where c_2, c_3 are positive constants which do not depend on f and ϱ.

In [3] similar results are proved when $P(\xi)\neq 0$ for $\xi\in\mathbf{R}^n\setminus\{0\}$ but $P(0)=0$ and (0.3), (0.4) is the Dirichlet problem, i.e. $B_j(x, D)=\dfrac{\partial^j}{\partial\nu^j}$ where ν is the normal vector to S_ϱ. Then instead of (0.2) and (0.5) the following estimations are valid: for any compact $K\subset R^n$

$$\|u\|_{H^{2m}(K)} \leqq c_1(K)\|f\|_{L^2_a(\mathbf{R}^n)} \tag{0.6}$$

and

$$\|u-u_\varrho\|_{H^{2m}(K)} \leqq c_2(K)g(\varrho)\|f\|_{L^2_a(\mathbf{R}^n)}, \tag{0.7}$$

where $c_1(K), c_2(K)$ are constants which do not depend on f and ϱ, $\lim_{\varrho\to+\infty} g(\varrho)=0$. Under certain conditions estimations of the form (0.2), resp.

$$\|u-u_\varrho\|_{H^{2m}(\mathbf{R}^n)} \leqq g(\varrho)\|f\|_{L^2_a(\mathbf{R}^n)} \tag{0.8}$$

can be shown where $\lim_{\varrho\to+\infty} g(\varrho)=0$.

In this paper it will be supposed that the differential operators P and Q satisfy the above mentioned conditions of works [1], resp. [3] such that estimations of the form (0.6), (0.7), resp. (0.2), (0.8) hold. Our aim is to consider a quasi-linear elliptic equation of the form

$$Au+g(x, u, \ldots, D^\beta u, \ldots)=f \quad \text{in} \quad \mathbf{R}^n \tag{0.9}$$

where $|\beta|\leqq 2m-1$ and to prove the existence of a solution of (0.9). Moreover, we are going to prove an estimation of type (0.7), resp. (0.8) for the quasi-linear equation (0.9).

In [4]—[8] there are proved existence theorems on quasi-linear and nonlinear elliptic equations in unbounded domains. These results, however, cannot be applied to the equation (0.9) in the case $P(0)=0$.

1. Existence of solutions

Theorem 1. *Suppose that for any* $f\in L_a^2(\mathbf{R}^n)$ *there exists a unique solution* u *of* (0.1) *and for this solution the estimation* (0.6) *holds. Let* $g\colon \mathbf{R}^{n+N}\to\mathbf{R}$ *be a continuous function* (*N denotes the number of multiindices* β *such that* $|\beta|\leqq 2m-1$) *satisfying the conditions:*

(1.1) $$g(x, u, \dots, u_\beta, \dots) = 0 \quad \text{if} \quad |x| > a;$$

(1.2) $$\lim_{|(u,\dots,u_\beta,\dots)|\to\infty} \frac{g(x, u, \dots, u_\beta, \dots)}{|(u, \dots, u_\beta, \dots)|} = 0 \quad \text{uniformly in } x;$$

(1.3) *the first partial dervatives of g are continuous and bounded.*

Then for any $f\in L_a^2(\mathbf{R}^n)$ *equation* (0.9) *has at least one solution* $u\in H_{\mathrm{loc}}^{2m}(\mathbf{R}^n)$, *vanishing at infinity.*

Proof. Denote by $A^{-1}f$ the unique solution of (0.1) which vanishes at infinity. Function u is the solution of (0.9) (vanishing at infinity) if and only if $v=Au$ is a solution of the equation

(1.4) $$v+G(v)=f$$

in $L_a^2(\mathbf{R}^n)$ where the operator G is defined by

$$G(v) = g(x, A^{-1}v, \dots, D^\beta A^{-1}v, \dots).$$

We shall first prove that G is a continuous and compact (nonlinear) operator in the Hilbert space $L_a^2(\mathbf{R}^n)$. By use of the mean value theorem and condition (1.3) we have the estimation

$$|G(v)-G(v^*)| \leqq c_1[|A^{-1}(v-v^*)| + \dots + |D^\beta A^{-1}(v-v^*)| + \dots]$$

(c_1 denotes a constant) and thus

(1.5) $$\Big\{\int_{B_a} |G(v)-G(v^*)|^2\Big\}^{1/2} \leqq$$

$$\leqq c_1\Big[\Big\{\int_{B_a} |A^{-1}(v-v^*)|^2\Big\}^{1/2} + \dots + \Big\{\int_{B_a} |D^\beta A^{-1}(v-v^*)|^2\Big\}^{1/2} + \dots\Big] \leqq c_2\|v-v^*\|_{L_a^2(\mathbf{R}^n)},$$

because in virtue of (0.6) $A^{-1}\colon L_a^2(\mathbf{R}^n)\to H^{2m}(B_a)$ is a bounded linear operator. Since A^{-1} is a bounded linear operator and by (1.2)

$$|g(x, u, \dots, u_\beta, \dots)| \leqq c_3|(u, \dots, u_\beta, \dots)|$$

(c_3 denotes a constant), thus

$$\|G(v)\|_{L^2(B_a)} \leqq c_4\|A^{-1}v\|_{H^{2m}(B_a)} \leqq c_5\|v\|_{L_a^2(\mathbf{R}^n)}.$$

Hence by use of condition (1.1) we find that for any $v \in L_a^2(\mathbf{R}^n)$, $G(v) \in L_a^2(\mathbf{R}^n)$ and thus by (1.5) $G\colon L_a^2(\mathbf{R}^n) \to L_a^2(\mathbf{R}^n)$ is a continuous operator.

From conditions (1.1)—(1.3) it follows that for any $v \in L_a^2(\mathbf{R}^n)$

$$(1.6) \qquad \frac{\partial}{\partial x_j} G(v) = \frac{\partial g}{\partial x_j}(x, A^{-1}v, \dots; D^\beta A^{-1}v, \dots) + \\ + \sum_{|\beta| \leq 2m-1} \frac{\partial g}{\partial u_\beta}(x, A^{-1}v, \dots, D^\beta A^{-1}v, \dots) \frac{\partial}{\partial x_j}(D^\beta A^{-1}v)$$

and $G(v) \in H_a^1(\mathbf{R}^n)$ (i.e. $G(v) \in H^1(\mathbf{R}^n)$ and $G(v)=0$ for $|x|>a$). From (1.6) it is also clear that G maps bounded subsets of $L_a^2(\mathbf{R}^n)$ into bounded subsets of $H_a^1(\mathbf{R}^n)$. Hence $G\colon L_a^2(\mathbf{R}^n) \to L_a^2(\mathbf{R}^n)$ is a compact operator.

Now we shall prove the equality

$$(1.7) \qquad \lim_{\|v\|_{L_a^2(\mathbf{R}^n)} \to \infty} \frac{\|G(v)\|_{L_a^2(\mathbf{R}^n)}}{\|v\|_{L_a^2(\mathbf{R}^n)}} = 0.$$

Denote $A^{-1}v$ by u then

$$(1.8) \qquad \frac{\|G(v)\|_{L_a^2(\mathbf{R}^n)}}{\|v\|_{L_a^2(\mathbf{R}^n)}} = \frac{\|g(x, u, \dots, D^\beta u, \dots)\|_{L_a^2(\mathbf{R}^n)}}{\|u\|_{H^{2m}(B_a)}} \cdot \frac{\|u\|_{H^{2m}(B_a)}}{\|v\|_{L_a^2(\mathbf{R}^n)}}.$$

In virtue of the boundedness of A^{-1} the second factor on the right hand side is bounded. Moreover, $\|u\|_{H^{2m}(B_a)} \to \infty$ as $\|v\|_{L_a^2(\mathbf{R}^n)} \to \infty$ since $v = A(u)$ and $A\colon H^{2m}(B_a) \to L^2(B_a)$ is a bounded linear operator. Thus to prove (1.7) we have only to show that

$$(1.9) \qquad \lim_{\|u\|_{H^{2m}(B_a)} \to \infty} \frac{\|g(x, u, \dots; D^\beta u, \dots)\|_{L_a^2(\mathbf{R}^n)}}{\|u\|_{H^{2m}(B_a)}} = 0.$$

For any positive number $b>0$

$$(1.10) \qquad \int_{B_a} |g(x, u(x), \dots, D^\beta u(x), \dots)|^2 \, dx = \\ = \int_{|(u(x), \dots, D^\beta u(x), \dots)| > b} |g(x, u(x); \dots, D^\beta u(x), \dots)|^2 \, dx + \\ + \int_{|(u(x), \dots, D^\beta u(x), \dots)| \leq b} |g(x, u(x), \dots, D^\beta u(x), \dots)|^2 \, dx.$$

By (1.2) for any $\varepsilon>0$ the number $b>0$ can be chosen such that

$$|g(x, u(x), \dots, D^\beta u(x), \dots)| \leq \varepsilon |(u(x), \dots, D^\beta u(x), \dots)|.$$

Thus

$$\int_{|(u(x),\,\dots,\,D^\beta u(x),\,\dots)|>b} |g(x, u(x), \dots, D^\beta u(x), \dots)|^2\,dx \leqq \tag{1.11}$$

$$\leqq \varepsilon^2 \int_{B_a} |(u(x), \dots, D^\beta u(x), \dots)|^2\,dx \leqq \varepsilon^2 \|u\|^2_{H^{2m}(B_a)}.$$

For a fixed $b>0$ the second term on the right in (1.10) is bounded because g is continuous and $|x|\leqq a$, $|(u(x), \dots, D^\beta u(x), \dots)|\leqq b$. Therefore from (1.10), (1.11) we have (1.9) and equality (1.7) is proved.

Since $G\colon L^2_a(\mathbf{R}^n)\to L^2_a(\mathbf{R}^n)$ is a continuous compact operator satisfying (1.7), thus by use of Schauder's fixed point theorem we can prove that the equation (1.4) has at least one solution $v\in L^2_a(\mathbf{R}^n)$ for any $f\in L^2_a(\mathbf{R})$. By (1.7) we can choose a number $\varrho_0>0$ such that

$$\|v\|_{L^2_a(\mathbf{R}^n)} > \varrho_0 \quad \text{implies} \quad \frac{\|G(v)\|_{L^2_a(\mathbf{R}^n)}}{\|v\|_{L^2_a(\mathbf{R}^n)}} < \frac{1}{2}.$$

Set $F(v)=f-G(v)$. Then the operator F is bounded in $L^2_a(\mathbf{R}^n)$, i.e.

$$\|v\|_{L^2_a(\mathbf{R}^n)} \leqq \varrho_0 \quad \text{implies} \quad \|F(v)\|_{L^2_a(\mathbf{R}^n)} \leqq \varrho_1,$$

since G is bounded in $L^2_a(\mathbf{R}^n)$. Let ϱ denote the number $\max\{\varrho_0, \varrho_1, 2\|f\|\}$. Then F maps the sphere $\{v\in L^2_a(\mathbf{R}^n)\colon \|v\|_{L^2_a(\mathbf{R}^n)}\leqq\varrho\}$ into itself, because $\|F(v)\|\leqq\varrho_1\leqq\varrho$ if $\|v\|\leqq\varrho_0$ and

$$\|F(v)\| \leqq \|f\| + \|G(v)\| \leqq \varrho/2 + \|v\|/2 \leqq \varrho \quad \text{if} \quad \varrho_0 \leqq \|v\| \leqq \varrho.$$

Moreover, F is a continuous and compact operator, hence by Schauder's fixed point theorem F has at least one fixed point. Thus equation (1.4) has at least one solution $v\in L^2_a(\mathbf{R}^n)$ and then the function $u=A^{-1}v\in H^{2m}_{\mathrm{loc}}(\mathbf{R}^n)$ is a solution of (0.1), vanishing at infinity.

Consider now the following boundary value problem in B_ϱ:

$$Au_\varrho + g(x, u_\varrho, \dots, D^\beta u_\varrho, \dots) = f \quad \text{in} \quad B_\varrho, \tag{1.12}$$

$$B_j(x, D)u_\varrho = 0 \quad \text{on} \quad S_\varrho,\ j = 1, \dots, m. \tag{1.13}$$

Theorem 2. *Assume that the conditions of Theorem 1 are fulfilled. Further suppose that if* $\varrho\geqq\varrho_0$ *then for any* $f\in L^2_a(\mathbf{R}^n)$ *the problem* (0.3), (0.4) *has a unique solution* $u_\varrho\in H^{2m}(B_\varrho)$ *and the estimation* (0.7) *holds. Then for any* $\varrho\geqq\varrho_0$ *and* $f\in L^2_a(\mathbf{R}^n)$ *the problem* (1.12), (1.13) *has at least one solution* $u_\varrho\in H^{2m}(B_\varrho)$.

Proof. Denote by $A_\varrho^{-1}f$ the unique solution $u_\varrho\in H^{2m}(B_\varrho)$ of the problem (0.3), (0.4). If $v_\varrho\in L^2_a(\mathbf{R}^n)$ is a solution of

$$v_\varrho + g(x, A_\varrho^{-1}v_\varrho, \dots, D^\beta A_\varrho^{-1}v_\varrho, \dots) = f \tag{1.14}$$

then $u_\varrho = A_\varrho^{-1} v_\varrho \in H^{2m}(B_\varrho)$ is a solution of (1.12), (1.13). Define an operator G_ϱ by the formula

$$G_\varrho(v_\varrho) = g(x, A_\varrho^{-1} v_\varrho, \ldots, D^\beta A_\varrho^{-1} v_\varrho, \ldots).$$

Then $G_\varrho \colon L_a^2(\mathbf{R}^n) \to L_a^2(\mathbf{R}^n)$ is a continuous and compact operator and

$$\lim_{\|v\|_{L_a^2(\mathbf{R}^n)} \to \infty} \frac{\|G_\varrho(v)\|_{L_a^2(\mathbf{R}^n)}}{\|v\|_{L_a^2(\mathbf{R}^n)}} = 0 \quad \text{uniformly for } \varrho \geqq \varrho_0. \tag{1.15}$$

This statement can be verified by means analogous to those used before in proving Theorem 1. We want only to show the proof of (1.15). Since

$$A_\varrho^{-1} v = (A_\varrho^{-1} - A^{-1}) v + A^{-1} v,$$

thus by estimations (0.6) and (0.7) $A_\varrho^{-1} \colon L_a^2(\mathbf{R}^n) \to H^{2m}(B_a)$ is a bounded linear operator and $\|A_\varrho^{-1}\|$ is uniformly bounded for $\varrho \geqq \varrho_0$:

$$\frac{\|A_\varrho^{-1} v\|_{H^{2m}(B_a)}}{\|v\|_{L_a^2(\mathbf{R}^n)}} \leqq c_1 \tag{1.16}$$

for any $v \in L_a^2(\mathbf{R}^n)$ and $\varrho \geqq \varrho_0$. Further

$$\|A_\varrho^{-1} v\|_{H^{2m}(B_a)} \to \infty \quad \text{uniformly for} \quad \varrho \geqq \varrho_0 \quad \text{as} \quad \|v\|_{L_a^2(\mathbf{R}^n)} \to \infty, \tag{1.17}$$

since $v = A(A_\varrho^{-1} v)$ and $A \colon H^{2m}(B_a) \to L^2(B_a)$ is a bounded linear operator which does not depend on ϱ. The equality

$$\frac{\|G_\varrho(v)\|_{L_a^2(\mathbf{R}^n)}}{\|v\|_{L_a^2(\mathbf{R}^n)}} = \frac{\|g(x, A_\varrho^{-1} v, \ldots, D^\beta A_\varrho^{-1} v, \ldots)\|_{L_a^2(\mathbf{R}^n)}}{\|A_\varrho^{-1} v\|_{H^{2m}(B_a)}} \cdot \frac{\|A_\varrho^{-1} v\|_{H^{2m}(B_a)}}{\|v\|_{L_a^2(\mathbf{R}^n)}}$$

and (1.9), (1.16), (1.17) imply (1.15).

Thus by use of Schauder's fixed point theorem we find that there exists a solution v_ϱ of (1.14) (see the proof of Theorem 1), hence $u_\varrho = A_\varrho^{-1} v_\varrho \in H^{2m}(B_\varrho)$ is a solution of (1.12), (1.13).

2. Theorem on approximation

Theorem 3. *Suppose that all conditions of Theorem 2 are fulfilled. Let* (ϱ_j) *be any sequence of numbers* $\varrho_j \geqq \varrho_0$ *such that* $\lim_{j \to \infty} \varrho_j = +\infty$ *and let* u_{ϱ_j} *be a solution of* (1.12), (1.13) *for* $\varrho = \varrho_j$. *Then the sequence* (ϱ_j) *has a subsequence* (ϱ_j^*) *such that for any compact* $K \subset \mathbf{R}^n$

$$\lim_{j \to \infty} \|u_{\varrho_j^*} - u^*\|_{H^{2m}(K)} = 0 \tag{2.1}$$

holds where $u^* \in H_{\text{loc}}^{2m}(\mathbf{R}^n)$ *is a solution of* (0.9) *vanishing at infinity.*

If the solution u of equation (0.9) *is unique then for the solutions* u_ϱ *of* (1.12), (1.13)

(2.2) $$\lim_{\varrho\to\infty} \|u_\varrho - u\|_{H^{2m}(K)} = 0$$

holds with arbitrary compact $K\subset\mathbf{R}^n$.

If estimations (0.2), (0.8) *are valid, too, then*

(2.3) $$\lim_{j\to\infty} \|u_{\varrho_j^*} - u^*\|_{H^{2m}(B_{\varrho_j^*})}$$

resp. (in the case of unicity)

(2.4) $$\lim_{\varrho\to\infty} \|u_\varrho - u\|_{H^{2m}(B_\varrho)} = 0$$

hold.

Proof. The solutions $v_\varrho\in L_a^2(\mathbf{R}^n)$ of the equation (1.14) constitute a bounded set in the Hilbert space $L_a^2(\mathbf{R}^n)$. If it were not true then there would exist a sequence $(v_{\varrho_j'})$ of solutions of (1.14) such that

(2.5) $$\lim_{j\to\infty} \|v_{\varrho_j'}\|_{L_a^2(\mathbf{R}^n)} = +\infty.$$

From (1.14) it is clear that

(2.6) $$\frac{v_{\varrho_j'}}{\|v_{\varrho_j'}\|_{L_a^2(\mathbf{R}^n)}} + \frac{G_{\varrho_j'}(v_{\varrho_j'})}{\|v_{\varrho_j'}\|_{L_a^2(\mathbf{R}^n)}} = \frac{f}{\|v_{\varrho_j'}\|_{L_a^2(\mathbf{R}^n)}}.$$

By (2.5) and (1.15) the term on the right and the second term on the left in (2.6) tend to the zero of $L_a^2(\mathbf{R}^n)$ as $j\to\infty$. The norm of the first term on the left equals one, thus from (2.6) we have a contradiction.

From the boundedness of the solutions v_ϱ of (1.14) and from (1.16) it follows the boundedness of the functions $u_\varrho = A_\varrho^{-1}v_\varrho$ in $H^{2m}(B_a)$.

Consider any sequence of numbers $\varrho_j\geqq\varrho_0$ such that $\lim_{j\to\infty}\varrho_j = +\infty$. The sequence (u_{ϱ_j}) of solutions of (1.12), (1.13) with $\varrho=\varrho_j$ is bounded in the norm of $H^{2m}(B_a)$. Hence (u_{ϱ_j}) has a subsequence $(u_{\varrho_j^*}) = (u_j')$ which tends to a function $u_0\in H^{2m-1}(B_a)$ in the norm of $H^{2m-1}(B_a)$:

(2.7) $$\lim_{j\to\infty} \|u_j' - u_0\|_{H^{2m-1}(B_a)} = 0.$$

In view of (1.3) and the mean value theorem it is clear that

$$|g(x, u_j', \dots, D^\beta u_j', \dots) - g(x, u_0, \dots, D^\beta u_0, \dots)| \leqq$$
$$\leqq c_1 \sum_{|\beta|\leqq 2m-1} |D^\beta u_j' - D^\beta u_0|$$

(c_1 denotes a constant). Thus

(2.8) $$\lim_{j\to\infty} \int_{B_a} |g(x, u_j', \dots, D^\beta u_j', \dots) - g(x, u_0, \dots, D^\beta u_0, \dots)|^2\,dx = 0.$$

Consider the functions $v'_j = Au'_j$. Then

$$v'_j + g(x, u'_j, \ldots, D^\beta u'_j, \ldots) = f \tag{2.9}$$

since the functions u'_j are solutions of the problem (1.12), (1.13) for $\varrho = \varrho_j^*$. Equalities (2.8), (2.9) imply that the sequence (v'_j) tends to a function $v^* \in L_a^2(\mathbf{R}^n)$ in the norm of $L_a^2(\mathbf{R}^n)$ and

$$v^* + g(x, u_0, \ldots, D^\beta u_0, \ldots) = f. \tag{2.10}$$

We shall prove that for any compact $K \subset \mathbf{R}^n$

$$\lim_{j \to \infty} \|u'_j - A^{-1} v^*\|_{H^{2m}(K)} = 0. \tag{2.11}$$

Since $u'_j = A_{\varrho_j^*}^{-1} v'_j$, thus

(2.12)

$$\|u'_j - A^{-1} v^*\|_{H^{2m}(K)} \leqq \|A_{\varrho_j^*}^{-1} v'_j - A^{-1} v'_j\|_{H^{2m}(K)} + \|A^{-1}(v'_j - v^*)\|_{H^{2m}(K)}.$$

The sequence (v'_j) is bounded in $L_a^2(\mathbf{R}^n)$ hence by (0.7) the first term on the right in (2.12) tends to zero as $j \to \infty$. Applying the estimation (0.6) to $A^{-1}(v'_j - v^*)$ we find that the second term on the right in (2.12) tends to zero, too. Thus (2.12) implies (2.11).

From (2.11), (2.7) it follows that

$$u_0 = A^{-1} v^* \quad \text{a. e. in} \quad B_a. \tag{2.13}$$

Denote $A^{-1}v^*$ by u^* then $u^* = u_0$ a.e. in B_a, $v^* = Au^*$ and by use of (2.10) we find that

$Au^* + g(x, u^*, \ldots, D^\beta u^*, \ldots) = f$, further u^* tends to zero at infinity. Equality (2.11) implies the estimation (2.1).

Equality (2.2) can be proved as follows. Assume that the solution u of (0.9) is unique but equality (2.2) is not valid. Then there exist a compact $K \subset \mathbf{R}^n$, a number $\varepsilon_0 > 0$ and a sequence $(u_{\tilde\varrho_j}) = (\tilde u_j)$ such that $\lim_{j \to \infty} \tilde\varrho_j = +\infty$ and

$$\|\tilde u_j - u\|_{H^{2m}(K)} \geqq \varepsilon_0, \quad j = 1, 2, \ldots. \tag{2.14}$$

Then by use of the first part of the proof we have that $(\tilde u_j)$ has a subsequence $(\tilde u'_j)$ such that

$$\lim_{j \to \infty} \|\tilde u'_j - \tilde u\|_{H^{2m}(K)} = 0 \tag{2.15}$$

where $\tilde u$ is a solution of (0.9), vanishing at infinity. Since the solution of (0.9) is unique, thus $\tilde u = u$ and (2.14) is impossible because of (2.15).

If the estimations (0.2), (0.8) are valid, too, then it is easily seen that

$$\lim_{j \to \infty} \|u'_j - A^{-1} v^*\|_{H^{2m}(B_{\varrho_j^*})} = 0$$

(see the proof of (2.11)). This equality implies (2.3). (2.4) can be proved similarly if the solution of (0.9) is unique.

Remark. In [1] and [2] there are formulated sufficient conditions on P and Q which guarantee that the conditions in Theorem 2 and in Theorem 3 are fulfilled (see the introduction).

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