Acta Sci. Math., 47 (1984), 233—237

Small sum sets and the Faber gap condition

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Our purpose in this paper is to analyze sets satisfying a small sum set condition in terms of the classical Faber gap condition and to prove a "shape" result for sets of positive upper Banach density.

For simplicity's sake we will take all sequences (before Theorem 4) to be increasing sequences of non-negative integers. As usual, $\mathbf{Z}^+$ stands for the set of positive integers.

Definition 1 (The Faber gap condition of order p). If $S=\{s_j\}_{j=1}^{\infty}$ and $p\in\mathbf{Z}^+$, then S is said to be an F_p set if $s_{n+p}-s_n\to\infty$.

Definition 2 (Lacunarity condition L_n). A set S is said to satisfy condition L_1 if for every infinite sequence $\{n_j\}_{j=1}^{\infty}$, $|\underline{\lim}\,(S-n_j)|$ is finite. Proceeding inductively, we say that S satisfies condition L_n if for every sequence $\{n_j\}_{j=1}^{\infty}$ the set $\underline{\lim}\,(S-n_j)$ satisfies condition L_{n-1}.

Definition 3 (Generalized Faber gap condition $F_p^{(n)}$). If S is an L_n set that is not L_{n-1} and for some fixed p every n fold iterated lim inf has no more than p elements, we say S has property $F_p^{(n)}$.

Definition 4. If $n\geqq 2$, then a set S is said to be a ϱ_n set if

$$\sup\{\min(|A_1|, |A_2|, \dots, |A_n|):\ A_1+A_2+\dots+A_n\subset S\}<\infty.$$

Examples of ϱ_n sets are constructed in [4] and [5]; see also [3]. Notice that any ϱ_n set $(n\geqq 2)$ is an L_{n-1} set.

Our first theorem shows that conditions F_p and $F_p^{(1)}$ coincide for all p.

Theorem 1. *S is an F_p set if and only if S is an $F_p^{(1)}$ set.*

Proof. Assume only for the sake of notation that $p\geqq 2$. We will show that $S=\{s_j\}_{j=1}^{\infty}$ is an F_p set but not an F_{p-1} set if and only if $\sup|\underline{\lim}\,(S-s_{j_i})|=p$, where the sup is taken over all subsequences $\{s_{j_i}\}_{i=1}^{\infty}\subset S$.

Received October 6, 1982 and in revised form March 10, 1983.

*) The second author was supported by NSF grant MCS-8102061.

First, suppose S is an F_p set, but not an F_{p-1} set. Then by the pigeon hole principle there is a subsequence $\{s_{j_i}\}_{i=1}^{\infty}$ such that for each t with $1 \leqq t \leqq p-1$, $s_{j_i+t}-s_{j_i}$ is constant for all i. It follows that

$$\{0, s_{j_1+1}-s_{j_1}, s_{j_1+2}-s_{j_1}, \ldots, s_{j_1+p-1}-s_{j_1}\}+\{s_{j_i}\}_{i=1}^{\infty} \subset S$$

which implies that $|\underline{\lim}\,(S-s_{j_i})| \geqq p$. On the other hand, if $|\underline{\lim}\,(S-s_{j_i})| > p$ for some $\{s_{j_i}\} \subset S$, then there exist $x_1 < x_2 < \ldots < x_{p+1}$ such that, for some $N \in \mathbf{Z}^+$ and each t with $1 \leqq t \leqq p+1$, $x_t+s_{j_i} \in S$ for all $i \geqq N$. Clearly then S is not an F_p set.

Now suppose $\sup|\underline{\lim}\,(S-s_{j_i})| = p$. If S were not an F_p set, then there would be $N \in \mathbf{Z}^+$ and a subsequence $\{s_{j_i}\}_{i=1}^{\infty} \subset S$ such that $s_{j_i+p}-s_{j_i} < N$ for all i. An argument similar to one we used above shows that $|\underline{\lim}\,(S-s_{j_i})| \geqq p+1$ which is a contradiction. In addition, if $\{s_{j_i}\}_{i=1}^{\infty} \subset S$ and $x_1 < x_2 < \ldots < x_p$ are such that for each t with $1 \leqq t \leqq p$, $x_t+s_{j_i} \in S$ for all sufficiently large i, then S is not an F_{p-1} set.

The next theorem relates the ϱ_2 property and the F_p property.

Theorem 2. *If S is a ϱ_2 set with* $\sup\{\min(|A_1|, |A_2|) \colon A_1+A_2 \subset S\} = p$, *then S is an F_p set.*

Proof. If S were not an F_p set, then there would exist $N \in \mathbf{Z}^+$ and a subsequence $\{s_{j_i}\}_{i=1}^{\infty} \subset S$ such that $s_{j_i+p}-s_{j_i} < N$ for all i. As in the proof of Theorem 1, there are infinitely many i's such that for each t with $1 \leqq t \leqq p$, $s_{j_i+t}-s_{j_i}$ is constant for all such i's. Thus,

$$\{0, s_{j_i+1}-s_{j_i}, s_{j_i+2}-s_{j_i}, \ldots, s_{j_i+p}-s_{j_i}\}+\{s_{j_i}\}_{i=1}^{\infty} \subset S$$

and so $\sup\{\min(|A_1|, |A_2|) \colon A_1+A_2 \subset S\} \geqq p+1$ which is a contradiction.

Next we relate the ϱ_n $(n \geqq 2)$ property and the iterative property $F_p^{(n)}$.

Theorem 3. *Let $n \geqq 3$. If S is a ϱ_n set with*

$$\sup\{\min(|A_1|, |A_2|, \ldots, |A_n|) \colon A_1+A_2+\ldots+A_n \subset S\} = p$$

and S is not an L_{n-2} set then S is an $F_p^{(n-1)}$ set.

Proof. Suppose S is as in the hypothesis and that, in contradiction of the conclusion, the cardinality of some $n-1$ fold iterated lim inf of S (with respect to sequences $\{s_{j_i}^{(k)}\}_{i=1}^{\infty}$, $0 \leqq k \leqq n-2$) is greater than p. Then there are $y_1 < y_2 < \ldots$ $\ldots < y_{p+1} \in \mathbf{Z}^+$ such that for each t with $1 \leqq t \leqq p+1$, $y_t+s_{j_i}^{(n-2)} \in \underline{\lim} \times$ $\times(\ldots(\underline{\lim}\,(\underline{\lim}\,(S-s_{j_i}^{(0)})-s_{j_i}^{(1)})\ldots-s_{j_i}^{(n-3)}))$ for all sufficiently large i. Next, for each t with $1 \leqq t \leqq p+1$, $y_t+s_{j_i}^{(n-2)}+s_{j_i}^{(n-3)} \in \underline{\lim}\,(\ldots(\underline{\lim}\,(S-s_{j_i}^{(0)})-s_{j_i}^{(1)})-\ldots-s_{j_i}^{(n-4)})$ for all sufficiently large i, etc.

Finally, we have, for each t with $1 \leqq t \leqq p+1$, $y_t+s_{j_i}^{(n-2)}+s_{j_i}^{(n-3)}+\ldots+s_{j_i}^{(1)}+$ $+s_{j_i}^{(0)} \in S$ for all sufficiently large i. This contradicts the hypothesis that S is ϱ_n and we are done.

Note. Theorem 3 tells us that if we have a ϱ_n set that is not L_{n-2} then whenever we take $n-2$ lim infs we have arrived at either an F_p set or a finite set.

The next theorem, of interest in itself, and whose proof may remind the reader of Lukomskaya's proof of van der Waerden's Theorem in [6], will help us relate density to the $F_p^{(n)}$ property. We need the following definitions.

Definition 5. If $E\subset \mathbf{Z}$, then the upper Banach density of E is defined by $\overline{Bd}(E)=\limsup_{|I|\to\infty}(|E\cap I|)/|I|$, where I ranges over all bounded intervals in $\mathbf{Z}$.

Theorem 4. *Let* $E=\{e_i\}_{i=1}^{\infty}$. *If* $\overline{Bd}(E)>0$ *then for each* $j\geqq 0$, *there exists* e_{i_j}, M_j *and* k_j *such that*

$$\{e_{i_j}, \dots, e_{i_j+M_j}\}+k_j \subset \{e_{i_{j+1}}, \dots, e_{i_{j+1}+M_{j+1}}\}$$

with $0<M_j<M_{j+1}$, *and, for* $j_1\neq j_2$,

$$\{e_{i_{j_1}}, \dots, e_{i_{j_1}+M_{j_1}}\}\cap\{e_{i_{j_2}}, \dots, e_{i_{j_2}+M_{j_2}}\} = \emptyset.$$

In addition, there exists $M>0$ *such that for each* j,

$$M_j+1 > \left|\frac{e_{i_j+M_j}-e_{i_j}}{M}\right|.$$

Proof. Say $\overline{Bd}(E)>1/N_0$ for some $N_0\in\mathbf{Z}^+$. Then there exist infinitely many integers x_0 such that $|E\cap[x_0, x_0+N_0]|>N_0/2N_0=1/2$. Form at most $2^{N_0+1}-1$ classes of such intervals $[x_0, x_0+N_0]$ according to the "shape" of $E\cap[x_0, x_0+N_0]$. Call such a class by the generic name C. At least one C_0 is infinite.

Next, choose $N_1\in\mathbf{Z}^+$ such that N_0 divides N_1 and $N_1/2N_0>N_0+1$. Now there exist infinitely many integers x_1 such that $E\cap[x_1, x_1+N_1]>N_1/2N_0$. We take such intervals that contain a member of some class C_0. Notice that by breaking up intervals of length N_1 into consecutive intervals of length N_0, we see that, except for finitely many intervals, each interval of length N_1 which has at least $N_1/2N_0$ members of E must contain at least one member of one class C_0 (for if not, all but finitely many such intervals of length N_1 would have fewer than $(N_1/N_0)(N_0/2N_0)=N_1/2N_0$ members). Since there exist infinitely many intervals of length N_1 which contain a member of some C_1 we must have infinitely many such intervals containing infinitely many members of the same class C_0. In addition, if we classify these intervals $[x_1, x_1+N_1]$ according to the "shape" of $E\cap[x_1, x_1+N_1]$ we see that at least one such class of intervals is infinite. Call such classes of intervals of length N_1 by the generic name C_1. Notice that since $N_1/2N_0>N_0+1$, we see that the intersection of E with any member of any C_1 has more elements than the intersection of E with any member of any C_0.

Now choose $N_2\in\mathbf{Z}^+$ such that N_1 divides N_2 and $N_2/2N_0>N_1+1$. We repeat the construction to obtain at least one infinite class C_2 of intervals of the form

$[x_2, x_2+N_2]$ all of whose members have the same "shape" when intersected with E and, for some class C_1, each of whose members contains a member of C_1. Continue in this manner. Notice that at the j^{th} stage, starting with $j=0$, only finitely many of the intervals $[x_j, x_j+N_j]$ which contain at least a proportion $1/2N_j$ of elements of E do not belong to some C_j. Also, no two classes C_j need be disjoint.

We thus may obtain, for each $j \geqq 0$, an interval $[x_j, x_j+N_j] \in C_j$ such that the intervals are pairwise disjoint and such that for each $j \geqq 0$ there is a $k_j \in \mathbf{Z}$ with

$$(E \cap [x_j, x_j+N_j]) + k_j \subset (E \cap [x_{j+1}, x_{j+1}+N_{j+1}])$$

where since $N_{j+1}/2N_0 > N_j+1$,

$$|E \cap [x_j, x_j+N_j]| < |E \cap [x_{j+1}, x_{j+1}+N_{j+1}]|.$$

If we write $\{e_{i_j}, \ldots, e_{i_j+M_j}\} = E \cap [x_j, x_j+N_j]$, the proof is complete.

Corollary. *If E is an L_n set for some n then $\overline{Bd}(E)=0$.*

Proof. If $\overline{Bd}(E)>0$, then if e_0 is the smallest element of $E \cap [x_0, x_0+N_0]$ and $\{n_j\}_{j=0}^{\infty} = \{e_0, e_0+k_0, e_0+k_0+k_1, \ldots\}$ then $\underline{\lim}\,(E-n_j)$ contains the set

$$B_1 = \{0, e_2-e_1, e_3-e_1, \ldots, e_{r_0}-e_1, e'_{r+1}-e'_1, \ldots$$
$$\ldots, e'_{r_1}-e'_1, e^{(2)}_{r+1}-e^{(2)}_1, \ldots, e^{(2)}_{r_2}-e^{(2)}_1, \ldots\}$$

where

$$E \cap [x_0, x_0+N_0] = \{e_1, \ldots, e_{r_0}\},$$
$$E \cap [x_1, x_1+N_1] = \{e'_1, \ldots, e'_{r_1}\},$$
$$\vdots$$
$$E \cap [x_j, x_j+N_j] = \{e^{(j)}_1, \ldots, e^{(j)}_{r_j}\},$$
$$\vdots$$

By construction $\overline{Bd}(B_1)>0$ because for each j, $\{e^{(j)}_1, \ldots, e^{(j)}_{r_j}\}$ contains at least a proportion $1/2N_0$ of the interval $[x_j, x_j+N_j]$.

Now, perform the construction another $n-1$ times to obtain a set B_n with $\overline{Bd}(B_n)>0$. Clearly then E is not an L_n set.

Definition 6. Let $N \in \mathbf{Z}^+$; a subset P of $\mathbf{Z}$ is a parallelepiped of dimension N if P has exactly 2^N elements and can be represented as a sum $P_1+\ldots+P_N$ of N two-element subsets.

It is easy to see that if E does not contain parallelepipeds of arbitrarily large dimension, then E is an L_n set for some n. Thus, an easy consequence of our corollary is that if $\overline{Bd}(E)>0$, then E contains parallelepipeds of arbitrarily large dimension; this fact for natural density was first pointed out in [1]. It is also proved in [1] that if E is a p-Sidon set, a $\Lambda(1)$ set, or a UC-set, then there is a $N \in \mathbf{Z}^+$ for which E contains no parallelepiped of dimension N; see [1] and [4] for some definitions. It is also known that such an analytically defined E cannot contain arithmetic progressions of arbitrary length, hence, it also follows from the work of E. Szemerédi [7] that such sets have natural density zero. On the other hand, it is quite easy to construct an L_n set containing arbitrarily long arithmetic progressions.

We conclude our paper with two examples which delineate the scope of Theorem 3.

Example 1. Let $A_1=\{3^{3^n}+3^{2^1}\colon n\in\mathbf{Z}^+\}$, $A_2=\{3^{5^n}+3^{2^2}\colon n\in\mathbf{Z}^+\}\cup\{3^{5^n}+3^{2^3}\colon n\in\mathbf{Z}^+\}$, ..., $A_t=\{3^{p_t^n}+3^{2^{t'}}\colon n\in\mathbf{Z}^+\}\cup\dots\cup\{3^{p_t^n}+3^{2^{t'+t-1}}\colon n\in\mathbf{Z}^+\}$ where p_t is the t^{th} prime and $t'=1+\sum_{i=1}^{t-1} i$. Let $S=\bigcup_{t=1}^{\infty} A_t$. Then S is clearly not ϱ_2 but since $S\subset\{3^{3^n}\colon n\in\mathbf{Z}^+\}+\{3^{2^n}\colon n\in\mathbf{Z}^+\}$, S is a ϱ_3 set.

In addition, it follows from [2, p. 76] that S is an L_1 set that is not F_p for any p. An appeal to Theorem 1 now confirms the following assertion: Given any natural number p there is a sequence $\{n_j\}_{j=1}^{\infty}$ such that $\infty>|\underline{\lim}\,(S-n_j)|>p$.

Example 2. Fix $q\in\mathbf{Z}^+$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be pairwise disjoint increasing sequences of positive integers. Let $A_1^1=\{3^{b_1}\}$ and $A_2^1=\{3^{c_1}\}$. Let $A_1^2=\{3^{b_2}, 3^{b_3}\}$ and $A_2^2=\{3^{c_2}, 3^{c_3}\}$. Let $A_1^3=\{3^{b_4}, 3^{b_5}, 3^{b_6}\}$ and $A_2^3=\{3^{c_4}, 3^{c_5}, 3^{c_6}\}$, etc. Finally, let

$$S=\bigcup_{n=1}^{\infty}\left[(\{3^{a_1}, 3^{a_2}, \dots, 3^{a_q}\}\cup A_1^n)+A_2^n\right].$$

By construction, S is L_1 (indeed it is F_{q+1}). Also if $\{d_n\}$ is any sequence such that $d_n\in A_2^n$ for each n, then $\{3^{a_1}, 3^{a_2}, \dots, 3^{a_q}\}\subset\underline{\lim}\,(S-d_n)$. It follows from say [4] that

$$\sup\{\min(|A_1|, |A_2|, |A_3|)\colon A_1+A_2+A_3\subset S\}$$

is bounded above for all q. Clearly q could have been chosen greater than this bound; thus $\sup\{\min(|A_1|, |A_2|, |A_3|)\colon A_1+A_2+A_3\subset S\}<\sup|\underline{\lim}\,(S-n_j)|=q+1$, for q sufficiently large.

Examples 1 and 2 show that the hypothesis in Theorem 3 that S not be an L_{n-2} set cannot be relaxed.

References

[1] J. J. F. Fournier and L. Pigno, Analytic and arithmetic properties of thin sets, *Pacific J. Math.*, **105** (1983), 115—141.

[2] R. E. Dressler and L. Pigno, Some lacunarity conditions for Fourier—Stieltjes transforms, *Ark. Math.*, **13** (1975), 73—77.

[3] R. E. Dressler, On sets which contain sum sets, *Acta Sci. Math.*, **42** (1980), 233—235.

[4] R. E. Edwards and K. A. Ross, p-Sidon sets, *J. Funct. Anal.*, **15** (1974), 404—427.

[5] G. W. Johnson and G. S. Woodward, On p-Sidon sets, *Indiana J. Math.*, **24** (1974), 161—168.

[6] A. Y. Khinchin, *Three pearls of number theory*, Graylock Press (Rochester, N. Y., 1952).

[7] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, *Acta Arith.*, **27** (1975), 199—245.

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