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A unified approach to fundamental theorems of approximation by sequences of linear operators and their dual versions

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Dedicated to Professor Béla Szőkefalvi-Nagy on the occasion of his 70th birthday on 29. July 1983, in high esteem

1. Introduction

The theorems of Jackson and Bernstein as well as those of Stečkin and Zamansky and their converses play a fundamental role in the theory of best approximation for periodic functions by trigonometric polynomials. These results have been generalized to the setting of abstract Banach spaces by Butzer and Scherer [7], [8], [9], who also proved corresponding approximation theorems for sequences of commutative bounded linear operators $\{T_n\}_{n=1}^{\infty}$ satisfying $\lim_{n\to\infty} T_n f = f$ in the norm topology as well as so called Jackson and Bernstein-type inequalities (cf. [10]).

This paper is concerned with the latter aspect. The aim is to weaken the assumption upon the sequence $\{T_n\}$ in the sense that $\lim_{n\to\infty} T_n f = f$ needs to hold only in a certain weak topology. This enables one to handle sequences of operators converging in the usual weak or weak* topology towards the identity operator.

Our main theorem (Theorem 1) subsumes not only the Butzer—Scherer theorem (Theorem 2) mentioned above, but also the corresponding results for sequences of dual operators (Theorem 3), contained in [17], [18]. This theory can be applied to classical linear approximation processes such as summation methods for Fourier series or semigroups of bounded linear operators. The further advantage is that it enables one to investigate processes defined by sequences of the dual operators.

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The paper is divided as follows. After a preliminary section containing definitions and elementary lemmas the general approximation theorem is established in Section 3. From this result the Butzer—Scherer theorem is deduced in Section 4. Its dual version together with applications to convolution integrals of periodic functions are dealt with in Section 5.

2. Preliminaries

Let us begin with some basic definitions and results concerning norm-determining sets and linear approximation processes.

Definition 1. Let X be a normed linear space with norm $\|\cdot\|_X$, and let X' denote its dual endowed with the usual norm $\|\cdot\|_{X'}$. For a linear subspace $\mathcal{M}$ in X' the characteristic $v(\mathcal{M})$ is defined as

$$v(\mathcal{M}) := \inf\{p_{\mathcal{M}}(f);\ f\in X,\ \|f\|_X = 1\} \tag{2.1}$$

where $p_{\mathcal{M}}(f)$ is given by

$$p_{\mathcal{M}}(f) := \sup\{|f'(f)|;\ f'\in\mathcal{M},\ \|f'\|_{X'} = 1\} \quad (f\in X). \tag{2.2}$$

If $v(\mathcal{M})>0$, then $\mathcal{M}$ is said to be norm-determining (for X).

It follows from the inequalities

$$v(\mathcal{M})\|f\|_X \leqq p_{\mathcal{M}}(f) \leqq \|f\|_X \quad (f\in X) \tag{2.3}$$

that $\|\cdot\|_X$ and $p_{\mathcal{M}}(\cdot)$ are equivalent norms for X, provided $\mathcal{M}$ is norm-determining. If, in addition, $\mathcal{M}$ is closed in X', then one has

Lemma 1 (cf. [19, p. 203]). *Let F be a subset of a normed linear space X, and let $\mathcal{M}$ be a closed norm-determining subspace of X'. If* $\sup\{|f'(f)|; f\in F\}<\infty$ *for each $f'\in\mathcal{M}$; then* $\sup\{\|f\|_X; f\in F\}<\infty$.

Next we introduce the concept of $\mathcal{M}$-weak convergence in X.

Definition 2. Let $\mathcal{M}$ be a norm-determining set for the linear space X. A sequence $\{f_n\}_{n=1}^{\infty}$ in X is said to be $\mathcal{M}$-weakly convergent to $f\in X$ ($\mathcal{M}$-$\lim_{n\to\infty} f_n=f$), if

$$\lim_{n\to\infty}\langle f', f_n-f\rangle = 0 \quad (f'\in\mathcal{M}).$$

Note that the $\mathcal{M}$-limit is uniquely determined, since $f'(f)=0$ for all $f'\in\mathcal{M}$ implies $f=0$ by (2.3). Moreover, if $\mathcal{M}$ is closed in X', then Lemma 1 yields that every $\mathcal{M}$-weak convergent sequence is bounded.

Choosing $\mathcal{M}=X'$ gives $v(X')=1$, and it follows that weak convergence is a particular case of $\mathcal{M}$-weak convergence. One may also take $\mathcal{M}=J(X)$, where

J is the canonical mapping from X into X''. Again one has $\nu(J(X))=1$, and $\mathcal{M}$-weak convergence in X' turns out to be w^*-convergence (cf. [19, pp. 208, 209]).

Let us finally mention that the only closed norm-determining subspace for a reflexive Banach space X is X' itself. This follows from the facts that a closed subspace of X' is also w^*-closed (cf. [13, p. 422]) and, on the other hand, a norm-determining subspace must be w^*-dense in X'. For further properties of norm-determining sets and $\mathcal{M}$-weak convergence see [1], [12], [19, Sec. 4.4].

Now we consider sequences of bounded linear operators from X into itself converging $\mathcal{M}$-weakly towards the identity.

Definition 3. Let X be a normed linear space, $\mathcal{M}$ a closed norm-determining subspace of X', and $\{T_n\}_{n=1}^{\infty}$ a sequence of bounded linear operators mapping X into itself with the properties

$$T_n T_m f = T_m T_n f \quad (f\in X;\ m, n\in \mathbf{N} := \{1, 2, \ldots\}), \tag{2.4}$$

$$\mathcal{M}\text{-}\lim_{n\to\infty} T_n f = f \quad (f\in X). \tag{2.5}$$

Then $\{T_n\}$ is called a commutative, $\mathcal{M}$-weak linear approximation process on X ($\mathcal{M}$-LAP).

Note that $\mathcal{M}$ will be assumed to be a closed norm-determining subspace of X' when speaking of an $\mathcal{M}$-LAP on X.

For an $\mathcal{M}$-LAP on a Banach space X the following inequalities hold; the first is a generalization of the well known uniform boundedness principle for sequences of strongly convergent operators.

Lemma 2. *Let $\{T_n\}_{n=1}^{\infty}$ be an $\mathcal{M}$-LAP on a Banach space X. Then*

$$\|T_n f\|_X \leqq M\|f\|_X\text{[1]} \quad (f\in X;\ n\in\mathbf{N}), \tag{2.6}$$

$$\|T_n f - f\|_X \leqq \frac{1}{\nu(\mathcal{M})}\sum_{k=0}^{\infty}\|T_{n2^k}f - T_{n2^{k+1}}f\|_X \quad (f\in X;\ n\in\mathbf{N}). \tag{2.7}$$

Proof. For fixed $f\in X$ the sequence $\{T_n f\}_{n=1}^{\infty}$ is an $\mathcal{M}$-weakly convergent sequence in X, and it follows by Lemma 1 that $\|T_n f\|_X \leqq M_f$, where M_f depends on f but not on n. Since X is a Banach space one can apply the classical uniform boundedness principle to deduce (2.6). Concerning (2.7), one has for $f'\in X'$, $f\in X$ that

$$\langle f', T_n f - f\rangle = \sum_{k=0}^{\infty}\langle f', T_{n2^k}f - T_{n2^{k+1}}f\rangle.$$

[1]) Throughout M denotes a positive constant, the value of which may be different at each occurrence, even in a given line. M is always independent of the quantities on the right margin.

Then (2.3) and (2.2) yield the assertion since

$$\nu(\mathcal{M})\|T_nf-f\|_X \leqq p_{\mathcal{M}}(T_nf-f) =$$

$$= \sup\left\{\left|\sum_{k=0}^{\infty}\langle f', T_{n2^k}f-T_{n2^{k+1}}f\rangle\right|;\ f'\in\mathcal{M},\ \|f'\|_{X'}=1\right\} \leqq$$

$$\leqq \sum_{k=0}^{\infty}\|T_{n2^k}f-T_{n2^{k+1}}f\|_X.$$

3. Order of approximation for $\mathcal{M}$-LAP's

In this section we consider $\mathcal{M}$-LAP's converging strongly towards the identity on certain subsets of X, and investigate the rate of convergence.

3.1. K-functional; Jackson and Bernstein-type inequalities

Definition 4. Let X be a linear space and Y a linear subspace of X with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The K-functional between X and Y is defined as

$$K(t,f;\ X,Y) := \inf\{\|f_1\|_X+t\|f_2\|_Y;\ f=f_1+f_2,\ f_1\in X,\ f_2\in Y\}\quad (f\in X;\ t>0).$$

For fixed $f\in X$ the K-functional is nondecreasing on $(0,\infty)$, and satisfies

$$K(\lambda t,f;\ X,Y) \leqq \max\{1,\lambda\}K(t,f;\ X,Y)\quad (\lambda, t>0). \tag{3.1}$$

Furthermore, $\lim_{t\to 0+} K(t,f;\ X,Y)=0$ for each $f\in X$ if and only if Y is dense in X.

Definition 5. Let X, Y be given as in Definition 4, $\mathcal{M}$ be a closed, norm-determining subspace of X', and $\{T_n\}_{n=1}^{\infty}$ an $\mathcal{M}$-LAP on X. If for some $\alpha\geqq 0$ there holds

$$\|T_ng-g\|_X \leqq Mn^{-\alpha}\|g\|_Y\quad (g\in Y;\ n\in\mathbf{N}), \tag{3.2}$$

then $\{T_n\}$ is said to satisfy a Jackson-type inequality of order α on X with respect to Y.

If $T_nf\in Y$ for all $f\in X$, $n\in\mathbf{N}$ and

$$\|T_nf\|_Y \leqq Mn^{\alpha}\|f\|_X\quad (f\in X;\ n\in\mathbf{N}), \tag{3.3}$$

then $\{T_n\}$ is said to satisfy a (weak) Bernstein-type inequality of order α on X with respect to Y.

In contrast to (3.3) one would speak of a *strong* Bernstein-type inequality, if $T_n f \in Y$ for all $f \in X$ and

$$\|g\|_Y \leqq M n^{\alpha} \|g\|_X \quad (g \in T_n(X);\ n \in \mathbf{N}). \tag{3.4}$$

In this respect see also the remarks following Theorem 1. Finally we need

Definition 6. The class of positive, nondecreasing functions φ defined on $(0, 1]$ with $\lim_{t\to 0+} \varphi(t)=0$ and $\varphi(1)<\infty$ is denoted by Φ.

The following condition on $\varphi \in \Phi$, $\alpha, \beta \geqq 0$ will be of interest, namely

$$\sum_{1 \leqq 2^j \leqq t^{-1}} 2^{\alpha j} \varphi(2^{-j}) = O\big(t^{-\alpha}\varphi(t)\big) \quad (t \to 0+), \tag{3.5}$$

$$\sum_{j=0}^{\infty} (t^{-1}2^j)^{\beta} \varphi(t2^{-j}) = O\big(t^{-\beta}\varphi(t)\big) \quad (t \to 0+). \tag{3.6}$$

For conditions which are equivalent to (3.5) or (3.6) we refer to the appendix.

Lemma 3. *If* $\varphi \in \Phi$, $\alpha, \beta \geqq 0$, *then* (3.5) *implies* $\lim_{t\to 0+} t^{-\alpha}\varphi(t)=\infty$, *and* (3.6) *implies* $\lim_{t\to 0+} t^{-\beta}\varphi(t)=0$. *In particular, if* (3.5) *and* (3.6) *are valid, then* $\alpha>\beta$. *Furthermore,* (3.5) *implies*

$$\sum_{j=0}^{k} 2^{\alpha j} \varphi(2^{-j}) = O\big(2^{\alpha k}\varphi(2^{-k})\big) \quad (k \to \infty). \tag{3.7}$$

Proof. One has by (3.5) for $t=2^{-k}$ that $\varphi(1) \leqq M 2^{\alpha k}\varphi(2^{-k})$. This yields, again by (3.5),

$$(m+1)\varphi(1) \leqq M \sum_{k=0}^{m} 2^{\alpha k}\varphi(2^{-k}) \leqq M 2^{\alpha m}\varphi(2^{-m}) \quad (m \in \mathbf{N}),$$

giving $\lim_{m\to\infty} 2^{\alpha m}\varphi(2^{-m})=\infty$. The first assertion now follows by choosing $m \in \mathbf{N}$ such that $2^{m-1} \leqq t^{-1} < 2^m$ since in this case $t^{-\alpha}\varphi(t) > 2^{(m-1)\alpha}\varphi(2^{-m})$. Concerning the second part, one has by the convergence of the series in (3.6) that $\lim_{j\to\infty} (2^j m)^{\beta}\varphi(2^{-j}m^{-1})=0$, at least for m large enough. If one takes $m \in \mathbf{N}$ such that $2^j m \leqq t^{-1} < 2^{j+1} m$, m fixed, then one can complete the proof as before. Finally, if $k \in \mathbf{P} := \{0, 1, 2 \dots\}$ satisfies $2^k \leqq t^{-1} < 2^{k+1}$, then

$$\sum_{j=0}^{k} 2^{\alpha j}\varphi(2^{-j}) = \sum_{1 \leqq 2^j \leqq t^{-1}} 2^{\alpha j}\varphi(2^{-j}) \leqq M t^{-\alpha}\varphi(t) \leqq M 2^{\alpha k}\varphi(2^{-k}) \quad (k \in \mathbf{P}),$$

which is (3.7).

3.2. The fundamental theorem for approximation processes. Our main results now read as follows.

Theorem 1. a) *Let X be a Banach space, Y a linear subspace of X and $\mathcal{M}$ a closed norm-determining subspace of X'. Further, let $\{T_n\}_{n=1}^{\infty}$ be an $\mathcal{M}$-LAP satisfying Jackson and Bernstein-type inequalities of order $\alpha>0$ on X with respect to Y, and let $\varphi\in\Phi$ be such that* (3.5) *holds. Then the following assertions are equivalent for $f\in X$:*

(i) $\|T_nf-f\|_X = O(\varphi(n^{-1}))\quad (n\to\infty)$,

(ii) $K(t^\alpha,f;\ X,Y) = O(\varphi(t))\quad (t\to 0+)$.

b) *Suppose, in addition, that Z is a Banach space continuously embedded in X*[2]) *such that $\{T_n\}$ satisfies Jackson and Bernstein-type inequalities of order $\beta\geqq 0$ on X with respect to Z and assume that* (3.6) *holds for $\varphi\in\Phi$. Then each of the following assertions is equivalent to those of part* a) *for $f\in X$:*

(iii) $f\in Z$ and $\|T_nf-f\|_Z = O(n^\beta\varphi(n^{-1}))\quad (n\to\infty)$,

(iv) $\|T_nf\|_Y = O(n^\alpha\varphi(n^{-1}))\quad (n\to\infty)$.

The proof of this theorem is based on the following four lemmas.

Lemma 4. *If $X, Y, \mathcal{M}$ and $\{T_n\}_{n=1}^{\infty}$ are given as in Theorem 1. a), then*

$$\|T_nf-f\|_X \leqq MK(n^{-\alpha},f;\ X,Y)\quad (f\in X;\ n\in\mathbf{N}). \tag{3.8}$$

Proof. For each representation $f=f_1+f_2$ with $f_1\in X, f_2\in Y$ one has in view of (2.6) and the Jackson-type inequality (3.2)

$$\|T_nf-f\|_X \leqq \|T_nf_1-f_1\|_X+\|T_nf_2-f_2\|_X \leqq M\|f_1\|_X+n^{-\alpha}\|f_2\|_Y.$$

Taking the infimum over all such representations yields (3.8).

Lemma 5. *Under the assumptions of Theorem 1. a) assertion* (ii) *implies*

$$\|T_nf\|_Y = O(n^\alpha\varphi(n^{-1}))\quad (f\in X;\ n\to\infty).$$

Proof. By (2.4) one has for each $k\in\mathbf{N}$

$$\begin{aligned} T_nf &= T_n(f-T_{2^k}f)-T_{2^k}(f-T_nf)+T_1f+\\ &\quad+\sum_{j=1}^{k}[T_{2^j}(f-T_{2^{j-1}}f)-T_{2^{j-1}}(f-T_{2^j}f)]. \end{aligned} \tag{3.9}$$

[2]) "Z continuously embedded in X" means that Z is a subspace of X and that the identity map is continuous, i.e., in case of normed linear spaces that $\|f\|_X\leqq M\|f\|_Z$ for all $f\in Z$.

So one deduces by the Bernstein-type inequality on X with respect to Y that

$$\|T_n f\|_Y \leqq M\{n^\alpha \|f-T_{2^k}f\|_X + 2^{\alpha k}\|f-T_n f\|_X + \|f\|_X + \sum_{j=1}^{k} [2^{\alpha j}\|f-T_{2^{j-1}}f\|_X + 2^{\alpha(j-1)}\|f-T_{2^j}f\|_X]\}. \tag{3.10}$$

This yields by (ii) the inequality

$$\|T_n f\|_Y \leqq M\{n^\alpha \varphi(2^{-k}) + 2^{\alpha k}\varphi(n^{-1}) + \|f\|_X + \sum_{j=1}^{k} [2^{\alpha j}\varphi(2^{-j+1}) + 2^{\alpha(j-1)}\varphi(2^{-j})]\}.$$

Choosing now $k\in\mathbf{N}$ such that $2^{k-1}\leqq n<2^k$, and using (3.7) gives

$$\|T_n f\|_Y \leqq M\{n^\alpha \varphi(n^{-1}) + \|f\|_X\} \quad (n\in\mathbf{N})$$

which in turn implies the assertion by Lemma 3.

Lemma 6. *Under the assumptions of Theorem 1. b) assertion* (i) *implies* (iii).

Proof. For $n, N\in\mathbf{N}$ one has by (2.4) for $f\in Z$

$$\sum_{k=0}^{N} \|T_{n2^k}f - T_{n2^{k+1}}f\|_Z \leqq \sum_{k=0}^{N} \{\|T_{n2^k}(f-T_{n2^{k+1}}f)\|_Z + \|T_{n2^{k+1}}(f-T_{n2^k}f)\|_Z\}.$$

Estimating the terms on the right hand side by the Bernstein-type inequality on X with respect to Z, (i) yields

$$\sum_{k=0}^{N} \|T_{n2^k}f - T_{n2^{k+1}}f\|_Z \leqq \leqq M\left\{\sum_{k=0}^{N} (n2^k)^\beta \varphi(n^{-1}2^{-k-1}) + (n2^{k+1})^\beta \varphi(n^{-1}2^{-k})\right\} \leqq Mn^\beta \varphi(n^{-1}) \quad (n, N\in\mathbf{N}), \tag{3.11}$$

the latter estimate follows from (3.6). Since Z is a Banach space this implies in particular that there exists a $g\in Z$ such that

$$\lim_{N\to\infty} \left\| \sum_{k=0}^{N} (T_{n2^k}f - T_{n2^{k+1}}f) - g \right\|_Z = 0 \quad (n\in\mathbf{N}).$$

On the other hand it follows from (i) that

$$\lim_{N\to\infty} \left\| \sum_{k=0}^{N} (T_{n2^k}f - T_{n2^{k+1}}f) - (T_n f - f) \right\|_X = 0 \quad (n\in\mathbf{N}).$$

So one obtains by the continuous embedding of Z in X that $T_n f - f = g\in Z$, yielding $f\in Z$, since $T_n f\in Z$ by the Bernstein-type inequality with respect to Z.

Furthermore, one has by (3.11)

$$\|T_n f-f\|_Z = \Big\|\sum_{k=0}^{\infty}(T_{n2^k}f-T_{n2^{k+1}}f)\Big\|_Z = O\big(n^\beta\varphi(n^{-1})\big) \quad (n\to\infty),$$

proving assertion (iii).

Lemma 7. *Under the assumptions of Theorem 1. b) there holds*

$$\|T_n f\|_Y \leqq Mn^{\alpha-\beta}\|f\|_Z \quad (f\in Z;\ n\in \mathbf{N}). \tag{3.12}$$

Proof. Since Z is a subset of X, one can deduce (3.10) for $f\in Z$ as in the proof of Lemma 5. Applying the Jackson-type inequality on X with respect to Z to the right side of that estimate, yields for each $k\in\mathbf{N}$

$$\|T_n f\|_Y \leqq M\{[n^\alpha 2^{-\beta k}+2^{\alpha k}n^{-\beta}+\sum_{j=1}^{k}(2^{\alpha j}2^{-\beta(j-1)}+2^{\alpha(j-1)}2^{-\beta j}]\|f\|_Z+\|f\|_X\}.$$

The desired result now follows by choosing $k\in\mathbf{N}$ such that $2^{k-1}\leqq n<2^k$, and the fact that Z is continuously embedded in X.

The inequality (3.12) could be regarded as a Bernstein-type inequality of order $\alpha-\beta$ on Z with respect to Y, if one disregards the general assumptions made in Definition 5 (e.g. one has not necessarily that Y is a subspace of Z or that $\{T_n\}$ is an $\mathcal{M}$-LAP on Z).

Proof of Theorem 1. The implication (ii)⇒(i) follows from Lemma 4. Conversely, let (i) be satisfied. Since $T_n f\in Y$ for all $f\in X$, one has by (i) and Lemma 5 that

$$K(t^\alpha,f;\ X,Y)\leqq \|f-T_n f\|_X+t^\alpha\|T_n f\|_Y\leqq M\{\varphi(n^{-1})+t^\alpha n^\alpha\varphi(n^{-1})\}$$

$$(t>0;\ n\in\mathbf{N}).$$

This yields (ii) by choosing $n\in\mathbf{N}$ such that $n-1<t^{-1}\leqq n$.

Concerning part b), we proceeded by proving (iv)⇒(i)⇒(iii)⇒(iv). Assume that (iv) holds. Using (2.7) and (2.4) one obtains

$$\|T_n f-f\|_X\leqq \frac{1}{\nu(\mathcal{M})}\sum_{j=0}^{\infty}\|T_{n2^j}f-T_{n2^{j+1}}f\|_X\leqq$$

$$\leqq\frac{1}{\nu(\mathcal{M})}\sum_{j=0}^{\infty}\{\|T_{n2^j}f-T_{n2^{j+1}}(T_{n2^j}f)\|_X+\|T_{n2^{j+1}}f-T_{n2^j}(T_{n2^{j+1}}f)\|_X\}.$$

Estimating the terms in curly brackets by the Jackson-type inequality on X with respect to Y, (iv) gives

$$\|T_n f - f\|_X \leqq M \sum_{j=0}^{\infty} \{(n2^{j+1})^{-\alpha} \|T_{n2^j} f\|_Y + (n2^j)^{-\alpha} \|T_{n2^{j+1}} f\|_Y\} \leqq$$

$$\leqq M \sum_{j=0}^{\infty} \{\varphi(n^{-1}2^{-j}) + \varphi(n^{-1}2^{-j-1})\} \leqq Mn^{-\beta} \sum_{j=0}^{\infty} (n2^j)^{\beta} \varphi(n^{-1}2^{-j}).$$

So (i) follows by (3.6). The implication (i)$\Rightarrow$(iii) was established in Lemma 6 and (iii)$\Rightarrow$(iv) can be deduced in the same way as Lemma 5, using (3.12). So the proof is complete.

3.3. Remarks. Note that one can always take $Z = X$ in Theorem 1. b). In this case the Jackson and Bernstein-type inequalities of order $\beta = 0$ are obviously valid, and one has the equivalence of (i), (ii) and (iv) under the assumption of part a) provided (3.6) holds for $\beta = 0$.

Theorem 1 could also be stated for families of bounded linear operators $\{T_t;\ t \in (0, 1]\}$ depending on a continuous parameter t satisfying

$$T_t T_s f = T_s T_t f \quad (f \in X;\ s, t \in (0, 1]), \quad \mathcal{M}\text{-}\lim_{t \to 0+} T_t f = f \quad (f \in X)$$

instead of (2.4) and (2.5). In this case one has to replace n everywhere by t^{-1} and $n \to \infty$ by $t \to 0+$. The only slight modification is that one uses condition (3.5) instead of (3.7) in the proof of Lemma 5.

The assumption of the commutativity (2.4) in Theorem 1 can be weakend to

$$(3.13) \qquad T_n T_{2n} f = T_{2n} T_n f \quad (f \in X;\ n \in \mathbf{N}).$$

Indeed, only this property was used in the proofs. Moreover, it is possible to establish part a) of Theorem 1 without any commutativity assumption, provided the operators T_n satisfy a strong Bernstein-type inequality (3.4) of order $\alpha > 0$ with respect to Y together with $T_n f \in T_m(X)$ for all $n, m \in \mathbf{N}$ with $m \geqq n$. Then one can apply the theory of best approximation to

$$(3.14) \qquad E(f;\ T_n(X)) := \inf \{\|f - g\|_X;\ g \in T_n(X)\}$$

to deduce

$$\|T_n f - f\|_X = O(n^{-\sigma}) \Rightarrow E(f;\ T_n(X)) = O(n^{-\sigma}) \Rightarrow K(t^{\alpha}, f;\ X, Y) = O(t^{\sigma}).$$

The final step, namely $K(t^{\alpha}, f; X, Y) = O(t^{\sigma}) \Rightarrow \|T_n f - f\|_X = O(n^{-\sigma})$, then follows by Lemma 4. The equivalence with assertion (iii) of Theorem 1. a) can also be proved in this frame. A similar approach in case of strong approximation processes (cf. Definition 7) can be found in [7], [8]; for results on best approximation see e.g. [7], [8], [9], [15].

Another method to obtain the equivalence of assertions (i) and (ii) for non-commutative operators is to assume the stability condition

$$(3.15) \qquad \|T_n g\|_Y \leqq M\|g\|_Y \quad (g\in Y;\ n\in \mathbf{N}).$$

In this regard see [3], [14].

4. Applications to particular approximation processes

4.1. Strong approximation processes. In this section we apply Theorem 1 to so called strong linear approximation processes as well as to their dual versions.

Definition 7. Let X be a normed linear space, and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of bounded linear operators from X into itself satisfying (2.4) together with

$$(4.1) \qquad \lim_{n\to\infty} \|T_n f - f\|_X = 0 \quad (f\in X).$$

Then $\{T_n\}$ is called a commutative strong linear approximation process on X (LAP).

Since (4.1) implies weak convergence of $T_n f$ to f, and weak convergence is a particular case of $\mathcal{M}$-weak convergence with $\mathcal{M}=X'$, one can apply Theorem 1 to deduce

Theorem 2. a) *Let X, Y, φ be given as in Theorem 1. a), and $\{T_n\}_{n=1}^{\infty}$ a* LAP *on X satisfying Jackson and Bernstein-type inequalities of order $\alpha>0$ on X with respect to Y. Then assertions* (i) *and* (ii) *of Theorem 1 are equivalent.*

b) *If, in addition, Z and φ are as in Theorem 1. b), and $\{T_n\}_{n=1}^{\infty}$ satisfies Jackson and Bernstein-type inequalities of order $\beta\geqq 0$ on X with respect to Z, then assertions* (i)—(iv) *of Theorem 1 are equivalent.*

This result in the more general setting of intermediate spaces but only for $\varphi(t)=t^{\sigma}$, $\sigma>0$ can be found in [7], [8], [9], [10].

4.2. Weak* approximation processes. The LAP $\{T_n\}$ of Theorem 2 maps X into the subspace Y. Hence the dual operators T_n' defined by

$$(4.2) \qquad \langle T_n' f', f\rangle := \langle f', T_n f\rangle \quad (f'\in Y';\ f\in X)$$

map Y' into X'. In order to have $\{T_n'\}$ as an $\mathcal{M}$-LAP on Y' for a suitable $\mathcal{M}$ in Y'', we need to make some further assumptions. It will turn out that in this case the Jackson and Bernstein-type inequalities posed upon $\{T_n\}$ imply those upon $\{T_n'\}$.

Lemma 8. *Let X, Y be Banach spaces, $Y\subset X$*[3]*), and $\{T_n\}_{n=1}^{\infty}$ a* LAP *on*

[3]) "$Y\subset X$" means that Y is a continuously embedded subspace of X.

X satisfying Jackson and Bernstein-type inequalities of order $\alpha \geqq 0$ on X with respect to Y. Furthermore, assume that

$$\lim_{n\to\infty} \|T_n g - g\|_Y = 0 \quad (g \in Y). \tag{4.3}$$

Then $X' \subset Y'$ and $\{T_n'\}_{n=1}^{\infty}$ is a $J(Y)$-LAP on Y' (J being the canonical map from Y into Y'') that satisfies Jackson and Bernstein-type inequalities of order α on Y' with respect to X'.

Proof. Since Y is continuously embedded and dense in X, (density holding in view of the Bernstein-type inequality and (4.1)), it follows that $X' \subset Y'$. Now one has for $f' \in Y', f \in Y$ that

$$|\langle T_n' f' - f', f\rangle| = |\langle f', T_n f - f\rangle| \leqq \|f'\|_{Y'} \|T_n f - f\|_Y = o(1) \quad (n \to \infty)$$

in view of (4.3). This shows that $T_n' f'$ converges in the w^*-sense to f' or equivalently, that $J(Y) - \lim_{n\to\infty} V_n' f' = f'$. Furthermore, the commutativity of the sequence $\{T_n\}$ implies that of $\{T_n'\}$, and it follows that $\{T_n'\}$ is a $J(Y)$-LAP on Y'. Note that $J(Y)$ is closed in Y'', since Y is a Banach space. Finally, if I denotes the identity operator in any space, the Jackson and Bernstein-type inequalities foı $\{T_n'\}$ on Y' with respect to X' follow from (3.2) and (3.3) since (cf. [19, p. 214])

$$\|T_n' - I\|_{[X', Y']} = \|T_n - I\|_{[Y, X]} \leqq M n^{-\alpha} \quad (n \in \mathbf{N}),$$

$$\|T_n'\|_{[Y', X']} = \|T_n\|_{[X, Y]} \leqq M n^{\alpha} \quad (n \in \mathbf{N}).$$

Since for a Banach space Y, $J(Y)$-convergence is the same as w^*-convergence, in the following we will speak of a w^*-LAP instead of a $J(Y)$-LAP.

Lemma 8 enables one to apply Theorem 1. a) to the sequence $\{T_n'\}$. In order to obtain a counterpart also of Theorem 1. b) we take a Banach space Z such that $Y \subset Z \subset X$, and assume that $\{T_n\}$ satisfies Jackson and Bernstein-type inequalities of order $\beta \in [0, \alpha)$ on X with respect to Z. Then $X' \subset Z' \subset Y'$ by Lemma 8, and it follows from Lemma 7 and Lemma 8 that there holds a Bernstein-type inequality of order $\alpha - \beta$ on Y' with respect to Z' for $\{T_n'\}$. Concerning the Jackson-type inequality one has

Lemma 9. *Let $X, Y, \{T_n\}_{n=1}^{\infty}$ be given as in Lemma 8, and assume that there is a Banach space Z such that $Y \subset Z \subset X$ and $\{T_n\}$ satisfies Jackson and Bernstein-type inequalities of order $\beta \in [0, \alpha)$ on X with respect to Z. Then $\{T_n\}$ and $\{T_n'\}$ satisfy Jackson-type inequalities of order $\alpha - \beta$ on Z with respect to Y and on Y' with respect to Z', respectively.*

Proof. Since $Y \subset Z$, one has by (4.3) for $f \in Y$ that

$$\|T_n f - f\|_Z = \left\| \sum_{j=0}^{\infty} T_{n2^j} f - T_{n2^{j+1}} f \right\|_Z \leqq$$
$$\leqq \sum_{j=0}^{\infty} \{\|T_{n2^j}(f - T_{n2^{j+1}} f)\|_Z + \|T_{n2^{j+1}}(f - T_{n2^j} f)\|_Z\}.$$

Estimating the right side first by means of the Bernstein-type inequality on X with respect to Z, and then by the Jackson-type inequality on X with respect to Y, yields

$$\|T_n f - f\|_Z \leqq M \sum_{j=0}^{\infty} \{(n2^j)^{\beta}(n2^{j+1})^{-\alpha} + (n2^{j+1})^{\beta}(n2^j)^{-\alpha}\}\|f\|_Y \leqq$$
$$\leqq M n^{-(\alpha-\beta)} \|f\|_Y \quad (n \in \mathbf{N}).$$

This is the first part of the assertion; the second follows by Lemma 8.

We are now in a position to apply Theorem 1 to the dual of a LAP.

Theorem 3. a) *Let* X, Y *be Banach spaces such that* $Y \subset X$, *and* $\{T_n\}_{n=1}^{\infty}$ *a* LAP *on* X *satisfying* (4.3) *as well as Jackson and Bernstein-type inequalities of order* $\alpha > 0$ *on* X *with respect to* Y *and let* $\varphi \in \Phi$ *be such that* (3.5) *holds. Then* $X' \subset Y'$, $\{T_n'\}$ *defined by* (4.2) *is a* w^*-LAP *on* Y', *and the following assertions are equivalent for* $f' \in Y'$:

(i) $\|T_n' f' - f'\|_{Y'} = O(\varphi(n^{-1})) \quad (n \to \infty)$,

(ii) $K(t^{\alpha}, f'; Y', X') = O(\varphi(t)) \quad (t \to 0+)$.

b) *Suppose that* Z *is another Banach space such that* $Y \subset Z \subset X$, *and* $\{T_n\}$ *satisfies Jackson and Bernstein-type inequalities of order* $\beta \in [0, \alpha)$ *on* X *with respect to* Z, *and assume, that* (3.6) *holds for the order* $\alpha - \beta$ *instead of for* β. *Then each of the following assertions is equivalent to those of part* a) *for* $f' \in Y'$:

(iii) $f' \in Z'$ *and* $\|T_n' f' - f'\|_{Z'} = O(n^{\alpha-\beta}\varphi(n^{-1})) \quad (n \to \infty)$,

(iv) $\|T_n' f'\|_{X'} = O(n^{\alpha}\varphi(n^{-1})) \quad (n \to \infty)$.

The equivalence of assertions (i), (ii) and (iii) is contained in [17], [18] where it was proved by methods parallel to Theorem 2 in the frame of intermediate spaces.

If the commutativity (2.4) or even (3.13) is dropped, the method of obtaining equivalence theorems by using a strong Bernstein-type inequality in connection with the best approximation (3.14) cannot be carried over to the dual case. This is due to the fact that when passing to best approximation in dual spaces the strong Bernstein inequality converts into a Jackson inequality (for dual best approximation) and, conversely, the Jackson inequality for best approximation converts into a Bernstein inequality. This is entirely different from the situation for $\mathcal{M}$-LAP as was shown in Lemma 8. (For best approximation in dual spaces see [9], [17].)

On the other hand the approach using the stability condition (3.15) can always be applied to the dual case, since the dual stability condition needed, namely $\|T_n'g'\|_{X'} \leqq M\|g'\|_{X'}$ for all $g' \in X'$, $n \in \mathbf{N}$, holds for every LAP $\{T_n\}$ in view of (2.6).

5. Applications to convolution integrals

5.1. Results for the space $C_{2\pi}$. In this section we consider LAP's generated by convolution integrals. Let $C_{2\pi}$ denote the space of all continuous, 2π-periodic, complex-valued functions f defined on the real axis $\mathbf{R}$, endowed with the supremum norm $\|f\|_\infty$. A sequence of functions $\{\chi_n\}_{n=1}^\infty$ in $C_{2\pi}$ is called an approximate identity, if $\int_{-\infty}^{\infty} \chi_n(u)\,du = 2\pi$ for all $n \in \mathbf{N}$, and $\lim_{n\to\infty} \int_{|u|\geqq\delta} |\chi_n(u)|\,du = 0$ for each $\delta > 0$. The convolution integrals of $f \in C_{2\pi}$ with χ_n are defined by

$$(5.1)\qquad (V_nf)(x) \equiv (f*\chi_n)(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\chi_n(u)\,du \quad (n\in\mathbf{N};\ x\in\mathbf{R}).$$

The V_n are bounded linear operators from $C_{2\pi}$ into itself satisfying

$$(5.2)\qquad V_nV_mf = V_mV_nf,\quad \lim_{n\to\infty}\|V_nf-f\|_\infty = 0 \quad (f\in C_{2\pi};\ m,n\in\mathbf{N}).$$

Hence $\{V_n\}$ is a LAP on $C_{2\pi}$.

As subspaces Y and Z we take the Banach spaces $C_{2\pi}^k := \{f\in C_{2\pi};\, f^{(k)}\in C_{2\pi}\}$[4] for different values of $k\in\mathbf{P}$, endoved with the norm

$$\|f\|_{\infty,k} := |f^\wedge(0)| + \|f^{(k)}\|_\infty,\text{[5]}$$

$f^\wedge(j)$ being the jth Fourier coefficient of f, namely

$$f^\wedge(j) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)e^{-iju}\,du \quad (j\in\mathbf{Z} := \{0, \pm1, \pm2, \ldots\}).$$

Note that $V_nf\in C_{2\pi}^k$ for each $f\in C_{2\pi}$ provided $\chi_n\in C_{2\pi}^k$.

Concerning the function $\varphi\in\Phi$ we restrict ourselves to the case $\varphi(t)=t^\sigma$ for some $\sigma>0$. Conditions (3.5) and (3.6) then read $\alpha>\sigma$ and $\beta<\sigma$, respectively.

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[4]) Derivatives are denote by $f^{(1)}, f^{(2)}, \ldots$, whereas the prime in f', g' indicates that these are elements of a dual space.

[5]) One may also take any equivalent norm, e.g. $\|f\|_\infty + \|f^{(k)}\|_\infty$, in particular $\|f\|_\infty$ instead of $\|f\|_{\infty,0}$.

Theorem 4. *Let* $\{\chi_n\}_{n=1}^{\infty}$ *be an approximate identity on* $C_{2\pi}$ *such that the associated convolution integrals* $\{V_n\}_{n=1}^{\infty}$ *(cf.* (5.1)) *satisfy Jackson and Bernstein-type inequalities of order* $\alpha>0$ *on* $C_{2\pi}$ *with respect to* $C_{2\pi}^r$ *for some (fixed)* $r\in\mathbf{N}$. *The following assertions are equivalent for* $f\in C_{2\pi}$, $s\in\mathbf{P}$, $\sigma>0$ *with* $\sigma s/r<\sigma<\alpha$:

(i) $\|V_n f-f\|_\infty = O(n^{-\sigma})$ $(n\to\infty)$,

(ii) $K(t^\alpha, f;\ C_{2\pi}, C_{2\pi}^r) = O(t^\sigma)$ $(t\to 0+)$,

(iii) $f\in C_{2\pi}^s$ *and* $\|V_n f-f\|_{\infty,s} = O(n^{-\sigma+\alpha s/r})$ $(n\to\infty)$,

(iv) $\|V_n f\|_{\infty,r} = O(n^{\alpha-\sigma})$ $(n\to\infty)$.

Proof. The equivalence of (i) and (ii) follows immediately by Theorem 2. a). To prove that of (iii) as well as (iv) with (i) or (ii) we have to show that the assumptions of Theorem 2. b) are satisfied. It suffices to verify Jackson and Bernstein-type inequalities of order $\beta=\alpha s/r$ on $C_{2\pi}$ with respect to $Z=C_{2\pi}^s$. This is achieved by

Lemma 10. *Let* $\{V_n\}$ *be given as above,* $r\in\mathbf{N}$, $s\in\mathbf{P}$ *with* $s<r$. *If* $\{V_n\}$ *satisfies a Jackson-type inequality of order* $\alpha>0$ *on* $C_{2\pi}$ *with respect to* $C_{2\pi}^r$, *then there holds a Jackson-type inequality of order* $\alpha s/r$ *on* $C_{2\pi}$ *with respect to* $C_{2\pi}^s$. *The same holds for the Bernstein-type inequality.*

Proof. Consider the integral operator I^ν defined for $\nu\in\mathbf{P}$ on $C_{2\pi}$ via the Fourier series

$$(5.3)\qquad (I^\nu f)(x)\sim f^\wedge(0)+\sum_{j=-\infty}^{\infty}{}' (ij)^{-\nu} f^\wedge(j)e^{-ijx}.\text{[6]}$$

It is a linear operator from $C_{2\pi}^\mu$ into $C_{2\pi}^{\mu+\nu}$, $\mu,\nu\in\mathbf{P}$, having the properties

$$(5.4)\qquad V_n I^\nu f = I^\nu V_n f \quad (f\in C_{2\pi};\ n\in\mathbf{N};\ \nu\in\mathbf{P}),$$

$$(5.5)\qquad (I^\nu f)^{(\mu)} = f^{(\mu-\nu)} \quad (f\in C_{2\pi}^{\mu-\nu};\ \mu,\nu\in\mathbf{N}, \mu\geqq\nu).$$

Furthermore, we need Landau's inequality, namely that

$$(5.6)\qquad \|f^{(\nu)}\|_\infty^\mu \leqq M_{\mu,\nu}\|f\|_\infty^{\mu-\nu}\|f^{(\mu)}\|_\infty^\nu \quad (f\in C_{2\pi}^\mu)$$

is valid for $\mu,\nu\in\mathbf{N}$ with $\nu\leqq\mu$ (cf. [16, p. 138]).

Now one has for $f\in C_{2\pi}^s$

$$\|V_n f-f\|_\infty^r = \|(V_n I^{r-s}f-I^{r-s}f)^{(r-s)}\|_\infty^r \leqq$$
$$\leqq M\|V_n I^{r-s}f-I^{r-s}f\|_\infty^s\,\|(V_n I^{r-s}f-I^{r-s}f)^{(r)}\|_\infty^{r-s}.$$

Estimating the first factor by the Jackson-type inequality on $C_{2\pi}$ with respect to $C_{2\pi}^r$ and using the equality $(V_n I^{r-s}f)^{(r)}=V_n((I^{r-s}f)^{(r)})$ together with (2.6) for the

[6]) The prime at Σ' indicates that the term for $j=0$ is to be omitted.

second, yields

$$\|V_n f-f\|_\infty^r \leqq M\{n^{-\alpha}\{|f^\wedge(0)|+\|(I^{r-s}f)^{(r)}\|_\infty\}^s\|(I^{r-s}f)^{(r)}\|_\infty^{r-s}.$$

The desired inequality now follows by (5.5).

To prove the second part of the lemma we use (5.6), (2.6) and the Bernstein-type inequality with respect to $C_{2\pi}^r$ to deduce

$$\|(V_n f)^{(s)}\|_\infty^r \leqq M\|V_n f\|_\infty^{r-s}\|(V_n f)^{(r)}\|_\infty^s \leqq M\|f\|_\infty^{r-s}n^{\alpha s}\|f\|_\infty^s.$$

The rest is obvious.

Remark. In view of the equivalence of the K-functional with the modulus of continuity (cf. [4]) one can express assertion (ii) of Theorem 4 in terms of Lipschitz spaces. Moreover, using the concept of fractional order derivatives (cf. [11]) the assumptions $r\in\mathbf{N}$, $s\in\mathbf{P}$ can be weakend to $r>0$ and $s\geqq 0$. Note that all results remain valid when $C_{2\pi}$ is replaced by $L_{2\pi}^p$, $1\leqq p<\infty$.

As particular examples of approximate identities $\{\chi_n\}$ one may take, e.g., the kernels $\{j_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ of Jackson and de la Vallée Poussin, namely

$$j_n(x) := \frac{3}{n(2n^2+1)}\left[\frac{\sin(nx/2)}{\sin(x/2)}\right]^4 \quad (n\in\mathbf{N};\ x\in\mathbf{R}),$$

$$v_n(x) := \frac{(n!)^2}{(2n)!}(2\cos(x/2))^{2n} \quad (n\in\mathbf{N};\ x\in\mathbf{R}).$$

For $\{j_n\}$ there hold Jackson and Bernstein-type inequalities of order 2 with respect to $C_{2\pi}^2$ (see [5], [6, p. 100]). Hence Theorem 4 holds for $\{f*j_n\}_{n=1}^\infty$ with $\alpha=r=2$ and $s<\sigma<2$. In case of $\{v_n\}$ Jackson and Bernstein-type inequalities of order 1 with respect to $C_{2\pi}^2$ are available ([5], [6, p. 113]). So Theorem 4 is valid for $\{f*v_n\}_{n=1}^\infty$ with $\alpha=1$, $r=2$ and $s/2<\sigma<1$.

5.2. Dual results. If $\{\chi_n\}_{n=1}^\infty$ is an approximate identity on $C_{2\pi}$, then one has in addition to (5.2) that

$$\lim_{n\to\infty}\|(V_n f)^{(k)}-f^{(k)}\|_\infty = \lim_{n\to\infty}\|V_n f-f\|_{\infty,k}=0 \quad (f\in C_{2\pi}^k),$$

i.e., $\{V_n\}$ satisfies the assumptions of Theorem 3, too. Before applying this theorem to the $\{V_n\}$ we compute the duals of $C_{2\pi}^k$ and V_n.

To this end we regard $C_{2\pi}$ and $(C_{2\pi}^k)'$ as subspaces of $\mathcal{D}_{2\pi}'$, the space of all 2π-periodic distributions (cf. [20, Chap. 11]). On $\mathcal{D}_{2\pi}'$ we consider the operators $(k\in\mathbf{Z})$

$$(W^k f')(x) := \sum_{j=-\infty}^{\infty}{}' (ij)^k f'^\wedge(j)e^{ijx} \quad (f'\in\mathcal{D}_{2\pi}'),$$

$$W_0^k f' := W^k f' + f'^\wedge(0) \quad (f'\in\mathcal{D}_{2\pi}'),$$

where the convergence is to be understood in the topology of $\mathcal{D}'_{2\pi}$, and the distributional Fourier coefficients are given by $f'^{\wedge}(j):=(2\pi)^{-1}\langle f'(x), e^{-ijx}\rangle$. For $k\in\mathbf{N}$ the operator W^k coincides on $C^k_{2\pi}$ with the usual kth order derivative, and W_0^{-k} is the same as I^k on $C_{2\pi}$.

Now we extend the definition of $C^k_{2\pi}$, $k\in\mathbf{P}$, to arbitrary $k\in\mathbf{Z}$ by

$$C^k_{2\pi} := \{f'\in\mathcal{D}'_{2\pi};\ W^k(f')\in C_{2\pi}\},\quad \|f'\|_{\infty,k} := |f'^{\wedge}(0)| + \|W^k(f')\|_{\infty},$$

and furthermore set

$$(C'_{2\pi})^k := \{f'\in\mathcal{D}'_{2\pi};\ W^k f'\in C'_{2\pi}\},\quad \|f'\|_{(C'_{2\pi})^k} := |f'^{\wedge}(0)| + \|W^k f'\|_{C'_{2\pi}}.$$

The norm for $(C'_{2\pi})^0=C'_{2\pi}$ introduced here is equivalent to that which is usually used.

Since W_0^{-k} is a linear homeomorphism from $C_{2\pi}$ onto $C^k_{2\pi}$, it follows that the dual operator $(W_0^{-k})'$ defined by

$$\langle (W_0^{-k})'f', f\rangle := \langle f', W_0^{-k}f\rangle \quad (f'\in\mathcal{D}'_{2\pi};\ f\in C^{\infty}_{2\pi})$$

is a linear homeomorphism from $(C^k_{2\pi})'$ onto $C'_{2\pi}$. Hence one can rewrite

$$(C^k_{2\pi})' = \{f'\in\mathcal{D}'_{2\pi};\ W_0^{-k}f'\in C'_{2\pi}\}.$$

Comparing this with the definition of $(C'_{2\pi})^{-k}$, and noting that $(W_0^{-k})'f'= =f'^{\wedge}(0)+(-1)^k W^{-k}f'$, one has that $(C^k_{2\pi})'=(C'_{2\pi})^{-k}$ in the set theoretical sense. Concerning the norms, there holds by the properties of $(W_0^{-k})'$

$$\|f'\|_{(C^k_{2\pi})'} \leqq M\|(W_0^{-k})'f'\|_{C'_{2\pi}} \leqq M\{|f'^{\wedge}(0)| + \|W^{-k}f'\|_{C'_{2\pi}}\} = M\|f'\|_{(C'_{2\pi})^{-k}}.$$

Since the converse inequality follows by the same arguments, we have proved (cf. [17])

Lemma 11. *The spaces* $(C^k_{2\pi})'$ *and* $(C'_{2\pi})^{-k}$ *are equal with equivalent norms, in notation* $(C^k_{2\pi})'\cong(C'_{2\pi})^{-k}$.

Note that this result is also valid for arbitrary $k\in\mathbf{R}$, working in the fractional frame.

Next we compute the dual operators V'_n assuming for simplicity $\{\chi_n\}_{n=1}^{\infty}$ to be an even approximate identity, i.e., $\chi_n(u)=\chi_n(-u)$ for all $n\in\mathbf{N}$, $u\in\mathbf{R}$. First we extend the domain of V_n from $C_{2\pi}$ to $\mathcal{D}'_{2\pi}$ via

$$(V_n f')(x) = \sum_{j=-\infty}^{\infty} f'^{\wedge}(j)\chi_n^{\wedge}(j)e^{ijx} \quad (f'\in\mathcal{D}'_{2\pi};\ n\in\mathbf{N})$$

which obviously coincides for $f'\in C_{2\pi}$ with (5.1). For the dual V_n' one has in view of $\chi_n^\wedge(j)=\chi_n^\wedge(-j)$ that

$$\langle V_n'f',f\rangle = \langle f',V_nf\rangle = \sum_{j=-\infty}^{\infty} f'^\wedge(-j)\chi_n^\wedge(j)f^\wedge(j) = \langle V_nf',f\rangle \quad (f\in C_{2\pi}^\infty),$$

hence $V_n'=V_n$. (If one drops the assumption χ_n to be even, then one would have $V_n'=V_n^-$, where $(V_n^-f)(x):=(f(\cdot)*\chi_n(-\cdot))(x)$.) Observing these various facts, we can formulate as a consequence of Theorem 3

Theorem 5. *Let $\{\chi_n\}_{n=1}^\infty$ be an even approximate identity on $C_{2\pi}$ such that the associated convolution integrals $\{V_n\}_{n=1}^\infty$ satisfy Jackson and Bernstein-type inequalities of order $\alpha>0$ on $C_{2\pi}$ with respect to $C_{2\pi}^r$ for some $r\in\mathbf{N}$. The following assertions are equivalent for $f'\in(C_{2\pi}^r)'\cong(C_{2\pi}')^{-r}$, $s\in\mathbf{P}$, $\sigma>0$ with $0<\alpha-\alpha s/r<\alpha$:*

(i) $\|V_nf'-f'\|_{(C_{2\pi}')^{-r}} = O(n^{-\sigma}) \quad (n\to\infty)$,

(ii) $K(t^\alpha,f';\ (C_{2\pi}')^{-r},C_{2\pi}') = O(t^\sigma) \quad (t\to 0+)$,

(iii) $f'\in(C_{2\pi}^s)'\cong(C_{2\pi}')^{-s}$ *and* $\|V_nf'-f'\|_{(C_{2\pi}')^{-s}} = O(n^{-\sigma+\alpha-\alpha s/r}) \quad (n\to\infty)$,

(iv) $\|V_nf'\|_{C_{2\pi}'} = O(n^{-\sigma+\alpha}) \quad (n\to\infty)$.

If one sets $g':=I^rf'$, then $g'\in C_{2\pi}'$, and each of the following is equivalent to the corresponding assertion for f':

(i)′ $\|V_ng'-g'\|_{C_{2\pi}'} = O(n^{-\sigma}) \quad (n\to\infty)$,

(ii)′ $K(t^\alpha;\ g';\ C_{2\pi}',(C_{2\pi}')^r) = O(t^\sigma) \quad (t\to 0+)$,

(iii)′ $g'\in(C^{s-r})'\cong(C_{2\pi}')^{r-s}$ and

$\|V_n(W^{r-s}g')-W^{r-s}g'\|_{C_{2\pi}'} = O(n^{-\sigma-\alpha-\alpha s/r}) \quad (n\to\infty)$,

(iv)′ $\|V_n(W^rg')\|_{C_{2\pi}'} = O(n^{-\sigma+\alpha}) \quad (n\to\infty)$.

Using the fact that $C_{2\pi}'$ is isometrically isomorphic to $BV[-\pi,\pi]$, the space of all complex-valued functions of bounded variation on $[-\pi,\pi]$, one can rewrite statements (i)′—(iv)′ in terms of functions $\mu\in BV[-\pi,\pi]$. Note that if $g'\in C_{2\pi}'$ corresponds to $\mu\in BV[-\pi,\pi]$, then V_ng' corresponds to $(\tilde\chi_n*d\mu)(x):=$ $:=(2\pi)^{-1}\int_{-\pi}^{\pi}\tilde\chi_n(x-u)\,d\mu(u)$, where $\tilde\chi_n(u):=\int_{-\pi}^{u}\chi_n(u)\,du$. We omit the details.

It would be of interest to find an approximation process that satisfies the assumptions of Theorem 3 but differs from its dual operator.

6. Appendix

The aim of this section is to express conditions (3.5) and (3.6) upon $\varphi\in\Phi$ by equivalent ones, which can be verified more easily. Although these results are implicitly contained in [2] we present the proofs for completeness.

Lemma 12. *Let* $\varphi\in\Phi$, $\alpha,\beta\geqq 0$. a) *If*

$$t^{-\alpha}\varphi(t)\leqq Ms^{-\alpha}\varphi(s)\quad(0<s\leqq t\leqq 1), \tag{6.1}$$

then assertion (3.5) *is equivalent to*

$$\varlimsup_{t\to 0+}\frac{\varphi(Ct)}{\varphi(t)}<C^{\alpha} \tag{6.2}$$

for some $C>1$.

b) *Under the assumption*

$$s^{-\beta}\varphi(s)\leqq Mt^{-\beta}\varphi(t)\quad(0<s\leqq t\leqq 1) \tag{6.3}$$

assertion (3.6) *is equivalent, for some* $C>1$, *to*

$$\varliminf_{t\to 0+}\frac{\varphi(Ct)}{\varphi(t)}>C^{\beta}. \tag{6.4}$$

Proof. a) It follows from (3.5) that there exists a constant $M>0$ such that

$$\sum_{2^{-r}t^{-1}\leqq 2^{j}\leqq t^{-1}}2^{\alpha j}\varphi(2^{-j})\leqq Mt^{-\alpha}\varphi(t)\quad(r\in\mathbf{N};\ 0\leqq t\leqq 2^{-r}).$$

Since the sum consists of at least r terms, each of which is $\geqq M2^{-\alpha r}t^{-\alpha}\varphi(2^{r}t)$ by (6.1) one obtains

$$r2^{-\alpha r}t^{-\alpha}\varphi(2^{r}t)\leqq Mt^{-\alpha}\varphi(t)\quad(r\in\mathbf{N};\ 0\leqq t\leqq 2^{-r}). \tag{6.5}$$

Choosing now $r\in\mathbf{N}$ greater than the constant M in (6.5) yields (6.2) with $C=2^{r}$.

Conversely, let $k\in\mathbf{N}$ be such that $2^{k-1}<C\leqq 2^{k}$. Then, in view of (6.1) and (6.2), one can find q, $t_0\in(0,1)$ such that

$$\frac{\varphi(t)}{2^{k\alpha}\varphi(t2^{-k})}\leqq q\quad\big(t\in(0,t_0)\big). \tag{6.6}$$

Now one has for $m_0, m\in\mathbf{N}$ satisfying $2^{k(m_0-1)}\leqq t_0^{-1}<2^{km_0}$, $2^{k(m-1)}\leqq t^{-1}<2^{km}$ that

$$\sum_{1\leqq 2^{j}\leqq t^{-1}}2^{\alpha j}\varphi(2^{-j})\leqq\Big(\sum_{j=0}^{km_0-1}+\sum_{j=km_0}^{km}\Big)2^{\alpha j}\varphi(2^{-j})\leqq$$

$$\leqq M+\sum_{\nu=m_0}^{m}\sum_{j=k\nu}^{k(\nu+1)-1}2^{\alpha j}\varphi(2^{-j})\leqq M+\sum_{\nu=m_0}^{m}\varphi(2^{-k\nu})\sum_{j=k\nu}^{k(\nu+1)-1}2^{\alpha j}=$$

$$=M+\sum_{\nu=m_0}^{m}\varphi(2^{-k\nu})2^{\alpha k\nu}\frac{2^{\alpha k}-1}{2^{\alpha}-1}\leqq M\{1+\sum_{\nu=m_0}^{m}2^{\alpha k\nu}\varphi(2^{-k\nu})\}\quad\big(t\in(0,t_0)\big).$$

Since $2^{-k\nu}<t_0$ for $\nu\geqq m_0$, one can estimate the latter sum by (6.6) to deduce

$$\sum_{\nu=m_0}^{m} 2^{\alpha k\nu}\varphi(2^{-k\nu}) \leqq 2^{\alpha km}\varphi(2^{-km}) \sum_{\nu=m_0}^{m} \frac{\varphi(2^{-k\nu})}{2^{(m-\nu)}\varphi(2^{-(m-\nu)k})} \leqq$$

$$\leqq 2^{\alpha km} \sum_{\nu=m_0}^{m} q^{m-\nu} \leqq M2^{\alpha km}.$$

This yields, in view of (6.1),

$$\sum_{1\leqq 2^j\leqq t^{-1}} 2^{\alpha j}\varphi(2^{-j}) \leqq M\{1+2^{\alpha km}\varphi(2^{-km})\} \leqq Mt^{-\alpha}\varphi(t) \quad (t\in(0, t_0))$$

which is assertion (3.5).

Concerning part b) set $\psi(t):=t^{-\beta}\varphi(t)$. Then, replacing t by $2^r t$ in (3.6), one has

$$\sum_{j=0}^{\infty} \psi(t2^{r-j}) \leqq M\psi(t2^r) \quad (r\in\mathrm{N};\ 0<t\leqq 2^{-r}).$$

By (6.3) it follows that $\psi(t2^{r-j})\leqq M\psi(t)$ for $j=0, 1, \ldots, r$, and so

$$(r+1)\psi(t) \leqq M\psi(t2^r) \quad (r\in\mathrm{N};\ 0<t\leqq 2^{-r}).$$

This in turn implies (6.4) by choosing $r\geqq M$ and $C=2^r$.

Conversely, if (6.4) holds and $k\in\mathrm{N}$ satisfies $2^{k-1}<C\leqq 2^k$, then

$$\frac{\psi(t2^{-k})}{\psi(t)} \leqq q < 1 \quad (t\in(0, t_0)) \tag{6.7}$$

for suitable $q, t_0\in(0, 1)$. Then one obtains by (6.3) and (6.7) that

$$\sum_{j=0}^{\infty} \psi(t2^{-j}) = \sum_{\nu=0}^{\infty} \sum_{j=k\nu}^{k(\nu+1)-1} \psi(t2^{-j}) \leqq M \sum_{\nu=0}^{\infty} k\psi(t2^{-k\nu}) \leqq$$

$$\leqq M\psi(t) \sum_{\nu=0}^{\infty} \frac{\psi(t2^{-k\nu})}{\psi(t)} \leqq M\psi(t) \sum_{\nu=0}^{\infty} q^\nu \quad (0<t\leqq t_0).$$

This gives (3.6), and so our proof is complete.

It should be mentioned that condition (6.1) is superfluous. Indeed, it can be shown that it follows from (3.5) as well as from (6.2). The proofs would then become more intricate. In this respect see [2], where also some further equivalent assertions to those of part a) and b) can be found.

Using Lemma 12 one can easily show that $\varphi(t):=(\log 1/t)^{-\gamma}, \gamma>0$, satisfies (3.5) for each $\alpha>0$, but that (3.6) is violated for each $\beta\geqq 0$. On the other hand, functions which behave like $e^{-1/t}$ are not admissible in our theorems since $\lim_{t\to 0+} t^{-\alpha}e^{-1/t}=0$ for each $\alpha\geqq 0$ which implies by Lemma 3 that (3.5) cannot hold. So one can handle the case where $T_n f$ converges very slowly towards f, but not the case where it converges very rapidly.

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