Acta Sci. Math., 47 (1984), 113—115

On minimal invariant manifolds and density of operator algebras

HEYDAR RADJAVI

Let $\mathfrak{X}$ be a complex Banach space and let $\mathfrak{A}$ be a subalgebra of $\mathscr{B}(\mathfrak{X})$, the algebra of all bounded linear operators on $\mathfrak{X}$. When is $\mathfrak{A}$ strongly dense in $\mathscr{B}(\mathfrak{X})$, i.e., dense in the topology of pointwise convergence? This question can sometimes be answered by examining various lattices associated with $\mathfrak{A}$. The first result of this sort was obtained by RICKART and YOOD [2, p. 62], a consequence of which is: if the only linear manifolds (i.e., not necessarily closed subspaces) of $\mathfrak{X}$ invariant under all members of $\mathfrak{A}$ are $\{0\}$ and $\mathfrak{X}$, then $\mathfrak{A}$ is strongly dense in $\mathscr{B}(\mathfrak{X})$. If we denote by $\mathscr{L}(\mathfrak{A})$ the lattice of all linear manifolds invariant under $\mathfrak{A}$, the hypothesis in this assertion amounts to saying that $\mathscr{L}(\mathfrak{A})$ is trivial, that is, the only nonzero element of $\mathscr{L}(\mathfrak{A})$ is $\mathfrak{X}$. We shall prove the following result.

Theorem 1. *Let $\mathfrak{A}$ be an algebra of bounded linear operators on the Banach space $\mathfrak{X}$. If the nonzero elements of $\mathscr{L}(\mathfrak{A})$ have a dense intersection, then $\mathfrak{A}$ is strongly dense in $\mathscr{B}(\mathfrak{X})$.*

Note that the hypothesis of the theorem implies that $\mathscr{L}(\mathfrak{A})$ has no *closed* members other than $\{0\}$ and $\mathfrak{X}$, i.e., $\mathfrak{A}$ is topologically transitive. It is not known whether topological transitivity for $\mathfrak{A}$ is sufficient for strong density if $\mathfrak{X}$ is a reflexive Banach space. (This is the Transitive Algebra Problem; see [3]).

Many examples of algebras satisfying the hypothesis of Theorem 1 exist. Here is a simple, nontrivial example. Fix an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for a separable, infinite-dimensional Hilbert space $\mathfrak{H}$. Let $\mathfrak{A}$ be the set of all those operators on $\mathfrak{H}$ whose matrices relative to $\{e_i\}_{i=1}^{\infty}$ are "column-finite", i.e., each of their columns have finitely many nonzero entries. It is easy to see that $\mathfrak{A}$ is in fact an algebra, that every operator in $\mathfrak{A}$ leaves the linear span $\mathscr{V}$ of $\{e_i\}_{i=1}^{\infty}$ invariant, and that, furthermore, $\mathscr{V}$ is contained in every invariant linear manifold of $\mathfrak{A}$. Another example is the subalgebra $\mathfrak{A}_0$ of the above $\mathfrak{A}$ consisting of finite-rank operators; $\mathscr{V}$ is still the intersection of all nonzero members of $\mathscr{L}(\mathfrak{A}_0)$.

Received July 19, 1982.

To prove Theorem 1, we shall need a lemma, which seems to be of independent interest, and which is itself an extension of the Rickart—Yood result referred to above. We first recall some definitions: An algebra $\mathfrak{A}$ of linear operators on any vectoı space $\mathscr{V}$ is called *strictly transitive* if for $x\neq 0$ and y in $\mathscr{V}$ there exists A in $\mathfrak{A}$ with $Ax=y$. More generally, $\mathfrak{A}$ is *strictly n-fold transitive* if for any independent vectors $x_1, \dots, x_n$ in $\mathscr{V}$ and arbitrary $y_1, \dots, y_n$ in $\mathscr{V}$ there exists A in $\mathfrak{A}$ with $Ax_i=y_i$ for $i=1, \dots, n$. If $\mathfrak{A}$ is strictly n-fold transitive for every n, then it is called *strictly dense*. The well-known theorem of JACOBSON [1] (see [2, p. 50]) states that 2-fold transitivity implies strict density. In general, 1-fold transitivity does not imply strict density.

Lemma 1. *Let $\mathfrak{A}$ be an algebra of bounded linear operators on the (not necessarily complete) complex normed linear space $\mathfrak{X}$. If $\mathfrak{A}$ is strictly transitive, then it is strictly dense.*

(Note that $\mathfrak{A}$ is not assumed to be closed in any topology.)

Proof. Suppose $\mathfrak{A}$ is strictly transitive but not strictly dense. Then by [1], $\mathfrak{A}$ is not 2-fold transitive. It follows (as in [2, p. 62]), that there exists a (not necessarily bounded) non scalar linear transformation T of $\mathfrak{X}$ onto $\mathfrak{X}$ that commutes with every A in $\mathfrak{A}$. Now

$$(T-\alpha I)A = A(T-\alpha I)$$

for every scalar α, and thus the nullspace and range of $T-\alpha I$ are invariant linear manifolds for $\mathfrak{A}$. It follows from the transitivity hypothesis that $T-\alpha I$ is bijective for every α. Thus $r(T)$ is a bijective linear transformation for every rational function r, and $r(T)A=Ar(T)$ for all A in $\mathfrak{A}$.

Fix a nonzero x_0 in $\mathfrak{X}$ and let $\mathfrak{X}_0$ be the linear manifold $\{r(T)x_0\colon r \text{ a rational function}\}$. Let $\mathfrak{A}_0=\{A\in\mathfrak{A}\colon Ax_0\in\mathfrak{X}_0\}$. Observe that $\mathfrak{X}_0$ is invariant under $\mathfrak{A}_0$. For each A in $\mathfrak{A}_0$ there is a rational function r_A such that $Ax_0=r_A(T)x_0$ (r_A is unique because of the bijectivity of $r(T)$ for nonzero r); thus it follows from

$$Ar(T)x_0 = r(T)Ax_0 = r(T)r_A(T)x_0 = r_A(T)r(T)x_0$$

that the restriction of A to $\mathfrak{X}_0$ is just that of $r_A(T)$. Conversely, by the transitivity hypothesis, every $r(T)$ coincides with $r_A(T)$ for some A in $\mathfrak{A}_0$. Hence the restriction of $\mathfrak{A}_0$ to $\mathfrak{X}_0$ is a field. Since this restriction consists of bounded operators on $\mathfrak{X}_0$, the Gelfand—Mazur theorem implies that T is a scalar on $\mathfrak{X}_0$: $T\,|\,\mathfrak{X}_0=\alpha I$, which contradicts the bijectivity of $T-\alpha I$ on $\mathfrak{X}$. This proves that $\mathfrak{A}$ is strictly dense.

Proof of Theorem 1. Let $\mathscr{V}$ be the intersection of all nonzero invariant linear manifolds of $\mathfrak{A}$. Then the restriction of $\mathfrak{A}$ to $\mathscr{V}$ is clearly strictly transitive and thus strictly dense. Also, $\mathfrak{A}$ has no closed invariant subspaces, because $\mathscr{V}$ is dense in

$\mathfrak{X}$. We use the notation and techniques described in [3, Chapter 8]. As in the proofs of Arveson's Lemma (Lemma 8.8) and Lemma 8.11 of [3], it suffices to show that each graph transformation for $\mathfrak{A}$ has an eigenvalue; this will imply the strong density of $\mathfrak{A}$ in $\mathscr{B}(\mathfrak{X})$.

Let $\{x\oplus T_1x\oplus\ldots\oplus T_nx\colon x\in\mathfrak{D}\}$ be an invariant graph subspace for $\mathfrak{A}^{(n+1)}$ for some positive integer n. We must show that each linear transformation T_i has an eigenvalue.

If T_i has nonzero null space, we are done. Otherwise, observe that the $\mathfrak{A}$-invariant linear manifolds $\mathfrak{D}$ and $T_i\mathfrak{D}$ both contain $\mathscr{V}$, by hypothesis. Hence the identities

$$AT_i = T_iA \quad \text{on } \mathfrak{D} \text{ and } \quad AT_i^{-1} = T_i^{-1}A \quad \text{on} \quad T_i\mathfrak{D}$$

(for all A in $\mathfrak{A}$) hold on $\mathscr{V}$. This implies that $T_i\mathscr{V}$ and $T_i^{-1}\mathscr{V}$ are also invariant under $\mathfrak{A}$, and thus contain $\mathscr{V}$. This yields $T_i\mathscr{V}=\mathscr{V}$. Now $\mathfrak{A}$ is strictly dense on $\mathscr{V}$ and commutes with the linear transformation T_i on $\mathscr{V}$. We conclude that T_i is a scalar on $\mathscr{V}$ and complete the proof.

We conclude with a question an affirmative answer to which would be a generalization of Theorem 1.

Question. If $\mathfrak{A}$ is a topologically transitive algebra of bounded linear operators on $\mathfrak{X}$ and if $\mathscr{L}(\mathfrak{A})$ has minimal nonzero elements, is $\mathfrak{A}$ necessarily strongly closed in $\mathscr{B}(\mathfrak{X})$?

References

[1] N. Jacobson, Structure theory of simple rings without finiteness assumptions, *Trans. Amer, Math. Soc.*, 57 (1944), 228—245.

[2] C. E. Rickart, *General Theory of Banach Algebras*, D. Van Nostrand (Princeton, 1960).

[3] H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Springer-Verlag (Berlin—Heidelberg—New York, 1973).

DALHOUSIE UNIVERSITY
HALIFAX, N.S., CANADA