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Moment theorems for operators on Hilbert space. II

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Introduction. The present note is a direct continuation of our previous investigation [1] about the momentlike problems of the existence of a contraction or a subnormal operator T on Hilbert space H such that $x_n=T^n x_0$ $(n=1, 2, \ldots)$ for some given sequence $\{x_n\}_{n=0}^{\infty}$ in H. The corresponding continuous problem was to find a continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions on H such that $T_0=I_H$ and $x_t=T_t x_0$ $(t\geqq 0)$ with some given continuous family $\{x_t\}_{t\geqq 0}$ in H. Our present object is to generalize these problems as follows.

Problems. Given a sequence $\{A_n\}_{n=0}^{\infty}$ of bounded linear operators on H it is natural to ask: under what condition does there exist an operator T on H with

$$A_n = T^n A_0 \quad (n = 1, 2, \ldots). \tag{1}$$

For a continuous family $\{A_t\}_{t\geqq 0}$ of operators on H a continuous semigroup $\{T_t\}_{t\geqq 0}$ of bounded linear operators with $T_0=I_H$ and

$$A_t = T_t A_0 \quad (t \geqq 0) \tag{2}$$

may be sought.

We shall treat only the following cases:

(A) (1) holds with a contraction T.

(B) (2) holds with a continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions such that $T_0=I_H$.

(C) (1) holds with a subnormal operator T.

Results.

Theorem A. *Problem* (A) *has a solution if and only if*

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(3) $$\|\sum_{n',n} A_{n'+n}h_{n',n}\|^2 \leqq \sum_{\substack{m\geqq n\\ m',n'}} (A_{m'-n+m}h_{m',m}, A_{n'}h_{n',n}) + \sum_{\substack{n<m\\ m',n'}} (A_{m'}h_{m',m}, A_{n'-m+n}h_{n',n})$$

holds for any finite sequence $\{h_{n',n}\}_{n'\geqq 0, n\geqq 0}$ *in* H.

Theorem B. *Problem* (B) *has a solution if and only if*

(4) $$\|\sum_{t',t} A_{t'+t}h_{t',t}\|^2 \leqq \sum_{\substack{t\leqq s\\ s',t'}} (A_{s'-t+s}h_{s's}, A_{t'}h_{t',t}) + \sum_{\substack{s<t\\ s',t'}} (A_{s'}h_{s',s}, A_{t'-s+t}h_{t',t})$$

holds for any finite sequence $\{h_{t',t}\}_{t'\geqq 0, t\geqq 0}$ *in* H.

Theorem C. *Let* $\{A_n\}_{n=0}^{\infty}$ *be a sequence of operators on the Hilbert space* H *such that*

(i) $\{\text{Range } A_n\}_{n=0}^{\infty}$ *spans the space* H.

(ii) $\|A_n\| \leqq \varkappa^n$ *holds for some constant* $\varkappa \geqq 0$ *and* $n=1, 2, \ldots$.

Under these assumptions Problem (C) *has a solution if and only if there exists a double sequence* $\{A_{n',n}\}_{n'=0, n=0}^{\infty}$ *of operators on* H *such that*

(iii) $$A_{0,n} = A_n \quad \textit{for} \quad n = 0, 1, 2, \ldots,$$

(iv) $$A_m^* A_{n',n} = A_{n'+m}^* A_n \quad \textit{for} \quad m, n', n = 0, 1, 2, \ldots, \quad \textit{and}$$

(v) $$\|\sum_{n',n} A_{n',n} h_{n',n}\|^2 \leqq \sum_{\substack{m',m\\ n',n}} (A_{n'+m}h_{m',m}, A_{m'+n}h_{n',n})$$

hold for any finite (double) sequence $\{h_{n',n}\}_{n'=0, n=0}$ *in* H.

Necessity. (A) Let U be a unitary dilation (see [2]) of T on some Hilbert space K containing H. Then

(5) $$PU^n h = T^n h \quad (h\in H;\ n = 0, 1, 2, \ldots),$$

where P is the orthogonal projection of K onto H. Let further $\{h_{n',n}\}_{n'=0, n=0}$ be any finite (double) sequence in H. Then by (1) and (5)

$$\|\sum_{n',n} A_{n'+n}h_{n',n}\|^2 = \|\sum_{n',n} T^n A_{n'}h_{n',n}\|^2 \leqq \|\sum_{n',n} U^n A_{n'}h_{n',n}\|^2 =$$

$$= \sum_{\substack{n\leqq m\\ m',n'}} (U^{m-n}A_{m'}h_{m',m}, A_{n'}h_{n',n}) + \sum_{\substack{m<n\\ m',n'}} (A_{m'}h_{m',m}, U^{n-m}A_{n'}h_{n',n}) =$$

$$= \sum_{\substack{n\leqq m\\ m',n'}} (T^{m-n}A_{m'}h_{m',m}, A_{n'}h_{n',n}) + \sum_{\substack{m<n\\ m',n'}} (A_{m'}h_{m',m}, T^{n-m}A_{n'}h_{n',n}) =$$

$$= \sum_{\substack{n\leqq m\\ m',n'}} (A_{m'-n+m}h_{m',m}, A_{n'}h_{n',n}) + \sum_{\substack{m<n\\ m',n'}} (A_{m'}h_{m',m}, A_{n'-m+n}h_{n',n}).$$

(B) Let U_t be a unitary dilation (see [2]) of the continuous semigroup $\{T_t\}_{t\geqq 0}$ of contractions on some Hilbert space K containing H. Then

$$PU_t h = T_t h \quad (h\in H;\ t\geqq 0),$$

where P is the orthogonal projection of K onto H. For any finite (double) sequence $\{h_{t',t}\}_{t'\geqq 0, t\geqq 0}$ in H (4) can be verified in the same manner as (3) was before.

(C) Let N be a normal extension of T on a Hilbert space K containing H. Then

$$PN^{*n'}N^n h = T^{*n'}T^n h \quad (h\in H;\ n', n = 0, 1, 2, \ldots), \tag{6}$$

where P denotes the orthogonal projection of K onto H. Let further

$$A_{n',n} = T^{*n'}T^n A_0 \quad (n', n = 0, 1, 2, \ldots).$$

Then by (1) and (6) we have $A_{0,n}=T^{*0}T^nA_0=T^nA_0=A_n$ for $n=0, 1, 2, \ldots$. Furthermore, for any h, k in H

$$(A_m^* A_{n',n}h, k) = (T^{*n'}T^n A_0 h, A_m k) =$$
$$= (T^{*(n'+m)}T^n A_0 h, A_0 k) = (N^{*(n'+m)}N^n A_0 h, A_0 k) =$$
$$= (N^n A_0 h, N^{n'+m} A_0 k) = (T^n A_0 h, T^{n'+m} A_0 k) =$$
$$= (A_n h, A_{n'+m} k) = (A_{n'+m}^* A_n h, k).$$

Finally, for any finite (double) sequence $\{h_{n',n}\}_{n'=0,n=0}$ in H we have

$$\|\sum_{n',n} A_{n',n}h_{n',n}\|^2 = \|P\sum_{n',n} N^{*n'}N^nA_0h_{n',n}\|^2 \leqq \|\sum_{n',n} N^{*n'}N^nA_0h_{n',n}\|^2 =$$
$$= \sum_{\substack{m',m\\ n',n}} (N^{n'+m}A_0h_{m',m}, N^{m'+n}A_0h_{n',n}) = \sum_{\substack{m',m\\ n',n}} (T^{n'+m}A_0h_{m',m}, T^{m'+n}A_0h_{n',n}) =$$
$$= \sum_{\substack{m',m\\ n',n}} (A_{n'+m}h_{m',m}, A_{m'+n}h_{n',n}).$$

These prove the properties (iii), (iv), (v) as required.

Sufficiency. (A) Let F_0 be the linear space of all double sequences $\{h_{n',n}\}$ $(n', n= =0, 1, 2, \ldots)$ in the Hilbert space H. In view of (3) one can define a semi-definite inner product $\langle\cdot, \cdot\rangle$ (for $\{h_{m',m}\}$, $\{k_{n',n}\}$ in F_0) by

$$\langle\{h_{m',m}\}, \{k_{n',n}\}\rangle := \sum_{\substack{n\leqq m\\ m',n'}} (A_{m'-n+m}h_{m',m}, A_{n'}k_{n',n}) + \sum_{\substack{m<n\\ m',n'}} (A_{m'}h_{m',m}, A_{n'-m+n}k_{n',n}).$$

Hence we obtain a Hilbert space F by factoring F_0 with respect to the null space of $\langle\cdot, \cdot\rangle$ and completing this factor space with respect to the norm arising from the new definite inner product (denoted also by $\langle\cdot, \cdot\rangle$). For simplicity the residue class of $\{h_{m',m}\}\in F_0$ in F is denoted by the same symbol. We have two natural operations on

F_0: for $\{h_{n',n}\}$ in F_0 we set

$$U_0\{h_{n',n}\} = \{h'_{n',n}\} \quad \text{where} \quad h'_{n',n} = h_{n',n-1} \quad \text{for} \quad n \geqq 1 \quad \text{and} \quad h'_{n',0} = 0;$$

$$V_0\{h_{n',n}\} = \sum_{n',n} A_{n'+n} h_{n',n}.$$

U_0 and V_0 induce an isometry U of F into F and a contraction V of F into H, as an easy calculation shows. We are going to show that $T = VUV^*$ is the desired contraction on H.

Indeed, for any $\{h_{m',m}\}$ in F_0 and h in H

$$\langle\{h_{m',m}\}, V^*A_j h\rangle = (V\{h_{m',m}\}, A_j h) = \sum_{m',m} (A_{m'+m} h_{m',m}, A_j h) =$$
$$= \langle\{h_{m',m}\}, \{k^{(j)}_{n',n}\}\rangle \quad (j = 0, 1, 2, \ldots),$$

where $k^{(j)}_{n',n} = h$ if $n=0$, $n'=j$, and 0 otherwise. Hence $V^*A_j h = \{k^{(j)}_{n',n}\}$, and thus $UV^*A_j h = \{k^{(j)'}_{n',n}\}$, where $k^{(j)'}_{n',n} = h$ if $n=1, n'=j$, and 0 otherwise. It follows that for h in H, $j=0, 1, 2, \ldots$,

$$TA_j h = VUV^*A_j h = \sum_{n',n} A_{n'+n} k^{(j)'}_{n',n} = A_{j+1} h.$$

This is actually identical with (1), and the proof is complete.

(B) The proof is completely analogous to the previous one. The continuity of the family of contractions is a direct consequence of the continuity of the original operator family and the construction given in the proof.

(C) Let $\{A_{n',n}\}$ $(n', n=0, 1, \ldots)$ be a double sequence of operators satisfying (i)—(v) of Theorem C on the Hilbert space H. Let K_0 be the linear space of all finite double sequences $\{h_{n',n}\}$ $(n', n=0, 1, \ldots)$ in H with semi-definite inner product of $\{h_{m',m}\}$ and $\{k_{n',n}\}$ in K_0 given by

$$\langle\{h_{m',m}\}, \{k_{n',n}\}\rangle := \sum_{\substack{m',m \\ n',n}} (A_{n'+m} h_{m',m}, A_{m'+n} k_{n',n}). \tag{7}$$

As in the proof of Theorem A we get a Hilbert space K in which the elements of K_0 may be considered to form a dense linear manifold. We also have two operations on K_0, namely for $\{h_{n',n}\}$ in K_0 we set

$$N_0\{h_{n',n}\} = \{h^{0;1}_{n',n}\} \quad \text{where} \quad h^{0;1}_{n',n} = h_{n',n-1} \quad \text{for} \quad n \geqq 1 \quad \text{and} \quad h^{0;1}_{n',0} = 0.$$

$$V_0\{h_{n',n}\} = \sum_{n',n} A_{n',n} h_{n',n}.$$

In view of (v), V_0 induces a contraction V of K into H.

By (7) for any $\{h_{m',m}\}$ in K_0 and h in H

$$\langle\{h_{m',m}\}, V^*A_j h\rangle = (V\{h_{m',m}\}, A_j h) =$$
$$= \sum_{m',m} (A_{m',m} h_{m',m}, A_j h) = \langle\{h_{m',m}\}, \{k^{(j)}_{n',n}\}\rangle \quad (j = 0, 1, 2, \ldots),$$

that is,

(8) $V^*A_jh=\{k_{n',n}(j)\}$ where $k_{n',n}(j)=h$ if $n=j, n'=0$, and 0 otherwise.

It follows that $VV^*A_jh=A_{0,j}h=A_jh$, and hence by (i)

(9) $VV^*=I_H$ (=the identity on H).

That N_0 induces a normal operator on H needs some further argument.

First of all we show that $N_0^*\{k_{n',n}\}=\{k_{n',n}^{1,0}\}$ where $k_{n',n}^{1,0}=k_{n'-1,n}$ for $n'\geqq 1$ and $k_{0,n}^{1,0}=0$. This follows from (7) since for any $\{h_{m',m}\}$ in K_0 we have

$$\langle\{h_{m',m}\}, N_0^*\{k_{n',n}\}\rangle = \langle N_0\{h_{m',m}\}, \{k_{n',n}\}\rangle =$$
$$= \sum_{\substack{m',m\\ n',n}} (A_{n'+m+1}h_{m',m}, A_{m'+n}k_{n',n}) = \langle\{h_{m',m}\}, \{k_{n',n}^{1,0}\}\rangle.$$

Let $h_{n',n}^{j',j}=h_{n'-j',n-j}$ for $n'\geqq j'$, $n\geqq j$, and 0 otherwise. Then

$$(N_0^*N_0)^{2^j}\{h_{n',n}\} = \{h_{n',n}^{2^j,2^j}\} \quad \text{for any} \quad \{h_{n',n}\} \quad \text{in} \quad K_0.$$

Hence

$$\|N_0\{h_{n',n}\}\|^2 = \langle N_0^*N_0\{h_{n',n}\}, \{h_{n',n}\}\rangle \leqq \|N_0^*N_0\{h_{n',n}\}\|\cdot\|\{h_{n',n}\}\| \leqq$$
$$\leqq \|(N_0^*N_0)^2\{h_{n',n}\}\|^{1/2}\cdot\|\{h_{n',n}\}\|^{3/2}$$

and by induction, for any $j=0, 1, 2, \ldots,$

$$\|N_0\{h_{n',n}\}\|^{2^{j+2}} \leqq \|(N_0^*N_0)^{2^j}\{h_{n',n}\}\|^2\cdot\|\{h_{n',n}\}\|^{2^{j+2}-2} =$$
$$= \|\{h_{n',n}^{2^j,2^j}\}\|^2\cdot\|\{h_{n',n}\}\|^{2^{j+2}-2} =$$
$$= \|\{h_{n',n}\}\|^{2^{j+2}-2}\sum_{\substack{m',m\\ n',n}} (A_{n'+m+2^j}h_{m',m}, A_{m'+n+2^j}h_{n',n}) \leqq$$
$$\leqq \|\{h_{n',n}\}\|^{2^{j+2}-2}\sum_{\substack{m',m\\ n',n}} \|A_{n'+m+2^j}\|\cdot\|A_{m'+n+2^j}\|\cdot\|h_{m',m}\|\cdot\|h_{n',n}\| \leqq$$
$$\leqq \|\{h_{n',n}\}\|^{2^{j+2}-2}\varkappa^{2^{j+1}}\Big(\sum_{m',m}\varkappa^{m'+m}\|h_{m',m}\|\Big)^2.$$

Letting $j\to\infty$ we get $\|N_0\{h_{n',n}\}\|\leqq\sqrt{\varkappa}\,\|\{h_{n',n}\}\|$ for any $\{h_{n',n}\}$ in K_0. Thus N_0 can be extended by continuity to a normal operator N on K.

We shall show that $T=VNV^*$ is the desired subnormal operator on H. In view of (9), we may regard N as a normal extension of T, only H must be identified (by the aid of V^*) with a subspace of K. Finally (8) implies

$$TA_jh = VNV^*A_jh = VN\{k_{n',n}(j)\} = V\{k_{n',n}^{0,1}(j)\} = A_{j+1}h$$

for any h in H and $j=0, 1, 2, \ldots,$ which amounts to (1).

The proof is complete.

We remark that

$$T^*A_jh = VN^*V^*A_jh = VN^*\{k_{n',n}(j)\} = V\{k^{1,0}_{n',n}\} = A_{1,j}h$$

for any h in H and $j=0, 1, 2, \ldots$.

The method of proof of Theorem C yields also the following.

Proposition. *Let* $\{A_{n',n}\}$ $(n', n=0, 1, 2, \ldots)$ *be a double sequence of operators on the Hilbert space* H *such that* $\{\text{Range } A_{0,n}\}_{n=0}^{\infty}$ *spans* H. *There exists a normal operator* T *on* H *with*

(1*) $$A_{n',n} = T^{*n'}T^nA_0 \quad for \quad n', n = 0, 1, 2, \ldots$$

if and only if

(iv**) $A^*_{n',n}A_{m',m}=A^*_{0,m'+n}A_{0,n'+m}$ *for* $m', m, n', n=0, 1, \ldots,$ *and*

(ii*) *there exists a constant* $\varkappa'\geqq 0$ *such that* $\|A_{n',n}\|\leqq\varkappa'^{n'+n}$ $(n', n=0, 1, 2, \ldots)$.

Proof. The necessity of the condition is simple so that we omit the details. Assume, on the contrary, that (ii*) and (iv*) are satisfied. Set $A_n=A_{0,n}$ for $n=0, 1, 2, \ldots$. Then we have equality in (v) of Theorem C, and therefore V is an isometry of K into H (see the proof of Theorem C) so that by (9) V is unitary between K and H. As a consequence, $T=VNV^*$ is unitarily equivalent to the normal operator N, hence is itself normal and by (10), satisfies (1*). The proof is complete.

References

[1] Z. Sebestyén, Moment theorems for operators on Hilbert space, *Acta Sci. Math.*, **44** (1982), 165—171.

[2] B. Sz.-Nagy, Extensions of linear transformations in Hilbert space with extend beyond this space, Appendix to F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Ungar (New York, 1960).

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